PARTIALLY HYPERBOLIC ENDOMORPHISMS WITH INVARIANT ANNULI

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ABSTRACT. These notes give a classification of certain invariant annuli for partially hyperbolic endomorphisms.

1. INTRODUCTION

This document establishes the following result stated in [HH20].

Theorem 1.1. *Suppose f is a partially hyperbolic endomorphism of a closed, oriented surface M and that there is an invariant annulus M_0 with the following properties:*

(1) *$f^k(M_0) = M_0$ and f^k restricted to M_0 is a covering map;*
(2) *the boundary components of M_0 are circles tangent to the center direction;*
(3) *no circle tangent to the center direction intersects the interior U_0 of M_0.*

Then, there is an embedding $h : U_0 \to S^1 \times \mathbb{R}$ such that the homeomorphism $h \circ f^k \circ h^{-1}$ from $h(U_0)$ to itself is of the form

$$h f^k h^{-1}(v, s) = (A(v), \phi(v, s))$$

where $A : S^1 \to S^1$ is an expanding linear map and $\phi : h(U_0) \to \mathbb{R}$ is continuous. Moreover, if $v \in \mathbb{T}^2$, then $h^{-1}(v \times \mathbb{R})$ is a curve tangent to E^c_f.

Roughly speaking, the proof is achieved by adapting the proof given in [HP19] by removing the stable direction from all of the arguments. As much as possible, the structure of the proofs including the numbering of propositions and lemmas was left unchanged.

There are a few subtleties introduced by moving to the endomorphism setting. As explained in [HH18], a partially hyperbolic endomorphism f_0 defined on $M = \mathbb{T}^2$ lifts to a diffeomorphism f defined on the universal cover $\tilde{M} = \mathbb{R}^2$ and the diffeomorphism has an invariant splitting $T\tilde{M} = E^u \oplus E^c$. The center bundle E^c is invariant under deck transformations and there is a branching foliation tangent to E^c. There is an f-invariant unstable foliation W^u on the universal cover. In general, the unstable bundle E^u and foliation W^u are not invariant under deck transformations.

In this document, an *unstable leaf* always refers to a leaf of the foliation W^u and an *unstable segment* or *unstable curve* always refers to a curve inside of an unstable leaf. Only briefly in section 7 do we define and use "unstable cone curves" which are more general and defined with respect to the unstable cone family.

2. Dynamics in dimension two

In the original paper, this covers properties of partially hyperbolic surface diffeomorphisms. The results are not needed in the current setting.

3. Branching foliations

We now list a number of properties which hold for all partially hyperbolic endomorphisms in dimension 2. These properties follow by adapting the branching foliation theory developed by Brin, Burago, and Ivanov [BBI04, BI08, BBI09].

A *branching foliation* on a Riemannian surface M is a collection $\mathscr{F}_0$ of immersed curves called *leaves* such that

(1) every leaf is complete under the Riemannian metric pulled back from M,
(2) no two leaves topologically cross,
(3) if a sequence of leaves converges in the compact-open topology, then the limit surface is also a leaf, and
(4) through every point of M there is at least one leaf.

Theorem 3.1. *Let f be a partially hyperbolic endomorphism of a closed surface M such that the unstable cone family $\mathscr{C}^u$ and the center direction E^c are oriented and f preserves these orientations. Then there is a branching foliation $\mathscr{F}_0^c$ on M such that each leaf is tangent to E^c.*

See [BI08] for further details and the proof of theorem 3.1. In the setting of the theorem, let $\tilde{M}$ be the universal cover of M. Lift the branching foliation $\mathscr{F}_0^c$ on M to a branching foliation $\mathscr{F}^c$ on $\tilde{M}$ by taking every possible lift of every leaf. In this paper, we almost exclusively work with the lifted branching foliation on the universal cover and theorem 3.1 may be restated as follows.

Corollary 3.2. *Let f be the lift of a partially hyperbolic endomorphism to the universal cover $\tilde{M}$. Then there is a branching foliation $\mathscr{F}^c$ tangent to E^c on $\tilde{M}$ such that if a deck transformation $\gamma : \tilde{M} \to \tilde{M}$ preserves the orientations of $\mathscr{C}^u$ and E^c on $\tilde{M}$, then γ takes leaves to leaves. Similarly, if f preserves these orientations, then f maps leaves to leaves.*

Note that for a (weakly) partially hyperbolic endomorphism in dimension two, $M = \mathbb{T}^2$, and the endomorphism lifts to a diffeomorphism on the universal cover $\tilde{M} = \mathbb{R}^2$.

