CLASSIFICATION OF SYSTEMS WITH CENTER-STABLE TORI

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ABSTRACT. This paper gives a classification of partially hyperbolic systems in dimension 3 which have at least one torus tangent to the center-stable bundle.

1. INTRODUCTION

A long-standing question in the study of partially hyperbolic dynamical systems was whether a system with one-dimensional center possessed a foliation tangent to that center direction. This question was recently answered by Rodriguez Hertz, Rodriguez Hertz, and Ures, who constructed a partially hyperbolic diffeomorphism on the 3-torus without such a center foliation [RHRHU16]. Crucial to their construction is a 2-torus embedded in the manifold tangent to the center-stable direction. In this paper, we give a classification of all 3-dimensional partially hyperbolic systems with center-stable tori.

Previous classification results relied on the notion of a leaf conjugacy between the center foliations of two different partially hyperbolic systems [HP17]. In certain manifolds, such as the 3-torus, the presence of a center-stable or center-unstable torus is the only potential obstruction to having an invariant center foliation [Pot15]. In the current setting, the lack of a center foliation in general means that it is not possible to use a global leaf conjugacy to classify the dynamics. Instead, we first remove all of the center-stable and center-unstable tori from the system leaving dynamics defined on an open manifold. Looking at the dynamics on each of the connected components, we show that it has the form of a topological skew product.

Before giving the full result, we state the definitions of partial hyperbolicity and related concepts. A diffeomorphism f of a closed connected manifold M is *partially hyperbolic* if there is a splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u$$

such that each subbundle is non-zero and invariant under the derivative Df and

$$\|Df v^s\| < \|Df v^c\| < \|Df v^u\| \quad \text{and} \quad \|Df v^s\| < 1 < \|Df v^u\|$$

for all $x \in M$ and unit vectors $v^s \in E^s(x)$, $v^c \in E^c(x)$, and $v^u \in E^u(x)$. There exist unique foliations W^s and W^u tangent to E^s and E^u, but in general there does not exist a foliation tangent to E^c. A *center-stable torus* is an embedded torus tangent

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to $E^{cs} = E^c \oplus E^s$, and a *center-unstable torus* is an embedded torus tangent to $E^{cu} = E^c \oplus E^u$. We also refer to these objects as cs and cu-tori.

The definition of partial hyperbolicity above is sometimes called *pointwise* partial hyperbolicity, in comparison to a stricter condition called *absolute* partial hyperbolicity. In dimensional three, it is not possible for an absolutely partially hyperbolic system to have a cs or cu-torus [BBI09, HP15]. Therefore, this paper only uses the pointwise definition of partial hyperbolicity.

Theorem 1.1. *Suppose f is a partially hyperbolic diffeomorphism of a closed, oriented 3-manifold M which has at least one center-stable or center-unstable torus. Then,*

(1) *there is a finite and pairwise disjoint collection $\{T_1, T_2, \dots, T_n\}$ of all center-stable and center-unstable tori,*
(2) *every connected component U_i of $M \setminus (T_1 \cup \dots \cup T_n)$ is homeomorphic to $\mathbb{T}^2 \times (0,1)$,*
(3) *there is $k \ge 1$ such that f^k maps each T_i to itself and each U_i to itself, and*
(4) *for each $U_i \subset M$, there is an embedding $h : U_i \to \mathbb{T}^2 \times \mathbb{R}$ such that the homeomorphism $h \circ f^k \circ h^{-1}$ from $h(U_i)$ to itself is of the form*

$$h f^k h^{-1}(v,s) = (A(v), \phi(v,s))$$

where $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a hyperbolic toral automorphism and $\phi : h(U_i) \to \mathbb{R}$ is continuous. Moreover, if $v \in \mathbb{T}^2$, then
   (a) *$h^{-1}(v \times \mathbb{R})$ is a curve tangent to E^c_f,*
   (b) *$h^{-1}(W^s_A(v) \times \mathbb{R})$ is a surface tangent to E^{cs}_f, and*
   (c) *$h^{-1}(W^u_A(v) \times \mathbb{R})$ is a surface tangent to E^{cu}_f.*

The first three items of the theorem state previously known results. The proof of item (1) is given in [Ham16b] and item (2) is a restatement of the work of Rodriguez Hertz, Rodriguez Hertz, and Ures to classify which 3-manifolds allow tori with hyperbolic dynamics [RHRHU11]. Item (3) follows as a consequence of items (1) and (2). Therefore, the substance and novelty of theorem 1.1 lies in item (4).

The theorem shows that there is a form of conjugacy between the dynamics on a region U_i, and a skew product over an Anosov map on $\mathbb{T}^2$. However, this is complicated by the fact that the dynamics on the cs and cu-tori may not be Anosov in general and could contain sinks or sources [Ham16a]. In such cases, the boundary components of $h(U_i)$ will not be the graphs of continuous functions from $\mathbb{T}^2$ to $\mathbb{R}$. The map h might be thought of as a "ragged leaf conjugacy" between f and $A \times \mathrm{id}$ since h maps the smooth subset U_i of M to a subset of $\mathbb{T}^2 \times \mathbb{R}$ with a ragged boundary. See figure 1. Nevertheless, the center direction E^c may still be accurately described in a neighbourhood of a cs or cu-torus. See section 7 for details.

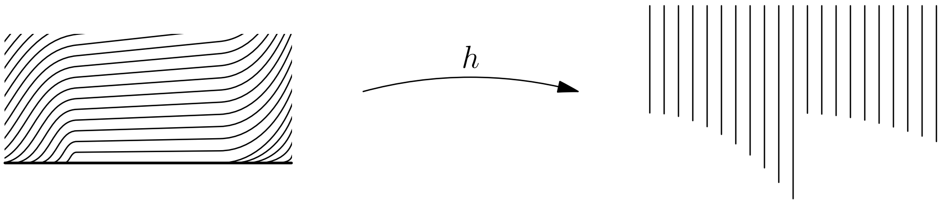

FIGURE 1. The "ragged leaf conjugacy" given by theorem 1.1. The left side of the figure depicts curves tangent to the center direction E^c in a region U_i as they approach a cs-torus which has a sink. This cs-torus is depicted by a thick line at the bottom of the left side. Each center curve is mapped by h to a vertical segment in $\mathbb{T}^2 \times \mathbb{R}$ and these segments are shown at right. As a consequence of results in section 7, the center curves must "skip over" the basin of the sink as they approach the cs-torus, and for a point $x \in U_i$, the distance along the center curve from x to the torus is discontinuous in x. As a result, the lengths of the vertical segments are also discontinuous.

Also note that the curves and surfaces given in item (4) are incomplete with respect to the Riemannian metric induced from M.

At a coarse level, the main steps of proving theorem 1.1 are similar those of previous classification results [Ham13a, Ham13b, HP14]. (See also the recent survey [HP17].) By the work of Brin, Burago, and Ivanov, there are branching foliations tangent to E^{cs} and E^{cu} on M [BI08, BBI09]. We restrict these branching foliations to one of the components U_i and consider the center curves given by intersecting leaves of the two branching foliations. By analyzing the interaction of the dynamics with the branching foliations, we show that these center curves correspond to fibers of the semiconjugacy given by Franks [Fra70]. Then, using an averaging method along center leaves, we construct the function h.

Two major complications to applying these steps in the current context are that U_i is not compact, and that the leaves of the branching foliations have tangencies with the boundary of U_i. To handle these complications, we consider the dynamics and the branching foliations both on the closure of U_i and on compact subsets in the interior of U_i. We also lift these compact subsets to the universal cover and mainly do analysis there.

