Ergodicity and Classification of Partially Hyperbolic Systems

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Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$

E^u expanding, E^s contracting, E^c dominated.

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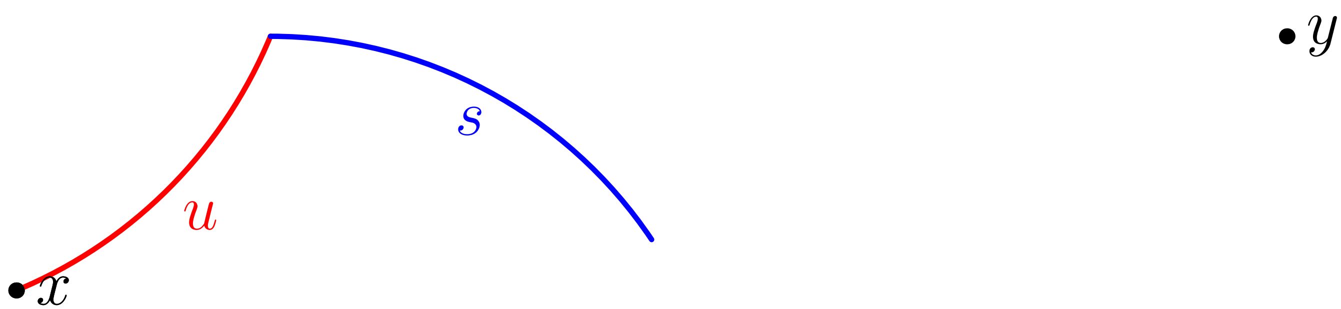

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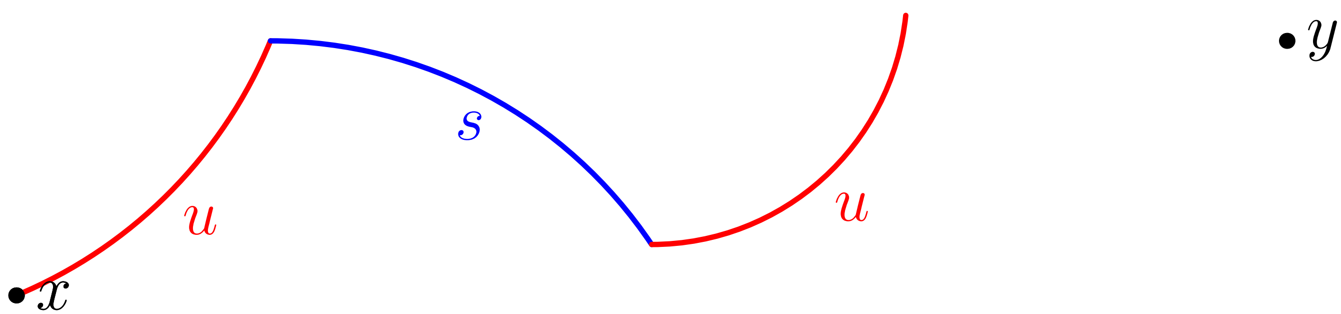

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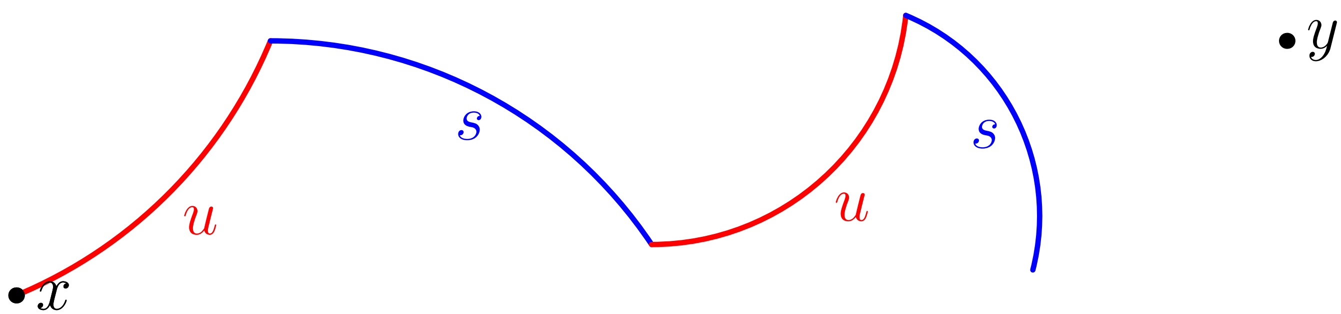