The branching foliation on $\tilde{M}$ has the following properties.

Proposition 3.3. *Not applicable to the two-dimensional setting.*

Proposition 3.4. *Each leaf $L \in \mathscr{F}^c$ is a properly embedded curve which separates $\tilde{M}$ into two half spaces. That is, $\tilde{M} \setminus L$ has two connected components L^+ and L^-.*

Proposition 3.5. *Each leaf of $\mathscr{F}^c$ intersects an unstable leaf in at most one point.*

Proposition 3.6. *There is a uniform constant $C > 0$ such that if J is an unstable segment, then* volume $U_1(J) \geq C \cdot \text{length}(J)$.

Here $U_1(J)$ consists of all points in $\tilde{M}$ at distance less than 1 from J.

Proposition 3.7. *Not applicable to the two-dimensional setting.*

Proposition 3.8. *There is a uniform constant $C > 0$ such that if J is a center segment lying inside a leaf of $\mathscr{F}^c$, then* volume $U_1(J) \geq C \cdot \text{length}(J)$.

These can be proved using the Poincaré-Bendixson theorem and adapting arguments proved in [BBI09, Section 3] to the two-dimensional case. See also [HH18, Proposition 2.8].

4. Regions between curves

From now on, assume that $f : M \to M$ is a partially hyperbolic endomorphism on a surface such that there is an invariant annulus M_0 with the following properties:

(1) $f(M_0) = M_0$ and f restricted to M_0 is a covering map;
(2) the boundary components of M_0 are circles tangent to the center direction;
(3) no circle tangent to the center direction intersects the interior of M_0.

We lift to the universal cover $\tilde{M} = \mathbb{R}^2$. Let $\Omega \subset \tilde{M}$ be a closed 2-dimensional submanifold with boundary such that each boundary component of Ω quotients down to a circle in $M = \mathbb{T}^2$ tangent to a center circle and such that no curve which intersects the interior of Ω quotients down to a center circle. This submanifold may then be thought of as a covering space for the annulus M_0. Then, Ω is diffeomorphic to $\mathbb{R} \times I$ where $I \subset \mathbb{R}$ is a compact interval.

It will at times be convenient to use coordinates on Ω and discuss linear maps from Ω to $\mathbb{R}$. Therefore, we simply assume that Ω is equal to $\mathbb{R} \times I$. That is, we treat $\mathbb{R} \times I$ as a subset of $\tilde{M}$ denoted by Ω. The Riemannian metric on $\tilde{M}$ inherited from M may differ from the standard Euclidean metric on $\mathbb{R} \times I$. However, distances and volumes measured with respect to the two metrics differ by at most a constant factor. Therefore, in our analysis, we freely assume that $\Omega = \mathbb{R} \times I$ is equipped with the Euclidean metric.

Since $\mathbb{Z}$ acts on Ω via deck transformations, we adopt the following notation: if $p = (v, s) \in \mathbb{R} \times I$ and $z \in \mathbb{Z}$, then $p + z = (v, s) + z = (v + z, s)$.

As we are assuming $f : M \to M$ maps the annulus to itself, it follows that there is a lift of f to the universal cover which leaves Ω invariant. We also denote this lifted map $\tilde{M} \to \tilde{M}$ by the letter f. Since $f|_\Omega$ quotients down to a map on the annulus, there is a linear map $A : \mathbb{R} \to \mathbb{R}$ such that if $p \in \mathbb{R} \times I$ and $z \in \mathbb{Z}$, then $f(p + z) = f(p) + Az$. By mapping an unstable segment forward by f inside of Ω, one can use a length versus volume argument to show that A is expanding.

That is, there is an integer λ such that $|\lambda| > 1$ and $Av = \lambda v$ for all $v \in \mathbb{R}$. This implies that there is a semiconjugacy between $f|_\Omega$ and A [Fra70]. We list several properties of this semiconjugacy.