To begin, section 2 gives a detailed description of the 2-dimensional dynamics possible on a cs or cu-torus. Section 3 introduces branching foliations and states a number of properties which hold for all partially hyperbolic systems in dimension three. Section 4 states properties specific to systems containing a cs or cu-torus and introduces a number of important propositions which are then proved in sections 5, 6, and 7. Section 8 then uses these to prove theorem 1.1. As part of the overall proof, we need a result on the structure of branching foliations on $\mathbb{T}^2 \times [0, 1]$ and this is given in an appendix.

2. Dynamics in dimension two

In order to understand 3-dimensional systems with cs and cu-tori, it is necessary to fully understand the 2-dimensional dynamics acting on these tori. In this section, assume $g : \mathbb{T}^2 \to \mathbb{T}^2$ is a weakly partially hyperbolic diffeomorphism with a splitting of the form $T\mathbb{T}^2 = E^c \oplus E^s$. To be precise, each of the one-dimensional subbundles E^c and E^s is invariant under the derivative Dg and

$$\|Dgv^s\| < \|Dgv^c\| \quad \text{and} \quad \|Dgv^s\| < 1$$

hold for all $x \in \mathbb{T}^2$ and unit vectors $v^s \in E^s(x)$ and $v^c \in E^c(x)$.

Lift g to a map on the universal cover. By a slight abuse of notation, we also denote the lifted map $\mathbb{R}^2 \to \mathbb{R}^2$ by the letter g. For the remainder of the section, we only consider the lifted dynamics on $\mathbb{R}^2$. This type of dynamical system is analyzed in detail in [Pot12, Section 4.A] and proofs of the next four propositions may be found there.

Proposition 2.1. *There is a hyperbolic linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ at finite distance from g.*

Proposition 2.2. *There is a unique $\mathbb{Z}^2$-invariant, g-invariant foliation W^c_g tangent to E^c_g.*

Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be the Franks semiconjugacy [Fra70]. That is, H is the unique continuous surjective map such that $AH(x) = Hg(x)$ and $H(x+z) = H(x)+z$ for all $x \in \mathbb{R}^2$ and $z \in \mathbb{Z}^2$. This implies that H is a finite distance from the identity map.

Proposition 2.3. *For $x, y \in \mathbb{R}^2$, $y \in W^c_g(x)$ if and only if $H(y) \in W^u_A(H(x))$.*

Proposition 2.4. *For $x, y \in \mathbb{R}^2$, the curves $W^c_g(x)$ and $W^s_g(y)$ intersect in exactly one point.*

The next result concerns unique integrability of the center and is proved in [PS07].

Proposition 2.5. *Any curve tangent to E^c_g lies inside a leaf of W^c_g.*

Remark. The proof given in [PS07] has a small typo which could be a source of confusion to the reader. In the equation $f^{-n}(J_1) \subset W^s_K(f^{-n}(J_1))$ near the end of section 4.2 of that paper, the J_1 on the left should actually be J_2.

We now state and prove several additional results which will be needed later in this paper.

Proposition 2.6. *For $x, y \in \mathbb{R}^2$, if $y \in W^s_g(x)$, then $H(y) \in W^s_A(H(x))$.*

Proof. As H is uniformly continuous, $d(g^n(x), g^n(y)) \to 0$ implies that

$$d(A^n H(x), A^n H(y)) = d(Hg^n(x), Hg^n(y)) \to 0.$$

This occurs exactly when $H(y) \in W^s_A(H(x))$. □

Note that proposition 2.3 has an "if and only if" condition and proposition 2.6 does not.

Let $\pi^u, \pi^s : \mathbb{R}^2 \to \mathbb{R}$ be linear maps such that $\ker \pi^u$ is the stable leaf of A which passes through the origin and $\ker \pi^s$ is the unstable leaf. Define $H^u = \pi^u \circ H$ and $H^s = \pi^s \circ H$.

Proposition 2.7. *For any stable leaf L of g, the restriction of H^s to L is a homeomorphism from L to $\mathbb{R}$.*

Proof. Using propositions 2.3 and 2.4 for $x, y \in L$,

$$H^s(y) = H^s(x) \quad \Leftrightarrow \quad H(y) \in W^u_A(H(x)) \quad \Leftrightarrow \quad y \in W^c_g(x) \quad \Leftrightarrow \quad y = x.$$

This shows that $H^s|_L$ is injective. If L accumulated on a point $y \in \mathbb{R}^2$, then $W^c_g(y)$ would intersect L in multiple points, contradicting proposition 2.4. Hence, L is properly embedded. If $\{x_n\}$ is a sequence in L such that $\|x_n\| \to \infty$, then $\|Hx_n\| \to \infty$. As proposition 2.6 implies that $H^u x_n$ is constant, it must be that $|H^s x_n| \to \infty$. From this, one may show that $H^s|_L$ is surjective. $\square$

In general, the restriction of H^u to a center leaf will not be a homeomorphism. However, it is still monotonic, as we show after first establishing a few lemmas.

Lemma 2.8. *There is $R > 0$ such that for any $x \in \mathbb{R}^2$, the set $\mathbb{R}^2 \setminus W^s_g(x)$ has connected components S^-_x and S^+_x satisfying*

$$\{y \in \mathbb{R}^2 : H^u(y) < H^u(x) - R\} \subset S^-_x \quad \textit{and} \quad \{y \in \mathbb{R}^2 : H^u(y) > H^u(x) + R\} \subset S^+_x.$$

Proof. By proposition 2.6 and the fact that H is a finite distance from the identity, there is a uniform constant $R_0 > 0$ such that $|\pi^u(p) - \pi^u(q)| < R_0$ for any two points p, q on the same stable leaf of g. Hence, the components S^-_x and S^+_x may be labelled so that

$$\{y \in \mathbb{R}^2 : \pi^u(y) < \pi^u(x) - R_0\} \subset S^-_x \quad \text{and} \quad \{y \in \mathbb{R}^2 : \pi^u(y) > \pi^u(x) + R_0\} \subset S^+_x.$$

As H^u is a finite distance from π^u, the desired result holds. $\square$

Lemma 2.9. *If $k > 0$ is even, then $f^k(S^-_x) = S^-_{f^k(x)}$ and $f^k(S^+_x) = S^+_{f^k(x)}$.*

Proof. Let λ denote the unstable eigenvalue of A. Then $AH = Hg$ implies that $\lambda H^u = H^u g$. As $\lambda^k > 1$ and H is surjective, there is a point $y \in \mathbb{R}^2$ such that both $H^u(y) > H^u(x) + R$ and $\lambda^k H^u(y) > \lambda^k H^u(x) + R$. Then $g^k(S^+_x)$ and $S^+_{g^k(x)}$ intersect at $g^k(y)$ and the two sets are therefore equal. $\square$

Lemma 2.10. *If $y \in S^+_x$, then $H^u(y) \geq H^u(x)$.*

Proof. Suppose $y \in S^+_x$ and $\delta := H^u(x) - H^u(y) > 0$. Then $g^k(y) \in S^+_{g^k(x)}$ and

$$H^u(y) = H^u(x) - \lambda^k \delta > H^u(x) - R$$

for large positive even k. This gives a contradiction. $\square$

Proposition 2.11. *Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ be a parameterization of a center leaf of g. Then $H^u \circ \alpha$ is monotonic. That is, up to possibly replacing α with the reverse parameterization, $H^u\alpha(s) \le H^u\alpha(t)$ holds for all $s \le t$.*

Proof. This follows from proposition 2.4 and lemma 2.10. □

Corollary 2.12. *For each $v \in \mathbb{R}^2$, $H^{-1}(v)$ is either a point or a compact curve lying inside a center leaf.*

Proof. This follows from propositions 2.3 and 2.11, and the fact that H is a finite distance from the identity. □

3. BRANCHING FOLIATIONS

We now list a number of properties which hold for all partially hyperbolic diffeomorphisms in dimension 3. These properties follow from the branching foliation theory developed by Brin, Burago, and Ivanov [BBI04, BI08, BBI09].

