Exercise
Mini-course on the Classification of Partially Hyperbolic Systems

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Disclaimer: While we have taken care in preparing questions, we have not solved all problems in complete detail. As such there may be errors or omissions in the problems as stated. Please, contact us if you have any questions.

1. Find a partially hyperbolic system which is leaf conjugate to an Anosov flow, but is not the time-one map of a flow.
2. Show that if a foliation of $\mathbb{T}^3$ has a compact leaf, it has a torus leaf.
3. Suppose $\mathcal{F}$ is a foliation on $\mathbb{R}^2$ defined by lifting a foliation on $\mathbb{T}^2$. Does it follow that every leaf of $\mathcal{F}$ lies at a finite distance from a line in $\mathbb{R}^2$?
4. What possibilities are there for a Reebless foliation on $\mathbb{S}^2 \times \mathbb{S}^1$?
5. Show that if E is a continuous and orientable line field in $\mathbb{T}^2$ then there exists a branching foliation $\mathcal{F}$ tangent to E and invariant under every diffeomorphism which preserves E.
6. Show that for any circle diffeomorphism and any 3-solvmanifold, one can embed the circle diffeomorphism into a partially hyperbolic system on the manifold.
7. Show that all dynamically coherent pointwise partially hyperbolic diffeomorphisms on the 3-torus are plaque expansive.
8. Show that a weakly partially hyperbolic system in dimension two is leaf conjugate to an Anosov diffeomorphism. What about higher dimensions with a codimension one unstable direction?

   A diffeomorphism $f : M \to M$ is *weakly partially hyperbolic* if there a continuous Tf-invariant splitting $TM = E^c \oplus E^u$ such that for all $p \in M$ and unit vectors $v^u \in E^u(p)$, $v^c \in E^c(p)$

   $$\|Tfv^c\| < \|Tfv^u\| \quad \text{and} \quad \|Tfv^u\| > 1.$$

9. Exactly which quotients of the 3-torus support partially hyperbolic diffeomorpisms? How about 3-nilmanifolds in general?
10. In the last question, which quotients have partially hyperbolic systems where the bundles E^u, E^c, and E^s are orientable?
11. If f is a Derived-from-Anosov system on the 3-torus, does it have periodic points of every period?

12. For any $j, k, l \geq 0$, can one find a partially hyperbolic diffeomorphism of $\mathbb{T}^3$ with exactly j, k, and l, invariant tori tangent to cs, cu, and us respectively?

    If not, what are the restrictions on j, k, and l?

13. If T is an Anosov torus on a manifold, show there is f partially hyperbolic such that $f(T) = T$. Further, show for any of cs, cu, and us, that f can be chosen so that T is tangent to the corresponding subbundle.

    An *Anosov torus* is a incompressible 2-torus in a 3-manifold M such that there is a diffeomorphism $f : M \to M$ where $f(T) = T$ and $f|_T$ is homotopic to a Anosov map on T. See the Hertz-Hertz-Ures paper "Tori with hyperbolic dynamics in 3-manifolds" `doi:10.3934/jmd.2011.5.185` for more details.

14. Suppose $f : M \to M$ is a diffeomorphism of a 3-manifold with distinct invariant 2-dimensional tori $S = f(S)$ and $T = f(T)$. Are the maps $f|_S$ and $f|_T$ π_1-conjugate? What if both S and T are Anosov tori, using the same map f in the definition above?