Proposition 4.1. *There is a unique continuous surjective map $H : \mathbb{R} \times I \to \mathbb{R}$ and a constant $C > 0$ such that if $p = (v, s) \in \mathbb{R} \times I$ and $z \in \mathbb{Z}$ then*

(1) $Hf(p) = AH(p)$,
(2) $H(p+z) = H(p) + z$, *and*
(3) $\|H(p) - v\| < C$.

Let $\pi : \mathbb{R} \times I \to \mathbb{R}$ be the projection onto the first coordinate. We can also write this as $\pi : \Omega \to \mathbb{R}$.

Now consider a branching foliation $\mathscr{F}^c$ on $\tilde{M}$ as in corollary 3.2. We only know a priori that $\mathscr{F}^c$ is invariant under those deck transformations which preserve the orientations of $\mathscr{C}^u$ and E^c. In particular, for a full-rank subgroup $Z_0 \subset \mathbb{Z}$ it holds that for any $z \in Z_0$, there is a deck transformation $\gamma_z : \tilde{M} \to \tilde{M}$ which preserves the orientations of $\mathscr{C}^u$ and E^c and such that $\gamma_z(p) = p + z$ for all $p \in \mathbb{R}^2 \times I = \Omega$. Therefore if L is a leaf of $\mathscr{F}^c$, then $\gamma_z(L)$ is a leaf of $\mathscr{F}^c$ as well. Replacing f by an iterate, we may freely assume that f preserves these orientations. Then the branching foliation $\mathscr{F}^c$ is invariant under f.

Let Ω° denote the interior of Ω. A major step is to relate the branching foliation $\mathscr{F}^c$ to the semiconjugacy H for points in Ω°.

Proposition 4.2. *For $p, q \in \Omega^\circ$, $H(p) = H(q)$ if and only if there is $L \in \mathscr{F}^c$ such that $p, q \in L$.*

Corollary 4.3. *Not applicable to the two-dimensional setting.*

After these results are established, they are used in section 7 to prove the following.

Proposition 4.4. *If $\gamma : \tilde{M} \to \tilde{M}$ is a deck transformation such that $\gamma(\Omega) = \Omega$, then γ preserves the orientations of $\mathscr{C}^u$ and E^c.*

This shows that Z_0 above may be taken as equal to $\mathbb{Z}$. Section 7 also proves the following characterization of the fibers of the semiconjugacy.

Proposition 4.5. *For every $v \in \mathbb{R}$, the pre-image $H^{-1}(v)$ is a compact segment tangent to E^c. Moreover, $H^{-1}(v)$ intersects each boundary component of Ω in either a point or a compact segment.*

Section 8 uses this to construct the topological conjugacy given in theorem 1.1.

5. CENTER LEAVES

This section gives the proof of proposition 4.2. Let f, Ω, H, and $\mathscr{F}^c$ be as in the previous section.

Proposition 5.1. *For any constant $D > 0$, there is $\ell > 0$ such that any unstable curve $J \subset \Omega$ of length at least ℓ contains points p and q with $|\pi(p) - \pi(q)| > D$.*

Proof. This follows from proposition 3.6. □

Let d_u be distance measured along an unstable leaf and define K^u as the largest subset of Ω for which the following property holds: if $p \in K^u$, $q \in W^u(p)$, and $d_u(p,q) < 1$, then $q \notin \partial\Omega$. In other words, K^u is the set of points at distance at least 1 from $\partial\Omega$ where the distance is measured along the unstable direction.

One can verify that K^u is a closed subset of Ω. In the lifted endomorphism setting, since the unstable foliation W^u is not in general invariant under deck transformations, the set K^u is not invariant either.

As f increases distances measured along the unstable direction, it follows that $f(K^u) \subset K^u$. Note that if J is a compact subset of Ω°, then there is an integer $N(J)$ such that $f^n(J) \subset K^u$ for all $n > N(J)$.

We now consider the intersection of K^u with the leaves of the branching foliation $\mathscr{F}^c$.

Proposition 5.2. *There is $R > 0$ such that if $L \in \mathscr{F}^c$ and $p, q \in K^u \cap L$ then $|\pi(p) - \pi(q)| < R$.*

We prove this by adapting techniques presented in [BBI09, HP14]. The proof is largely topological in nature, instead of involving the dynamics acting on Ω. Therefore, we defer the proof of proposition 5.2 to the appendix. Some adjustments to the proof were made due to the fact that K^u is not invariant under deck transformations.