A *branching foliation* on a Riemannian 3-manifold M is a collection $\mathscr{F}_0$ of immersed surfaces called *leaves* such that

(1) every leaf is complete under the Riemannian metric pulled back from M,
(2) no two leaves topologically cross,
(3) if a sequence of leaves converges in the compact-open topology, then the limit surface is also a leaf, and
(4) through every point of M there is at least one leaf.

Theorem 3.1. *Let f be a partially hyperbolic diffeomorphism of a closed 3-manifold M such that E^u, E^c, and E^s are oriented and f preserves these orientations. Then there is a branching foliation $\mathscr{F}_0^{cs}$ on M such that each leaf is tangent to E^{cs}.*

See [BI08] for further details and the proof of theorem 3.1. In the setting of the theorem, let $\tilde{M}$ be the universal cover of M. Lift the branching foliation $\mathscr{F}_0^{cs}$ on M to a branching foliation $\mathscr{F}^{cs}$ on $\tilde{M}$ by taking every possible lift of every leaf. In this paper, we almost exclusively work with the lifted branching foliation on the universal cover and theorem 3.1 may be restated as follows.

Corollary 3.2. *Let f be the lift of a partially hyperbolic diffeomorphism to the universal cover $\tilde{M}$. Then there is a branching foliation $\mathscr{F}^{cs}$ tangent to E^{cs} on $\tilde{M}$ such that if a deck transformation $\gamma : \tilde{M} \to \tilde{M}$ preserves the orientations of E^u, E^c, and E^s as bundles over $\tilde{M}$, then γ takes leaves to leaves. Similarly, if f preserves these orientations, then f maps leaves to leaves.*

The branching foliation on $\tilde{M}$ has the following properties, as proved in [BBI09, Section 3].

Proposition 3.3. *Each leaf $L \in \mathscr{F}^{cs}$ is saturated by stable leaves. That is, if $x \in L$, then $W^s(x)$ is a subset of L.*

Proposition 3.4. *Each leaf $L \in \mathscr{F}^{cs}$ is a properly embedded plane which separates $\tilde{M}$ into two half spaces. That is, $\tilde{M} \setminus L$ has two connected components L^+ and L^-.*

Proposition 3.5. *Each leaf of $\mathscr{F}^{cs}$ intersects an unstable leaf in at most one point.*

Proposition 3.6. *There is a uniform constant $C > 0$ such that if J is an unstable segment, then* volume $U_1(J) \geq C \cdot \text{length}(J)$.

Here $U_1(J)$ consists of all points in $\tilde{M}$ at distance less than 1 from J.

Applying theorem 3.1 to f^{-1}, there is also a branching foliation tangent to E^{cu}. Let $\mathscr{F}^{cu}$ denote the lifted branching foliation on $\tilde{M}$.

Proposition 3.7. *If L^{cs} is a leaf of $\mathscr{F}^{cs}$ and L^{cu} is a leaf of $\mathscr{F}^{cu}$, then any connected component of the intersection $L^{cs} \cap L^{cu}$ is a topological line which is properly embedded in $\tilde{M}$.*

See [HP15, Lemma 6.1] for a proof of proposition 3.7.

Proposition 3.8. *There is a uniform constant $C > 0$ such that if J is a center segment lying inside a leaf of $\mathscr{F}^{cs}$, then* volume $U_1(J) \geq C \cdot \text{length}(J)$.

Proof. Suppose J is a compact curve tangent to E^c lying in a leaf L of $\mathscr{F}^{cs}$. Further suppose J intersects a stable leaf in two distinct points. Then there is a circle which is the concatenation of a center segment and a stable segment, and as L is diffeomorphic to $\mathbb{R}^2$, the Jordon curve theorem implies that this circle bounds a disk, D. We may assume that E^s is oriented so that everywhere along the boundary of D, E^s is either tangent to the boundary or transerve to the boundary and pointing into D. Then, as in the proof of the Poincaré–Bendixson theorem, one can show that the flow along the stable direction either has a closed orbit or a fixed point in D. Since the stable foliation only consists of lines, this gives a contradiction.

Hence, $J \cap W^s(x) = \{x\}$ for all $x \in J$. By proposition 3.5,

$$J \cap \bigcup_{y \in W^s(x)} W^u(y) = \{x\}$$

for all $x \in J$ as well. Using this, one may adapt the proof of proposition 3.6 given in [BBI09] to apply to center segments. ☐

4. Regions between tori

From now on, assume that $f : M \to M$ is a partially hyperbolic diffeomorphism on a closed oriented 3-manifold and that there is at least one cs or cu-torus. We first consider dynamics on the closed manifold M, but later in this section we lift to the universal cover.

Proposition 4.1. *No cs-torus intersects a cu-torus.*

Proof. As E^{cs} and E^{cu} are transverse, such an intersection would consist of circles tangent to E^c. A center circle is ruled out by proposition 2.3. ☐

Proposition 4.2. *No two distinct cs-tori intersect.*

Note that since E^{cs} is not uniquely integrable, it is a priori possible for two cs-tori to intersect without coinciding. This possibility is explored in [Ham16b] where a proof of proposition 4.2 is given. The proof relies heavily on branching foliations and other tools specific to dimension three, and is much more involved than the simple proof of proposition 4.1 above.

Let $\mathscr{T} \subset M$ be the union of all cs and cu-tori. By the above propositions, $\mathscr{T}$ consists of disjoint tori and [Ham16b] shows that there are only finitely many. Let $\{M_i\}$ be the collection of compact manifolds with boundary obtained by cutting M along $\mathscr{T}$. Since $f(\mathscr{T}) = \mathscr{T}$, there is an iterate of f^k which maps each M_i to itself and each torus in $\mathscr{T}$ to itself. For simplicity, we replace f by an iterate and assume $k = 1$. Throughout the proof of theorem 1.1, we freely replace f by an iterate when convenient.

Proposition 4.3. *Each M_i is diffeomorphic to* $\mathbb{T}^2 \times [0,1]$.

Proof. This is basically a restatement of the main result of [RHRHU11]. Since M supports a partially hyperbolic diffeomorphism, it is irreducible. Hence, M_i is irreducible. Let T be a boundary component of M_i. Proposition 2.1 implies that $f|_T$ is homotopic to a hyperbolic toral automorphism and [RHRHU11, Theorem 2.2] implies that T is incompressible. As such, $f|_{M_i}$ and T satisfy the conditions of [RHRHU11, Theorem 1.2], which implies that M_i is diffeomorphic to $\mathbb{T}^2 \times [0,1]$. □

We have now established items (1)–(3) of theorem 1.1. The rest of the paper focuses on proving item (4). To do this, we lift to the universal cover $\tilde{M}$. Let $\Omega \subset \tilde{M}$ be a closed 3-dimensional submanifold with boundary such that each boundary component of Ω quotients down to a cs or cu-torus in M and such that no surface which intersects the interior of Ω quotients down to a cs or cu-torus. This submanifold may then be thought of as a covering space for one of the M_i. Proposition 4.3 implies that Ω is diffeomorphic to $\mathbb{R}^2 \times I$ where $I \subset \mathbb{R}$ is a compact interval.