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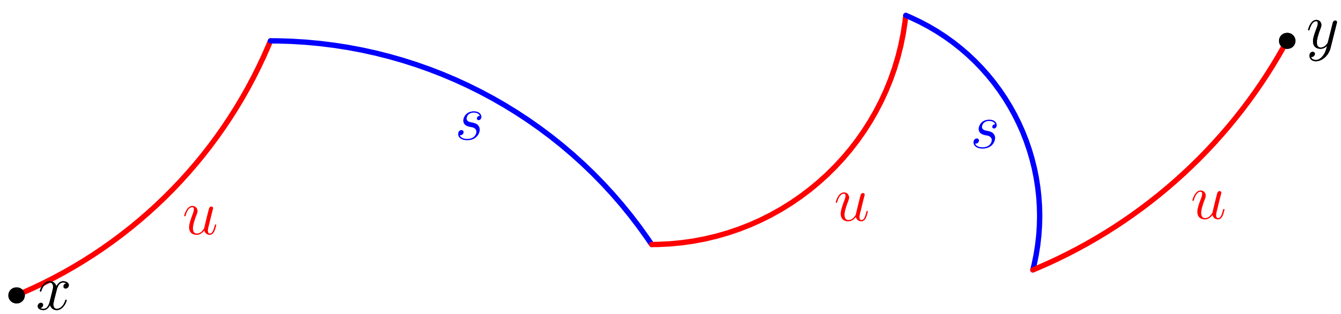

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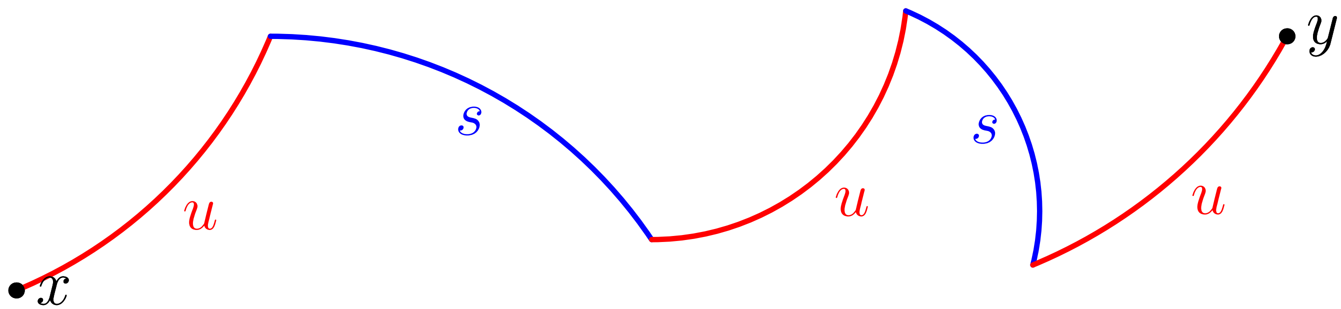

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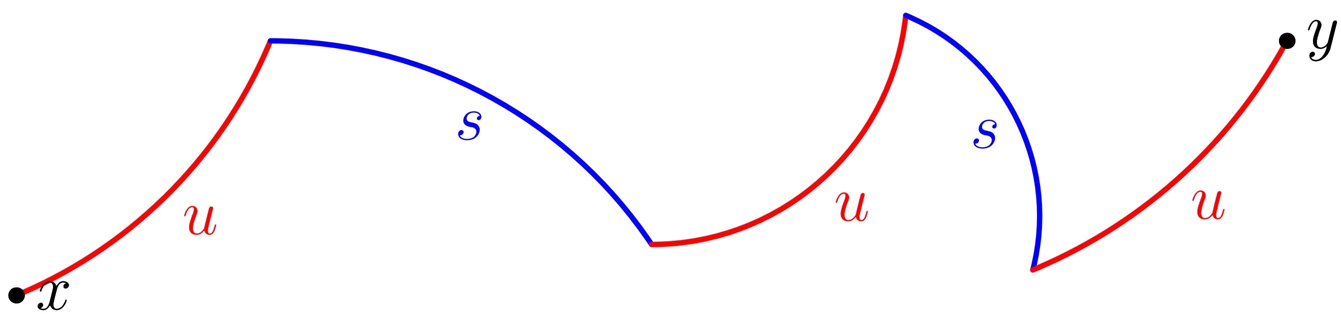

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(Rodriguez-Hertz, Rodriguez-Hertz, Ures).

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Question. How does this relate to classification results?

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If $f : M_B \to M_B$ is partially hyperbolic, and B is Anosov, the center foliation of f is equivalent to the orbits of ϕ and there is n such that f^n fixes every center leaf.

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  - If U is a connected component of $\mathbb{S}^1 \setminus K$, then $p^{-1}(U)$ is an ergodic component of f^n and is homeomorphic to $\mathbb{T}^2 \times U$.