Lemma 5.3. *No unstable leaf intersects both boundary components of Ω.*

Proof. Note that there is a uniform lower bound on the distance between points in the two boundary components of Ω. If an unstable segment J had endpoints on both boundary components, one could find n such that the length of $f^{-n}(J)$ was smaller than this lower bound, and this would give a contradiction. □

Lemma 5.4. *This lemma is not needed in the two-dimensional setting.*

Proposition 5.5. *In the two-dimensional setting, this is simply a restatement of proposition 5.2: there is $R > 0$ such that if $L \in \mathscr{F}^c$ and $p, q \in K^u \cap L$ then $|\pi(p) - \pi(q)| < R$.*

For a point $p \in \Omega$, define

$$K^-(p) = \{q \in K^u : \pi(q) \le \pi(p) - R\}$$

and

$$K^+(p) = \{q \in K^u : \pi(q) \ge \pi(p) + R\}.$$

Replacing f by f^2 if necessary, we assume that A has a positive eigenvalue. The fact that f is at finite distance from $A \times \mathrm{id}$ then implies that $K^+(f^n(p))$ intersects $f^n(K^+(p))$ for all n.

Proposition 5.6. *If $L \in \mathscr{F}^c$ and $p \in K^u \cap L$, then $\tilde{M} \setminus L$ has connected components L^- and L^+ such that $K^-(p) \subset L^-$ and $K^+(p) \subset L^+$.*

Proof. Proposition 5.5 shows that L is disjoint from both $K^-(p)$ and $K^+(p)$. Therefore, it is enough to show that each of L^- and L^+ intersects at least one of $K^-(p)$ or $K^+(p)$.

Suppose instead that $K^-(p) \cup K^+(p) \subset L^-$. Then for any $n \geq 0$, $K^+(f^n(p))$ intersects $f^n(K^+(p))$ and is therefore a subset of $f^n(L^-)$. Similarly for $K^-(f^n(p))$. Since $p \in K^u \cap L$, the open set $\Omega^\circ \cap L^+$ is non-empty. Let J be a small unstable segment lying in $\Omega^\circ \cap L^+$. Then $f^n(J) \subset K^u$ for all large n. Since $f^n(J) \cap f^n(L^-)$ is empty, the length of $\pi f^n(J)$ is bounded by $2R$ for all large n. However, proposition 5.1 shows that there is no uniform bound on the length of $\pi f^n(J)$ and gives a contradiction. □

Proposition 5.7. *For $p, q \in K^u$, the following are equivalent:*

- $\sup_{n\geq 0} |\pi f^n(p) - \pi f^n(q)| < \infty$, *and*
- *there is $L \in \mathscr{F}^c$ such that $p, q \in L$.*

Proof. One direction follows from proposition 5.5 and the fact that $f(K^u) \subset K^u$. To prove the other direction, suppose $p \in L_p \in \mathscr{F}^c$ and $q \in L_q \in \mathscr{F}^c$. Let L_p^- and L_p^+ be as in the previous proposition and let L_q^- and L_q^+ be the corresponding sets associated to L_q. Assume q does not lie on L_p. Then q lies either in L_p^- or L_p^+. Without loss of generality, assume $q \in L_p^-$. Since L_q is the boundary of L_q^+, it follows that $L_p^- \cap L_q^+ \cap \Omega^\circ$ is a non-empty open set. Consider a small unstable curve J in this set. Then $f^n(J) \subset K^u$ and

$$f^n(J) \cap \big(K^+(f^n(p)) \cup K^-(f^n(q))\big) = \varnothing$$

for all large n. The assumption that $|\pi f^n(p) - \pi f^n(q)|$ is uniformly bounded implies that the length of $\pi f^n(J)$ is uniformly bounded. Proposition 5.1 again gives a contradiction. □

Proof of proposition 4.2. By the properties of the semiconjugacy,

$$\begin{aligned} H(p) = H(q) \quad &\Leftrightarrow \quad \sup_{n\geq 0} |\pi A^n H(p) - \pi A^n H(q)| < \infty \\ &\Leftrightarrow \quad \sup_{n\geq 0} |\pi f^n(p) - \pi f^n(q)| < \infty. \end{aligned}$$

Since $f^n(p)$ and $f^n(q)$ are in K^u for all large n, the result follows from proposition 5.7. □

6. Finding a hidden torus

This section is not needed in the two-dimensional setting.