It will at times be convenient to use coordinates on Ω and discuss linear maps from Ω to $\mathbb{R}$. Therefore, we simply assume that Ω is equal to $\mathbb{R}^2 \times I$. That is, we treat $\mathbb{R}^2 \times I$ as a subset of $\tilde{M}$ denoted by Ω. The Riemannian metric on $\tilde{M}$ inherited from M may differ from the standard Euclidean metric on $\mathbb{R}^2 \times I$. However, distances and volumes measured with respect to the two metrics differ by at most a constant factor. Therefore, in our analysis, we freely assume that $\Omega = \mathbb{R}^2 \times I$ is equipped with the Euclidean metric.

Since $\mathbb{Z}^2$ acts on Ω via deck transformations, we adopt the following notation: if $p = (v, s) \in \mathbb{R}^2 \times I$ and $z \in \mathbb{Z}^2$, then $p + z = (v, s) + z = (v + z, s)$.

As we are assuming $f : M \to M$ maps each M_i to itself, it follows that there is a lift of f to the universal cover which leaves Ω invariant. We also denote this lifted map $\tilde{M} \to \tilde{M}$ by the letter f. Since $f|_\Omega$ quotients down to a map on M_i, there is a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that if $p \in \mathbb{R}^2 \times I$ and $z \in \mathbb{Z}^2$, then $f(p+z) = f(p) + Az$. By proposition 2.1, A is hyperbolic. This implies that there is a semiconjugacy between $f|_\Omega$ and A [Fra70]. We list several properties of this semiconjugacy.

Proposition 4.4. *There is a unique continuous surjective map $H : \mathbb{R}^2 \times I \to \mathbb{R}^2$ and a constant $C > 0$ such that if $p = (v, s) \in \mathbb{R}^2 \times I$ and $z \in \mathbb{Z}^2$ then*

(1) $Hf(p) = AH(p)$,
(2) $H(p+z) = H(p) + z$, *and*
(3) $\|H(p) - v\| < C$.

As in section 2, let $\pi^u : \mathbb{R}^2 \to \mathbb{R}$ be a linear map such that $\ker \pi^u$ is the stable leaf of A which passes through the origin. Define $H^u = \pi^u \circ H$.

Now consider a branching foliation $\mathscr{F}^{cs}$ on $\tilde{M}$ as in corollary 3.2. We only know a priori that $\mathscr{F}^{cs}$ is invariant under those deck transformations which preserve the orientations of the subbundles. In particular, for a full-rank subgroup $Z_0 \subset \mathbb{Z}^2$ it holds that for any $z \in Z_0$, there is a deck transformation $\gamma_z : \tilde{M} \to \tilde{M}$ which preserves the orientations of the subbundles and such that $\gamma_z(p) = p + z$ for all $p \in \mathbb{R}^2 \times I = \Omega$. Therefore if L is a leaf of $\mathscr{F}^{cs}$, then $\gamma_z(L)$ is a leaf of $\mathscr{F}^{cs}$ as well. Replacing f by an iterate, we may freely assume that f preserves the orientations of the subbundles on $\tilde{M}$. Then the branching foliation $\mathscr{F}^{cs}$ is invariant under f.

Let Ω° denote the interior of Ω. A major step is to relate the branching foliation $\mathscr{F}^{cs}$ to the semiconjugacy H for points in Ω°.

Proposition 4.5. *For $p, q \in \Omega^\circ$, $H^u(p) = H^u(q)$ if and only if there is $L \in \mathscr{F}^{cs}$ such that $p, q \in L$.*

This is proved first in section 5 for the specific case where at least one component of $\partial\Omega$ is tangent to E^{cu}. Section 6 proves the result in the case where both components of $\partial\Omega$ are tangent to E^{cs}. The proof of the latter case relies on the proof of the former, and this significantly complicates the exposition. However, we know of no simpler method to establish proposition 4.5.

An analogous statement also holds for $H^s = \pi^s \circ H$ and any branching foliation $\mathscr{F}^{cu}$ tangent to E^{cu}.

Corollary 4.6. *For $p, q \in \Omega^\circ$, $H^s(p) = H^s(q)$ if and only if there is $L \in \mathscr{F}^{cu}$ such that $p, q \in L$.*

After these results are established, they are used in section 7 to prove the following.

Proposition 4.7. *If $\gamma : \tilde{M} \to \tilde{M}$ is a deck transformation such that $\gamma(\Omega) = \Omega$, then γ preserves the orientations of E^u, E^c, and E^s as bundles over $\tilde{M}$.*

This shows that Z_0 above may be taken as equal to $\mathbb{Z}^2$. Section 7 also proves the following characterization of the fibers of the semiconjugacy.

Proposition 4.8. *For every $v \in \mathbb{R}^2$, the pre-image $H^{-1}(v)$ is a compact segment tangent to E^c. Moreover, $H^{-1}(v)$ intersects each boundary component of Ω in either a point or a compact segment.*

Section 8 uses this to construct the topological conjugacy given in theorem 1.1.

5. CENTER-STABLE LEAVES

This section gives the proof of proposition 4.5 under certain assumptions. These assumptions are removed in the next section. Let f, Ω, H, and $\mathcal{F}^{cs}$ be as in the previous section. By abuse of notation, we consider π^u and π^s as both linear maps from $\mathbb{R}^2$ to $\mathbb{R}$ and as maps from $\Omega = \mathbb{R}^2 \times I$ to $\mathbb{R}$ which depend only on the $\mathbb{R}^2$ coordinate.

Proposition 5.1. *For any constant $D > 0$, there is $\ell > 0$ such that any unstable curve $J \subset \Omega$ of length at least ℓ contains points p and q with $|\pi^u(p) - \pi^u(q)| > D$.*

Proof. This result is analogous to [Ham13b, Lemma 4.7] and the proof given there applies here with only minor modifications. □

Define $\partial^{cs}\Omega$ as the union of those components of $\partial\Omega$ which are tangent to E^{cs}. Let d_u be distance measured along an unstable leaf and define K^u as the largest subset of Ω for which the following property holds: if $p \in K^u$, $q \in W^u(p)$, and $d_u(p,q) < 1$, then $q \notin \partial^{cs}\Omega$. In other words, K^u is the set of points at distance at least 1 from $\partial^{cs}\Omega$ where the distance is measured along the unstable direction.

Note that there are three possibilities. If both components of $\partial\Omega$ are tangent to E^{cs}, then E^u is transverse to $\partial\Omega$ and K^u lies in the interior Ω° of Ω. If both components of $\partial\Omega$ are tangent to E^{cu}, then $\partial^{cs}\Omega$ is empty and $K^u = \Omega$. Finally, if one component of $\partial\Omega$ is tangent to E^{cu} and the other is tangent to E^{cs}, then K^u contains one boundary component and not the other. Fortunately, most of the arguments in the remainder of the paper do not depend on which case we are in.

One can verify that K^u is a closed subset of Ω and is invariant under any deck transformation which fixes Ω. As f increases distances measured along the unstable direction, it follows that $f(K^u) \subset K^u$. Note that if J is a compact subset of Ω°, then there is an integer $N(J)$ such that $f^n(J) \subset K^u$ for all $n > N(J)$.

We now consider the intersection of K^u with the leaves of the branching foliation $\mathcal{F}^{cs}$.

