Ergodic components of partially hyperbolic systems

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Why? Partial hyperbolicity.

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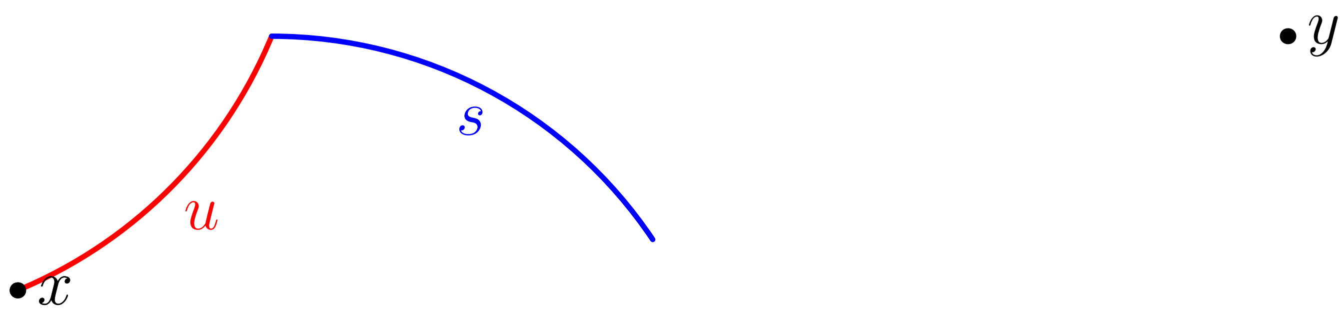

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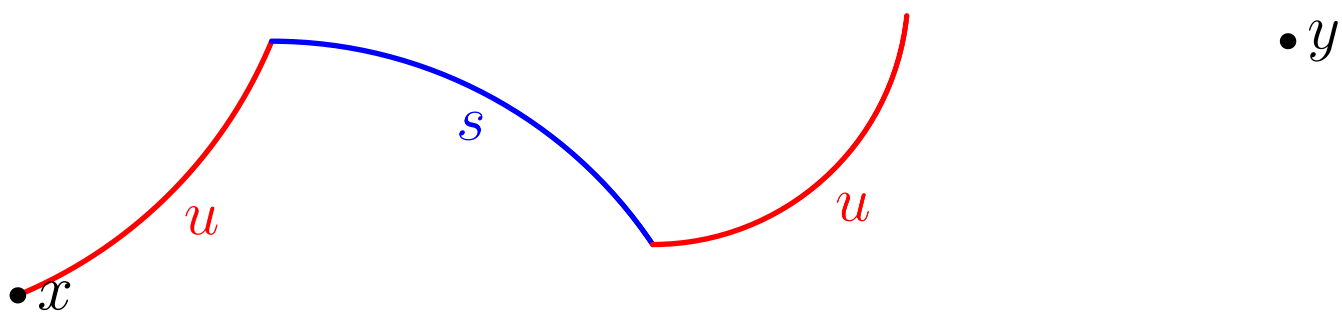

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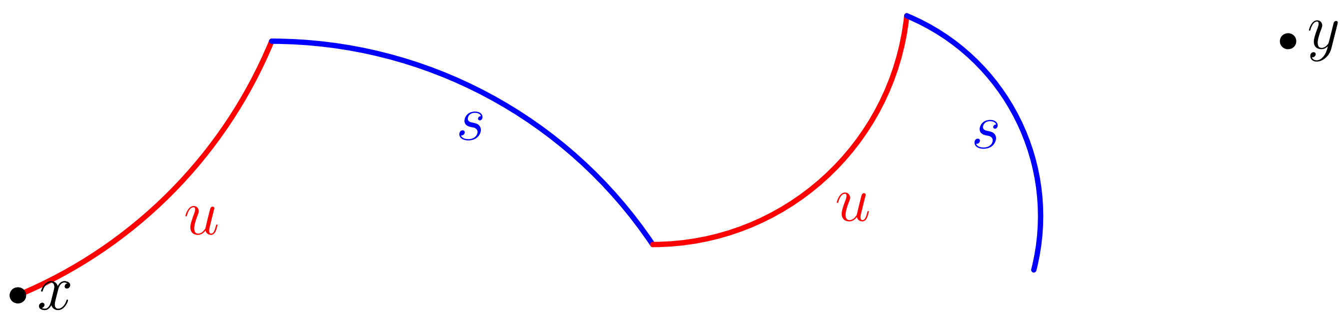