7. Fibers of the semiconjugacy

This section gives the proofs of propositions 4.4 and 4.5. Let f, Ω, and H be as in section 4.

Lemma 7.1. *An unstable curve intersects $\partial\Omega$ in at most one point.*

Proof. Suppose J is an unstable segment which intersects $\partial\Omega$ at both endpoints. By lemma 5.3, both endpoints must lie on the same boundary component. Then for large n, $f^{-n}(J)$ would be an arbitrarily small unstable curve connecting two points on the same center curve. This is ruled out by the uniform transversality of E^c and E^u. □

Lift the unstable cone family $\mathscr{C}^u$ to $\tilde{M} = \mathbb{R}^2$. We call a C^1 curve J an *unstable cone curve* if its tangent vectors lie inside this cone family.

Since $\mathscr{F}^c$ is a (branching) foliation by lines, a Poincaré-Bendison argument shows that any curve transverse to the center direction must intersect a center leaf in at most one point. This generalizes proposition 3.5.

Lemma 7.2. *If J is an unstable cone curve in Ω, then $H|_J$ is a homeomorphism to its image.*

Here, the curve J may be bounded or unbounded and may or may not include its endpoints.

Proof. First, consider the case where J is in the interior of Ω. By proposition 4.2, the fibers of H are center leaves and, by the argument above, each center leaf intersects an unstable cone curve at most once, so $H|_J$ is injective. Since H is continuous, this implies that $H|_J$ is a homeomorphism to its image.

Note that if $\phi : [0,1) \to \mathbb{R}$ is a continuous function and $\phi|_{(0,1)}$ is an embedding, then ϕ itself must also be an embedding. Therefore, in the case where J intersects $\partial\Omega$ in a point, the fact that the restriction of H to $J \setminus \partial\Omega$ is injective implies that H is injective on all of J.

□

Proof of proposition 4.4. For a point $x \in \Omega^\circ$, let $J \subset \Omega^\circ$ be a short unstable cone curve passing through x. Using that $H|_J$ is injective, define the orientation for E^u near x so that H increases along J. This gives a well-defined continuous orientation of E^u on all of Ω°.

Suppose $\gamma : \tilde{M} \to \tilde{M}$ is a deck transformation mapping $\Omega = \mathbb{R} \times I$ to itself. By the properties of the semiconjugacy, there is $z \in \mathbb{Z}$ such that $H\gamma(x) = H(x) + z$ for all $x \in \Omega^\circ$. Hence, γ preserves the orientation of E^u. By assumption, the original closed manifold M is orientable. Therefore, γ preserves the orientation of $T\tilde{M}$ and must also preserve the orientation of E^c. □

Lemma 7.3. *For $x \in \partial\Omega$, the set $H(W^u(x) \cap \Omega)$ is equal either to $(-\infty, H(x)]$ or $[H(x), +\infty)$.*

Proof. On the closed manifold M, the set of unstable cone curves of length exactly one is a compact set under the C^1 topology. For $x \in \partial\Omega$, let $J_1(x) \subset W^u(x)$ be the compact unstable segment which starts at x, is directed into Ω, and has length exactly one. By lemma 7.2, $H(J_1(x))$ always has positive length. By the compactness above, one can further show that there is $\delta > 0$ such that the length of $H(J_1(x))$ is greater than δ for all x. One may show that

$$\text{length}\, Hf^n J_1(x) > \lambda^n \delta$$

for all $x \in \partial\Omega$ and $n \geq 0$ where $\lambda > 1$ is the unstable eigenvalue of A. Lemma 7.1 implies that $f^n J_1(f^{-n}(x))$ is a subset of $W^u(x) \cap \Omega$ for all n. Thus, the length of $H(W^u(x) \cap \Omega)$ is unbounded. ☐

Lemma 7.4. *Let S be a connected component of $\partial\Omega$ and let*

$$U = \Omega \cap \bigcup_{x \in S} W^u(x).$$

For any $v \in \mathbb{R}$, $H^{-1}(v) \cap U$ is a topological ray. That is, there is a proper topological embedding $\beta : [0, +\infty) \to U$ such that the image is $H^{-1}(v) \cap U$.