Proposition 5.2. *There is a non-zero map $\pi : \Omega \to \mathbb{R}$ of the form $\pi = a\pi^u + b\pi^s$ with constants $a, b \in \mathbb{R}$ such that if $L \in \mathcal{F}^{cs}$ and $p, q \in K^u \cap L$ then $|\pi(p) - \pi(q)| < 1$.*

We prove this by adapting techniques presented in [BBI09, HP14]. The proof is largely topological in nature, instead of involving the dynamics acting on Ω. Therefore, we defer the proof of proposition 5.2 to the appendix.

Lemma 5.3. *No stable or unstable leaf intersects both boundary components of Ω.*

Proof. Note that there is a uniform lower bound on the distance between points in the two boundary components of Ω. If a stable or unstable segment J had endpoints on both boundary components, one could find $n \in \mathbb{Z}$ such that the length of $f^n(J)$ was smaller than this lower bound, and this would give a contradiction. $\square$

Lemma 5.4. *If $\partial^{cs}\Omega \neq \partial\Omega$, then for any $c > 0$, there is $L \in \mathscr{F}^{cs}$ and $p, q \in K^u \cap L$ such that $|\pi^s(p) - \pi^s(q)| > c$.*

Proof. If $\partial^{cs}\Omega \neq \partial\Omega$, then Ω has a boundary component S tangent to E^{cu}. By lemma 5.3, S is contained in K^u. For any $x \in S$, there is a leaf $L \in \mathscr{F}^{cs}$ through x. By proposition 2.5 where g is given by the restriction of f^{-1} to S, the intersection of S and L contains $W^c_g(x)$. The result then follows by proposition 2.3 $\square$

Assume for the remainder of the section that the conclusion of lemma 5.4 holds.

Proposition 5.5. *There is $R > 0$ such that if $L \in \mathscr{F}^{cs}$ and $p, q \in K^u \cap L$ then $|\pi^u(p) - \pi^u(q)| < R$.*

Proof. Define a linear map $\pi^A : \Omega \to \mathbb{R}$ by

$$\pi^A(v, s) = \pi(A(v), s)$$

for $(v, s) \in \Omega = \mathbb{R}^2 \times I$ where π is given by proposition 5.2.

If $L \in \mathscr{F}^{cs}$ and $p, q \in K^u \cap L$, then $f(K^u) \subset K^u$ implies that $f(p), f(q) \in K^u \cap f(L)$ and therefore $|\pi f(p) - \pi f(q)| < 1$. Since f is a finite distance from $A \times \text{id}$ on Ω, there is $C > 0$ such that $|\pi^A(p) - \pi^A(q)| < C$ for all such p and q. If both constants a and b are non-zero in proposition 5.2, then π and π^A are linearly independent and π^u is a linear combination of π and π^A. From this, the result would follow and therefore we may assume that one of a or b is zero. The conclusion of lemma 5.4 implies that the latter case must hold. $\square$

For a point $p \in \Omega$, define

$$K^-(p) = \{q \in K^u : \pi^s(q) \leq \pi^s(p) - R\}$$

and

$$K^+(p) = \{q \in K^u : \pi^s(q) \geq \pi^s(p) + R\}.$$

Replacing f by f^2 if necessary, we assume that A has positive eigenvalues. The fact that f is at finite distance from $A \times \text{id}$ then implies that $K^+(f^n(p))$ intersects $f^n(K^+(p))$ for all n.

Proposition 5.6. *If $L \in \mathscr{F}^{cs}$ and $p \in K^u \cap L$, then $\tilde{M} \setminus L$ has connected components L^- and L^+ such that $K^-(p) \subset L^-$ and $K^+(p) \subset L^+$.*

Proof. Proposition 5.5 shows that L is disjoint from both $K^-(p)$ and $K^+(p)$. Therefore, it is enough to show that each of L^- and L^+ intersects at least one of $K^-(p)$ or $K^+(p)$.

Suppose instead that $K^-(p) \cup K^+(p) \subset L^-$. Then for any $n \geq 0$, $K^+(f^n(p))$ intersects $f^n(K^+(p))$ and is therefore a subset of $f^n(L^-)$. Similarly for $K^-(f^n(p))$. Since $p \in K^u \cap L$, the open set $\Omega^\circ \cap L^+$ is non-empty. Let J be a small unstable segment lying in $\Omega^\circ \cap L^+$. Then $f^n(J) \subset K^u$ for all large n. Since $f^n(J) \cap f^n(L^-)$ is empty, the length of $\pi^u f^n(J)$ is bounded by $2R$ for all large n. However, proposition 5.1 shows that there is no uniform bound on the length of $\pi^u f^n(J)$ and gives a contradiction. □

Proposition 5.7. *For $p, q \in K^u$, the following are equivalent:*

- $\sup_{n\geq 0} |\pi^u f^n(p) - \pi^u f^n(q)| < \infty$, *and*
- *there is $L \in \mathscr{F}^{cs}$ such that $p, q \in L$.*

Proof. One direction follows from proposition 5.5 and the fact that $f(K^u) \subset K^u$. To prove the other direction, suppose $p \in L_p \in \mathscr{F}^{cs}$ and $q \in L_q \in \mathscr{F}^{cs}$. Let L_p^- and L_p^+ be as in the previous proposition and let L_q^- and L_q^+ be the corresponding sets associated to L_q. Assume q does not lie on L_p. Then q lies either in L_p^- or L_p^+. Without loss of generality, assume $q \in L_p^-$. Since L_q is the boundary of L_q^+, it follows that $L_p^- \cap L_q^+ \cap \Omega^\circ$ is a non-empty open set. Consider a small unstable curve J in this set. Then $f^n(J) \subset K^u$ and

$$f^n(J) \cap \big(K^+(f^n(p)) \cup K^-(f^n(q))\big) = \varnothing$$

for all large n. The assumption that $|\pi^u f^n(p) - \pi^u f^n(q)|$ is uniformly bounded implies that the length of $\pi^u f^n(J)$ is uniformly bounded. Proposition 5.1 again gives a contradiction. □

Proof of proposition 4.5. By the properties of the semiconjugacy,

$$\begin{aligned} H^u(p) = H^u(q) \quad &\Leftrightarrow \quad \sup_{n\geq 0} |\pi^u A^n H(p) - \pi^u A^n H(q)| < \infty \\ &\Leftrightarrow \quad \sup_{n\geq 0} |\pi^u f^n(p) - \pi^u f^n(q)| < \infty. \end{aligned}$$

Since $f^n(p)$ and $f^n(q)$ are in K^u for all large n, the result follows from proposition 5.7. □

This proof was conditional on the conclusion of lemma 5.4 and therefore we have only established proposition 4.5 in the case where $\partial^{cs}\Omega \neq \partial\Omega$. The next section gives a replacement for lemma 5.4 in the case where $\partial^{cs}\Omega = \partial\Omega$ and will therefore finish the proof of proposition 4.5.

6. FINDING A HIDDEN TORUS

The goal of this section is to prove the following.

Lemma 6.1. *If $\partial^{cs}\Omega = \partial\Omega$, then for any $c > 0$, there is $L \in \mathscr{F}^{cs}$ and p, $q \in K^u \cap L$ such that $|\pi^s(p) - \pi^s(q)| > c$.*

Note that up to replacing f with f^{-1}, lemma 6.1 is equivalent to the following.