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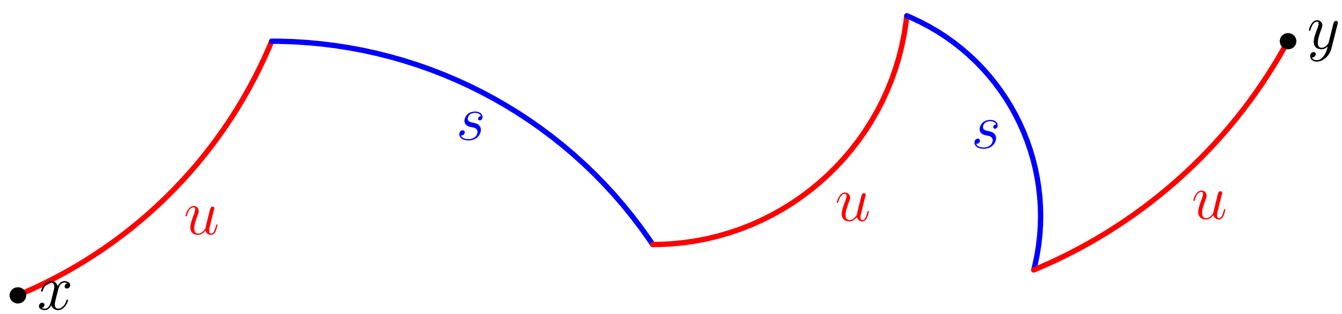

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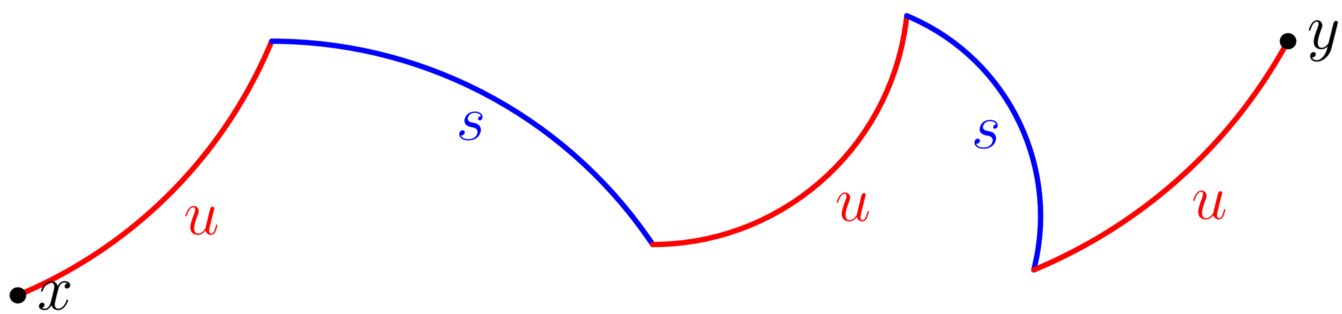

Conjectures are true when $\dim(E^c) = 1$.

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Long history of related work by

Birkhoff, Hopf, Anosov, Sinai, Brin, Pesin, Grayson, Pugh, Shub, Burns, Dolgopyat,Wilkinson, Rodriguez-Hertz, Rodriguez-Hertz, Ures, Avila, Crovisier, and others.

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In some sense, these are the only ways to construct non-ergodic perturbations.

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Also want to include suspensions of Anosov diffeomorphisms.

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on the manifold

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For a suspension, $B = A$.

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More general examples exist. Say where A, B on $N = \mathbb{T}^3$ given by

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

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Note that every AB-prototype is a volume-preserving non-ergodic partially hyperbolic system.

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These need to be included in our taxonomy.

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Leaf conjugacy is a technical but natural notion due to Hirsch-Pugh-Shub.

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Definition. f is an **infra-AB-system** if there is an iterate f^n ($n \geq 1$) which lifts to an AB-system on a finite cover.

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Further, we can classify the ergodic properties of AB-systems and infra-AB-systems completely.

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Theorem. Suppose f is leaf conjugate to the time-one map of an Anosov flow with $\dim E^{uu} = 1$.

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    $p^{-1}(z)$ is an f^n-invariant submanifold tangent to $E^u \oplus E^s$.
  - If I is a connected component of U then

    $p^{-1}(I)$ is an ergodic component of f^n

    and is homeomorphic to $N \times I$.