Proof. Since S is invariant under deck transformations, if $t \in H(S)$, then $t + z \in H(S)$ for all $z \in \mathbb{Z}$. Using this, one may show that $H|_S$ is surjective. This implies that $H^{-1}(v) \cap U$ is non-empty. Let S be parameterized by a function $\alpha : \mathbb{R} \to S$.

As S is connected, exactly one of the two cases in lemma 7.3 holds for all $x \in S$. Without loss of generality, assume the case $H(W^u(x) \cap \Omega) = (-\infty, H(x)]$ holds. The Franks semiconjugacy $H|_S \circ \alpha$ from $\mathbb{R}$ to $\mathbb{R}$ is monotonic (in the non-strict sense); this holds because it is constructed as the C^0 limit of monotonic functions [Fra70]. That is, up to composing α by an affine map on $\mathbb{R}$, we may assume α is defined so that $H\alpha(t) \geq v$ exactly when $t \geq 0$. Define a map $\beta : [0, +\infty) \to U$ by setting $\beta(t)$ to be the unique point in $W^u(\alpha(t))$ which satisfies $H\beta(t) = v$. Proving that β is continuous reduces to proving the following claim.

> **Claim.** Suppose $h : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ is a continuous function with the properties that $x_1 \leq x_2$ implies $h(x_1, 0) \leq h(x_2, 0)$ and $y_1 < y_2$ implies $h(x, y_2) < h(x, y_1)$. Then, any level set of h is the graph of a continuous function.

The proof of the claim is left to the reader. In fact, the proof is highly similar in form to steps used in proving the implicit function theorem.

It is clear that β is injective. Suppose a sequence $\{t_k\}$ tending to $+\infty$ is such that $\beta(t_k)$ converges to a point $x \in U$. Since $W^u(x)$ intersects S, one may use an unstable foliation chart in a neighbourhood of x, to derive a contradiction. This shows that β is proper. ☐

Proof of proposition 4.5. Consider $v \in \mathbb{R}$. By lemma 7.4, there is at least one point $x \in \Omega^\circ$ such that $H(x) = v$. Let L be a leaf of $\mathscr{F}^c$ passing through x. By proposition 3.4, L is a properly embedded line. Since H is proper, $H^{-1}(v)$ is a compact subset of Ω, and the ends of L must eventually leave the interior of Ω. As such, there is a compact center segment $J \subset L$ such that the endpoints of J lie on $\partial\Omega$ and all other points of J lie in the interior of Ω. Lemma 7.4 then describes the exact shape of J near the boundary of Ω. In particular, one sees that the two endpoints of J cannot lie on the same boundary component, and so each of the two boundary components contains exactly one endpoint of J.

Now suppose another connected component of $L \cap \Omega$ intersected the interior of Ω. This would lead to a center segment J' disjoint from J but such that $H(J) = H(J')$ and where each boundary component of Ω contained exactly one endpoint of J'. Lemma 7.4 would then imply that J and J' coincide near the boundary of Ω, a contradiction. Thus $H^{-1}(v)$ consists of J together with the pre-images of v on the two boundary components of Ω. Since H restricted to a boundary component S is a proper monotonic map, the pre-image $H|_S^{-1}(v)$ is either a point or a compact segment. Using lemma 7.4, the result follows. □

Corollary 7.5. *There is a uniform upper bound on the length of a fiber $H^{-1}(v)$.*

Proof. As H is proper and commutes with deck transformations, there is a uniform upper bound on the diameter of $H^{-1}(v)$ independent of $v \in \mathbb{R}$. There is then a uniform upper bound on the volume of $U_1(H^{-1}(v))$ and the result follows from proposition 3.8. □

8. Building the ragged leaf conjugacy

Lemma 8.1. *There is a continuous function $p : \Omega^\circ \to (0,1)$ such that for any center segment of the form $J = H^{-1}(v) \cap \Omega^\circ$, the restriction $p|_J$ is a C^1 embedding. Moreover, with respect to arc length, $p|_J$ has a uniform speed independent of the choice of J.*

Proof. Let S_0 and S_1 be the two boundary components of Ω. For $x \in \Omega^\circ$, define

$$p_0(x) = \frac{\operatorname{dist}(x, S_0)}{\operatorname{dist}(x, S_0) + \operatorname{dist}(x, S_1)}.$$

Extend p_0 to a continuous function $p_0 : \tilde{M} \to [0,1]$ by requiring it to be locally constant outside of Ω.