Lemma 6.2. *If $\partial^{cs}\Omega = \varnothing$, then for any $c > 0$, there is $L \in \mathscr{F}^{cu}$ and $x, y \in K^s \cap L$ such that $|\pi^u(x) - \pi^u(y)| > c$.*

Here, K^s is the set of points at distance at least 1 from $\partial\Omega$ where distance is measured along the stable direction. The advantage in proving lemma 6.2 in place of lemma 6.1 is that, since $\partial^{cs}\Omega = \varnothing$, all of the results of the previous section are known to hold for $\mathscr{F}^{cs}$ and we may use those properties of $\mathscr{F}^{cs}$ when proving results for $\mathscr{F}^{cu}$.

We prove lemma 6.2 by contradiction. Therefore, for the remainder of the section, assume that $\partial\Omega$ is tangent to E^{cu}, that $\mathscr{F}^{cu}$ is a branching foliation tangent to E^{cu}, and that there is $D_0 > 0$ such that if $L \in \mathscr{F}^{cu}$ and $x, y \in K^s \cap L$, then $|\pi^u(x) - \pi^u(y)| < D_0$. We will use a result in [Ham16b] to establish the existence of a surface lying in Ω° which quotients down to a cs-torus in the original compact 3-manifold M. Since Ω was chosen so that no such torus exists, this will provide the needed contradiction.

Lemma 6.3. *There is $D > 0$ such that if $L \in \mathscr{F}^{cu}$ and $x, y \in K^s \cap L$, then $|H^u(x) - H^u(y)| < D$.*

Proof. This follows from the inequality with D_0 above and the fact that H^u and π^u are at finite distance. □

Lemma 6.4. *There is $\ell > 0$ such that if $x \in \Omega^\circ$, $y \in W^u(x)$, and $d_u(x, y) > \ell$, then $|H^u(x) - H^u(y)| > D$.*

Proof. Let $C > 1$ be such that for any $x \in \Omega$ and $y \in W^u(x)$, there is an integer k such that $1 \le d_u(f^k(x), f^k(y)) \le C$. Define

$$X = \{(x, y) : x \in \Omega,\ y \in W^u(x), \text{ and } 1 \le d_u(x, y) \le C\}.$$

Under the assumptions of the current section, $\partial^{cs}\Omega = \varnothing$ and therefore the last section shows that the conclusions of proposition 4.5 hold for $\mathscr{F}^{cs}$. As such, proposition 3.5 implies that $H^u(x) - H^u(y)$ is non-zero for all $(x, y) \in X$. As X may be quotiented down to a compact set, there is $\delta > 0$ such that $|H^u(x) - H^u(y)| > \delta$ for all $(x, y) \in X$.

The semiconjugacy relation $Hf = AH$ implies that $H^u(f(x)) = \lambda H^u(x)$ where $\lambda > 1$ is the unstable eigenvalue of A. Choose n such that $\lambda^n\delta > D$. Then $|H^u(x) - H^u(y)| > \delta$ implies $|H^u f^n(x) - H^u f^n(y)| > \lambda^n\delta > D$. To conclude the proof, choose $\ell > 0$ so that $d_u(x, y) > \ell$ implies $d_u(f^{-n}(x), f^{-n}(y)) > 1$. □

Corollary 6.5. *If $x \in K^s$, $y \in W^u(x)$, and $d_u(x, y) > \ell$, then $y \notin K^s$.*

Proof. Otherwise, the results above give $D < |H^u(x) - H^u(y)| < D$. □

Lemma 6.6. *There is a continuous function $g : \Omega \to [0, 1]$ such that*

(1) *g is invariant under deck transformations;*
(2) *$g(\partial\Omega) = \{0, 1\}$; and*
(3) *if $0 < g(x) < 1$, then $x \in K^s$.*

Proof. Let S_0 and S_1 be the two boundary components of Ω. Define

$$K_i = \{x \in W^s(y) \cap \Omega : y \in S_i,\ d_s(x,y) \le 1\}.$$

By lemma 5.3, no stable manifold intersects both S_0 and S_1 and therefore K_0 and K_1 are disjoint. Define

$$g(x) = \frac{\operatorname{dist}(x,K_0)}{\operatorname{dist}(x,K_0)+\operatorname{dist}(x,K_1)}.$$ □

Now Ω quotients down to a subset $M_0 \subset M$ and g quotients down to a function from M_0 to [0,1]. Applying [Ham16b, Theorem 2.5], there is a compact cs-submanifold in the interior of M_0. This contradicts the assumptions on Ω given in section 4 and completes the proof of lemma 6.2. Since the two statements are equivalent, this also proves lemma 6.1. Now, substituting lemma 6.1 in place of lemma 5.4 in the previous section, one sees that proposition 4.5 holds in full generality.

7. Fibers of the semiconjugacy

This section gives the proofs of propositions 4.7 and 4.8. Let f, Ω, and H be as in section 4. Recall that $\partial^{cs}\Omega$ is the union of those boundary components of Ω which are tangent to E^{cs}.

Lemma 7.1. *An unstable curve intersects $\partial^{cs}\Omega$ in at most one point.*

Proof. Suppose J is an unstable segment which intersects $\partial^{cs}\Omega$ at both endpoints. By lemma 5.3, both endpoints must lie on the same boundary component. Then for large n, $f^{-n}(J)$ would be an arbitrarily small unstable curve connecting two points on the same center-stable surface. This is ruled out by the uniform transversality of E^{cs} and E^u. □

Lemma 7.2. *If J is an unstable curve in Ω, then $H^u|_J$ is a homeomorphism to its image.*

Here, the curve J may be bounded or unbounded and may or may not include its endpoints.

Proof. First, consider the case where J is in the interior of Ω. By proposition 4.5, the fibers of H^u are cs-leaves and, by proposition 3.5, each cs-leaf intersects an unstable leaf at most once, so $H^u|_J$ is injective. Since H^u is continuous, this implies that $H^u|_J$ is a homeomorphism to its image.

Note that if $\phi : [0,1) \to \mathbb{R}$ is a continuous function and $\phi|_{(0,1)}$ is an embedding, then ϕ itself must also be an embedding. Therefore, in the case where J intersects $\partial\Omega$ in a point, the fact that the restriction of H^u to $J \setminus \partial\Omega$ is injective implies that H^u is injective on all of J.

The last possibility is if J lies in a component of $\partial\Omega$ which is tangent to E^{cu}. This case follows from proposition 2.7. □

Proof of proposition 4.7. For a point $x \in \Omega^\circ$, let $J \subset \Omega^\circ$ be a short unstable segment passing through x. Using that $H^u|_J$ is injective, define the orientation for E^u near x so that H^u increases along J. This gives a well-defined continuous orientation of E^u on all of Ω°.