Now suppose $x \in \Omega^\circ$ and let $L \in \mathscr{F}^c$ be the center leaf through x. Let $\alpha : \mathbb{R} \to L$ be a parameterization by arc length of this center curve. By proposition 3.7, $\alpha(\mathbb{R})$ is a complete curve properly embedded in $\tilde{M}$. Let $T > 0$ be the upper bound given by corollary 7.5 and for any $t \in \mathbb{R}$, define

$$p(\alpha(t)) = \frac{1}{2T} \int_{t-T}^{t+T} p_0(\alpha(s))\, ds.$$

If $\alpha(t) \in \Omega^\circ$, then neither $\alpha(t-T)$ nor $\alpha(t+T)$ lies in Ω°. Up to possibly reversing the parameterization, it follows that $\alpha(t-T) = 0$ and $\alpha(t+T) = 1$ and by the Fundamental Theorem of Calculus

$$\frac{d}{dt} p(\alpha(t)) = \frac{1}{2T}.$$ □

Lemma 8.2. *If $x, y \in \Omega^\circ$ with $y \in W^u(x)$, then $H(x) = H(y)$ if and only if $x = y$.*

Proof. Assume $H(x) = H(y)$. By proposition 4.2, x and y lie on the same leaf L of $\mathscr{F}^c$. The uniqueness given by proposition 3.5 implies that $x = y$. □

Proof of theorem 1.1. Define $h = H \times p : \Omega^\circ \to \mathbb{R}^2$. This is a continuous map and injective by proposition 4.5 and lemma 8.1. We now show that h is an open map. Consider $x \in \Omega^\circ$ and assume that E^u, and E^c are oriented. With respect to these orientations, define unit speed flow φ^u along the unstable direction. Define φ^c as the unit speed flow along the fibers of H with the direction given by the orientation of E^c. Define $i : [-\epsilon,\epsilon]^2 \to \Omega^\circ$ by

$$i(t_1,t_2) = \varphi^c_{t_2}\varphi^u_{t_1}(x)$$

where ϵ is small enough that the range of i is contained in Ω°. By lemmas 8.1 and 8.2, the range of $h \circ i$ contains $h(x)$ in its interior. Taking ϵ to zero, one can show that for any neighbourhood V of x, $h(V)$ is a neighbourhood of $h(x)$ and therefore h is open. It follows that h is a homeomorphism to its image, $U := h(\Omega^\circ)$.

Note that we used E^u here, which in the endomorphism case is not invariant under deck transformations. However, E^u is only used to establish that h is open and does not affect the proof that h is invariant under deck transformations.

As H is a semiconjugacy, the homeomorphism $hfh^{-1} : U \to U$ is of the form

$$hfh^{-1}(v,s) = (A(v),\phi(v,s))$$

where $A : \mathbb{R} \to \mathbb{R}$ is the expanding linear map and $\phi : U \to \mathbb{R}$ is some continuous function. By construction, the maps H, p, and therefore h are $\mathbb{Z}$-equivariant. Thus, h quotients down to an embedding $U_0 \to \mathbb{T}^2 \times \mathbb{R}$ which satisfies the properties listed in item (4) of theorem 1.1. □

Appendix A. Bounds on foliations

This appendix proves proposition 5.2. Let $\mathscr{F}^c$, $\Omega = \mathbb{R} \times I$, and $Z_0 \subset \mathbb{Z}$ be as in section 4. Let S be a leaf of $\mathscr{F}^c$ which intersects the interior Ω° of Ω. Here, we use S instead of L to keep closer to the notation of [BBI09]. As S is properly embedded, the complement $\tilde{M} \setminus S$ consists of two open connected components S_+ and S_- where the oriented E^u bundle points into S_+. Define $S_+ + Z_0 = \{p+z : p \in S_+, z \in Z_0\}$ and similarly for $S_- + Z_0$.