Suppose $\gamma : \tilde{M} \to \tilde{M}$ is a deck transformation mapping $\Omega = \mathbb{R}^2 \times I$ to itself. By the properties of the semiconjugacy, there is $z \in \mathbb{Z}^2$ such that $H\gamma(x) = H(x) + z$ for all $x \in \Omega^\circ$. Hence, γ preserves the orientation of E^u. An analogous argument shows that γ preserves the orientation of E^s. By assumption, the original closed 3-manifold M is orientable. Therefore, γ preserves the orientation of $T\tilde{M}$ and must also preserve the orientation of E^c. □

Lemma 7.3. *For $x \in \partial^{cs}\Omega$, the set $H^u(W^u(x) \cap \Omega)$ is equal either to $(-\infty, H^u(x)]$ or $[H^u(x), +\infty)$.*

Proof. For $x \in \partial^{cs}\Omega$, let $J_1(x)$ be the compact unstable segment which starts at x, is directed into Ω, and has length exactly one. By lemma 7.2, $H^u(J_1(x))$ always has positive length. By a compactness argument, there is $\delta > 0$ such that the length of $H^u(J_1(x))$ is greater than δ for all x. As in the proof of lemma 6.4, one may show that

$$\operatorname{length} H^u f^n J_1(x) > \lambda^n \delta$$

for all $x \in \partial^{cs}\Omega$ and $n \geq 0$ where $\lambda > 1$ is the unstable eigenvalue of A. Lemma 7.1 implies that $f^n J_1(f^{-n}(x))$ is a subset of $W^u(x) \cap \Omega$ for all n. Thus, the length of $H^u(W^u(x) \cap \Omega)$ is unbounded. □

Lemma 7.4. *Let S be a connected component of $\partial^{cs}\Omega$ and let*

$$U = \Omega \cap \bigcup_{x \in S} W^u(x).$$

For any $v \in \mathbb{R}^2$, $H^{-1}(v) \cap U$ is a topological ray. That is, there is a proper topological embedding $\beta : [0, +\infty) \to U$ such that the image is $H^{-1}(v) \cap U$.

Proof. First, note that $H^{-1}(v) = (H^u)^{-1}(q) \cap (H^s)^{-1}(r)$ for some pair of numbers $q, r \in \mathbb{R}$. By corollary 4.6, there is a leaf $L^{cu} \in \mathcal{F}^{cu}$ such that $(H^s)^{-1}(r) \cap U = L^{cu} \cap U$. Therefore, we may restrict our attention to this leaf.

Consider the intersection of L^{cu} and S. The semiconjugacy $H : \Omega \to \mathbb{R}^2$ when restricted to S agrees with the semiconjugacy, also denoted H, studied in section 2. In particular, $H|_S$ is surjective. This implies that $L^{cu} \cap S$ is non-empty. Every connected component of $L^{cu} \cap S$ is a center line. By proposition 2.4, any stable leaf lying in S intersects every connected component of $L^{cu} \cap S$. By proposition 3.5, L^{cu} intersects a stable curve in at most one point and so $L^{cu} \cap S$ has exactly one connected component. Thus, $L^{cu} \cap S$ is a properly embedded center curve and may be parameterized by a function $\alpha : \mathbb{R} \to L^{cu} \cap S$.

As S is connected, exactly one of the two cases in lemma 7.3 holds for all $x \in S$. Without loss of generality, assume the case $H^u(W^u(x) \cap \Omega) = (-\infty, H^u(x)]$ holds. By proposition 2.11, α is monotonic. Up to composing α by an affine map on $\mathbb{R}$, we may assume α is defined so that $H^u\alpha(t) \geq q$ exactly when $t \geq 0$. Define a map

$\beta : [0,+\infty) \to U$ by setting $\beta(t)$ to be the unique point in $W^u(\alpha(t))$ which satisfies $H^u\beta(t) = q$. Proving that β is continuous reduces to proving the following claim.

> **Claim.** Suppose $h : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ is a continuous function with the properties that $x_1 \le x_2$ implies $h(x_1, 0) \le h(x_2, 0)$ and $y_1 < y_2$ implies $h(x, y_2) < h(x, y_1)$. Then, any level set of h is the graph of a continuous function.

The proof of the claim is left to the reader. In fact, the proof is highly similar in form to steps used in proving the implicit function theorem.

It is clear that β is injective. Suppose a sequence $\{t_k\}$ tending to $+\infty$ is such that $\beta(t_k)$ converges to a point $x \in U$. Since $W^u(x)$ intersects S, one may use an unstable foliation chart in a neighbourhood of x, to derive a contradiction. This shows that β is proper. □

Proof of proposition 4.8. Consider $v \in \mathbb{R}^2$. By lemma 7.4, there is at least one point $x \in \Omega^\circ$ such that $H(x) = v$. Let L^{cs} be a leaf of $\mathscr{F}^{cs}$ passing through x and L^{cu} a leaf of $\mathscr{F}^{cu}$. Let L be the connected component of $L^{cs} \cap L^{cu}$ which passes through x. By proposition 3.7, L is a properly embedded line. Since H is proper, $H^{-1}(v)$ is a compact subset of Ω, and the ends of L must eventually leave the interior of Ω. As such, there is a compact center segment $J \subset L$ such that the endpoints of J lie on $\partial\Omega$ and all other points of J lie in the interior of Ω. Lemma 7.4 then describes the exact shape of J near the boundary of Ω. In particular, one sees that the two endpoints of J cannot lie on the same boundary component, and so each of the two boundary components contains exactly one endpoint of J.

Now suppose another connected component of $L^{cs} \cap L^{cu}$ intersected the interior of Ω. This would lead to a center segment J' disjoint from J but such that $H(J) = H(J')$ and where each boundary component of Ω contained exactly one endpoint of J'. Lemma 7.4 would then imply that J and J' coincide near the boundary of Ω, a contradiction. Thus $H^{-1}(v)$ consists of J together with the pre-images of v on the two boundary components of Ω. By corollary 2.12 and lemma 7.4, the result follows. □

Corollary 7.5. *There is a uniform upper bound on the length of a fiber $H^{-1}(v)$.*

Proof. As H is proper and commutes with deck transformations, there is a uniform upper bound on the diameter $H^{-1}(v)$ independent of $v \in \mathbb{R}^2$. There is then a uniform upper bound on the volume of $U_1(H^{-1}(v))$ and the result follows from proposition 3.8. □

8. BUILDING THE RAGGED LEAF CONJUGACY

Lemma 8.1. *There is a continuous function $p : \Omega^\circ \to (0,1)$ such that for any center segment of the form $J = H^{-1}(v) \cap \Omega^\circ$, the restriction $p|_J$ is a C^1 embedding. Moreover, with respect to arc length, $p|_J$ has a uniform speed independent of the choice of J.*

Proof. Let S_0 and S_1 be the two boundary components of Ω. For $x \in \Omega^\circ$, define

$$p_0(x) = \frac{\operatorname{dist}(x, S_0)}{\operatorname{dist}(x, S_0) + \operatorname{dist}(x, S_1)}.$$

Extend p_0 to a continuous function $p_0 : \tilde{M} \to [0,1]$ by requiring it to be locally constant outside of Ω.

Now suppose $x \in \Omega^\circ$ and let $L^{cs} \in \mathscr{F}^{cs}$ and $L^{cu} \in \mathscr{F}^{cu}$ be leaves of the foliations such that $x \in L^{cs} \cap L^{cu}$. Let $\alpha : \mathbb{R} \to L^{cs} \cap L^{cu}$ be a parameterization by arc length of this center curve. By proposition 3.7, $\alpha(\mathbb{R})$ is a complete curve properly embedded in $\tilde{M}$. Let $T > 0$ be the upper bound given by corollary 7.5 and for any $t \in \mathbb{R}$, define

$$p(\alpha(t)) = \frac{1}{2T} \int_{t-T}^{t+T} p_0(\alpha(s))\, ds.$$

If $\alpha(t) \in \Omega^\circ$, then neither $\alpha(t-T)$ nor $\alpha(t+T)$ lies in Ω°. Up to possibly reversing the parameterization, it follows that $\alpha(t-T) = 0$ and $\alpha(t+T) = 1$ and by the Fundamental Theorem of Calculus

$$\frac{d}{dt} p(\alpha(t)) = \frac{1}{2T}. \qquad \square$$

Lemma 8.2. *If $x, y, z \in \Omega^\circ$ with $y \in W^u(x)$ and $z \in W^s(y)$, then $H(x) = H(z)$ if and only if $x = y = z$.*