Lemma A.1. *The set $\Omega \setminus \partial\Omega$ is contained in both $S_+ + Z_0$ and $S_- + Z_0$.*

Proof. As the branching foliation is complete in the compact-open topology, one can show that the boundary $\partial\tilde{X}$ of $\tilde{X} = S_+ + Z_0$ is a union of leaves of $\mathscr{F}^c$. (See the proof of Lemma 3.10 in [BBI09] for details.)

Consider the manifold $\hat{M}$ defined by the quotient $\tilde{M}/Z_0$. Then $\tilde{X}$ quotients down to a subset $\hat{X} \subset \hat{M}$ and $\partial\tilde{X}$ quotients down to $\partial\hat{X}$. In particular, $\partial\hat{X}$ is closed subset of $\hat{M}$ and the orientation of E^u shows that $\partial\hat{X}$ does not accumulate on itself.

Let $\hat{\Omega}$ be the quotient of Ω to $\hat{M}$. Its boundary consists of two center circles.

The boundary of the intersection $\hat{\Omega} \cap \hat{X}$ is tangent to the center direction and consists of circles. If a boundary component intersects the interior of $\hat{\Omega}$, this would contradiction the assumption on Ω.

Note that $\hat{M}$ may be a finite cover of M, but an embedded circle tangent to the center direction will quotient down to an immmersed circle on M that contains and embedded center circle as a subset. □

Lemma A.2. *Up to a change of orientation, we may assume that* $\pi(z) \geq 0$ *implies* $S_+ + z \subset S_+$ *and* $\pi(z) \leq 0$ *implies* $S_+ \subset S_+ + z$ *for* $z \in Z_0$.

Proof. This is shown by adapting the proofs of Lemmas 3.8 to 3.12 in [BBI09]. □

At this point, we need to adjust the proof slightly due to the fact that K^u may not be invariant under deck transformations. However, one can see from the definition of K^u given in section 5 that there is a uniform lower bound on the distance between points in K^u and $\partial\Omega$. Consequently, the closed set defined by $K = \overline{K^u + \mathbb{Z}}$ satisfies $K^u \subset K \subset \Omega \setminus \partial\Omega$ and is invariant under deck transformations.

Let Q be the compact set defined by intersecting K with a set of the form $[0, N] \times I$ for some large N. Since Z_0 is a full rank subgroup of $\mathbb{Z}$, N may be chosen large enough that any $x \in K$ can be written as $x = q + z$ with $q \in Q$ and $z \in Z_0$.

Lemma A.3. *There is* $z_0 \in Z_0$ *such that* $Q \subset S_+ - z_0$ *and* $Q \subset S_- + z_0$.

Proof. By lemma A.1, $\{S_+ - z : z \in Z_0\}$ is an open cover of K, and so some finite subset $\{S_+ - z_1, \dots, S_+ - z_n\}$ covers the compact set Q. Take z_0 such that $\pi(z_0) \geq \pi(z_i)$ for all i. The case for S_- is analogous. □

By abuse of notation, if $p = (v, s) \in \mathbb{R} \times I$ define $\pi(p) = \pi(v)$.

Lemma A.4. *There is* $r > 0$ *such that* $\pi(x) > r$ *implies* $x \in S_+$ *and* $\pi(x) < -r$ *implies* $x \in S_-$ *for all* $x \in K$.

Proof. Choose $r > 0$ such that $r - \pi(q) > \pi(z_0)$ for all $q \in Q$. Any $x \in K$ may be written as $x = q + z$ with $q \in Q$ and $z \in Z_0$. If $\pi(x) > r$, then $\pi(z - z_0) \geq 0$ and $x \in Q + z \subset S_+ - z_0 + z \subset S_+$. □

Lemma A.5. *There is* $R > 0$ *such that if* $p, q \in K$ *lie on the same leaf of* $\mathscr{F}^c$, *then* $|\pi(p) - \pi(q)| < R$.

Proof. Without loss of generality, shift p and q by an element of Z_0 and assume $q \in Q$. Let S' be the leaf containing both p and q. Since S' intersects $S_+ - z_0$ and leaves do not topologically cross, S' is disjoint from $S_- - z_0$ and so $\pi(p), \pi(q) > -r - \pi(z_0)$. Similarly, $\pi(p), \pi(q) < r + \pi(z_0)$. Take $R = 2(r + \pi(z_0))$. □

This concludes the proof of proposition 5.2.

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