Proof. Assume $H(x) = H(z)$. By proposition 4.5, x and z lie on the same leaf L of $\mathscr{F}^{cs}$. By proposition 3.3, L is saturated by stable curves and therefore the point y also lies in L. The uniqueness given by proposition 3.5 implies that $x = y$. A similar argument shows that $y = z$. $\square$

Proof of theorem 1.1. Define $h = H \times p : \Omega^\circ \to \mathbb{R}^3$. This is a continuous map and injective by proposition 4.8 and lemma 8.1. We now show that h is an open map. Consider $x \in \Omega^\circ$ and assume that E^u, E^c, and E^s are oriented. With respect to these orientations, define unit speed flows φ^s and φ^u along the stable and unstable directions. Define φ^c as the unit speed flow along the fibers of H with the direction given by the orientation of E^c. Define $i : [-\epsilon, \epsilon]^3 \to \Omega^\circ$ by

$$i(t_1, t_2, t_3) = \varphi^c_{t_3} \varphi^s_{t_2} \varphi^u_{t_1}(x)$$

where ϵ is small enough that the range of i is contained in Ω°. By lemmas 8.1 and 8.2, the range of $h \circ i$ contains $h(x)$ in its interior. Taking ϵ to zero, one can show that for any neighbourhood V of x, $h(V)$ is a neighbourhood of $h(x)$ and therefore h is open. It follows that h is a homeomorphism to its image, $U := h(\Omega^\circ)$.

As H is a semiconjugacy, the homeomorphism $hfh^{-1} : U \to U$ is of the form

$$hfh^{-1}(v, s) = (A(v), \phi(v, s))$$

where $A : \mathbb{R}^2 \to \mathbb{R}^2$ is the hyperbolic linear map and $\phi : U \to \mathbb{R}$ is some continuous function. By construction, the maps H, p, and therefore h are $\mathbb{Z}^2$-equivariant.

Thus, h quotients down to an embedding $U_i \to \mathbb{T}^2 \times \mathbb{R}$ which satisfies the properties listed in item (4) of theorem 1.1. □

APPENDIX A. BOUNDS ON FOLIATIONS

This appendix proves proposition 5.2. Let $\mathscr{F}^{cs}$, $\Omega = \mathbb{R}^2 \times I$, and $Z_0 \subset \mathbb{Z}^2$ be as in section 4. Let S be a leaf of $\mathscr{F}^{cs}$ which intersects the interior Ω° of Ω. Here, we use S instead of L to keep closer to the notation of [BBI09]. As S is properly embedded, the complement $\tilde{M} \setminus S$ consists of two open connected components S_+ and S_- where the oriented E^u bundle points into S_+. Define $S_+ + Z_0 = \{p+z : p \in S_+, z \in Z_0\}$ and similarly for $S_- + Z_0$.

Lemma A.1. *The set $\Omega \setminus \partial^{cs}\Omega$ is contained in both $S_+ + Z_0$ and $S_- + Z_0$.*

Proof. As the branching foliation is complete in the compact-open topology, one can show that the boundary $\partial\tilde{X}$ of $\tilde{X} = S_+ + Z_0$ is a union of leaves of $\mathscr{F}^{cs}$. (See the proof of Lemma 3.10 in [BBI09] for details.)

Consider the manifold $\hat{M}$ defined by the quotient $\tilde{M}/Z_0$. Then $\tilde{X}$ quotients down to a subset $\hat{X} \subset \hat{M}$ and $\partial\tilde{X}$ quotients down to $\partial\hat{X}$. In particular, $\partial\hat{X}$ is closed subset of $\hat{M}$ and the orientation of E^u shows that $\partial\hat{X}$ does not accumulate on itself.

Let $\hat{\Omega}$ be the quotient of Ω to $\hat{M}$. Its boundary consists of two tori. We claim that each torus is either contained in $\hat{X}$ or disjoint from $\hat{X}$. Indeed, let T be one of the tori and suppose $\hat{X} \cap T$ is a non-empty proper subset. If T is tangent to E^{cs}, then $\hat{X} \cap T$ is saturated by stable leaves. If T is tangent to E^{cu}, then $\hat{X} \cap T$ is saturated by center leaves. In either case, the results in section 2 imply that $\partial\hat{X}$ contains a topological line immersed in T. This line accumulates on itself and gives a contradiction.

Hence, if $\partial\hat{X}$ intersects the interior of $\hat{\Omega}$, it must have a connected component lying entirely in $\hat{\Omega}$. This component would be a cs-torus, which would contradict the assumptions given on Ω in section 4. This shows that $\tilde{X}$ contains Ω°. As $\tilde{X}$ is saturated by stable leaves, it also contains any component of $\partial\Omega$ tangent to E^{cu}. □

Lemma A.2. *There is a non-zero linear map $\pi : \mathbb{R}^2 \to \mathbb{R}$ such that $\pi(z) \geq 0$ implies $S_+ + z \subset S_+$ and $\pi(z) \leq 0$ implies $S_+ \subset S_+ + z$ for $z \in Z_0$.*

Proof. This is shown by adapting the proofs of Lemmas 3.8 to 3.12 in [BBI09]. □

The set K^u defined in section 5 is a closed subset of $\Omega \setminus \partial^{cs}\Omega$. Let Q be the compact set defined by intersecting K^u with a cube of the form $[0,N] \times [0,N] \times I$ for some large N. Since Z_0 is a full rank subgroup of $\mathbb{Z}^2$, N may be chosen large enough that any $x \in K^u$ can be written as $x = q + z$ with $q \in Q$ and $z \in Z_0$.

Lemma A.3. *There is $z_0 \in Z_0$ such that $Q \subset S_+ - z_0$ and $Q \subset S_- + z_0$.*

Proof. By lemma A.1, $\{S_+ - z : z \in Z_0\}$ is an open cover of K^u, and so some finite subset $\{S_+ - z_1, \dots, S_+ - z_n\}$ covers the compact set Q. Take z_0 such that $\pi(z_0) \geq \pi(z_i)$ for all i. The case for S_- is analogous. □

By abuse of notation, if $p = (v, s) \in \mathbb{R}^2 \times I$ define $\pi(p) = \pi(v)$.

Lemma A.4. *There is $r > 0$ such that $\pi(x) > r$ implies $x \in S_+$ and $\pi(x) < -r$ implies $x \in S_-$ for all $x \in K^u$.*

Proof. Choose $r > 0$ such that $r - \pi(q) > \pi(z_0)$ for all $q \in Q$. Any $x \in K^u$ may be written as $x = q + z$ with $q \in Q$ and $z \in Z_0$. If $\pi(x) > r$, then $\pi(z - z_0) \geq 0$ and $x \in Q + z \subset S_+ - z_0 + z \subset S_+$. □

Lemma A.5. *There is $R > 0$ such that if $p, q \in K^u$ lie on the same leaf of $\mathscr{F}^{cs}$, then $|\pi(p) - \pi(q)| < R$.*

Proof. Without loss of generality, shift p and q by an element of Z_0 and assume $q \in Q$. Let S' be the leaf containing both p and q. Since S' intersects $S_+ - z_0$ and leaves do not topologically cross, S' is disjoint from $S_- - z_0$ and so $\pi(p), \pi(q) > -r - \pi(z_0)$. Similarly, $\pi(p), \pi(q) < r + \pi(z_0)$. Take $R = 2(r + \pi(z_0))$. □

Up to rescaling π so that $R < 1$, this concludes the proof of proposition 5.2.

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