Lecture 4

Thm (HP) $f: M \to M$ p.h. $\pi_1(M)$ solvable

then:
- $\to M = \mathbb{T}^3$ and f leaf conj to Anosov
- $\to M = \mathbb{T}^3$ or Nil and f leaf conj to skew prod.
- $\to M =$ Sol and f leaf conj to suspensi[on] of Anosov
- $\to \exists\, T$ torus tangent to E^{cs} or E^{cu}.

(mod finite cover + iterate)

First 3 lectures: (Brin-Burago-Ivanov/Parwani[...]

"One can restrict to those manifolds"

$f: \mathbb{T}^3 \to \mathbb{T}^3$ p.h. $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$

$f_*: \pi_1(\mathbb{T}^3) \to \pi_1(\mathbb{T}^3)$ is Anosov

$$|\lambda_1| \leq |\lambda_2| < 1 < |\lambda_3|$$

<u>Thm</u>: f is dynamically coherent.

Recall: $\exists$ f-invariant branching foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ tg to E^{cs} and E^{cu}.

Goal: Show that they don't branch.

1) Classify ~~Reebless~~ foliations
w.o. tori

<u>Ideas</u>: Plante, Roussarie, Gabai, ...

$\mathcal{F}_1, \mathcal{F}_2$ br. foliations are ALMOST PARALL

$\exists R$ – $\forall L \in \tilde{\mathcal{F}}_1 \; \exists L' \in \tilde{\mathcal{F}}_2$ s.t. $d_H(L, L') < R$

– $\forall L \in \tilde{\mathcal{F}}_2 \; \exists L' \in \tilde{\mathcal{F}}_1$ s.t. $d_H(L, L') < R$

Branching foliations are almost parallel to true foliations. (BI)

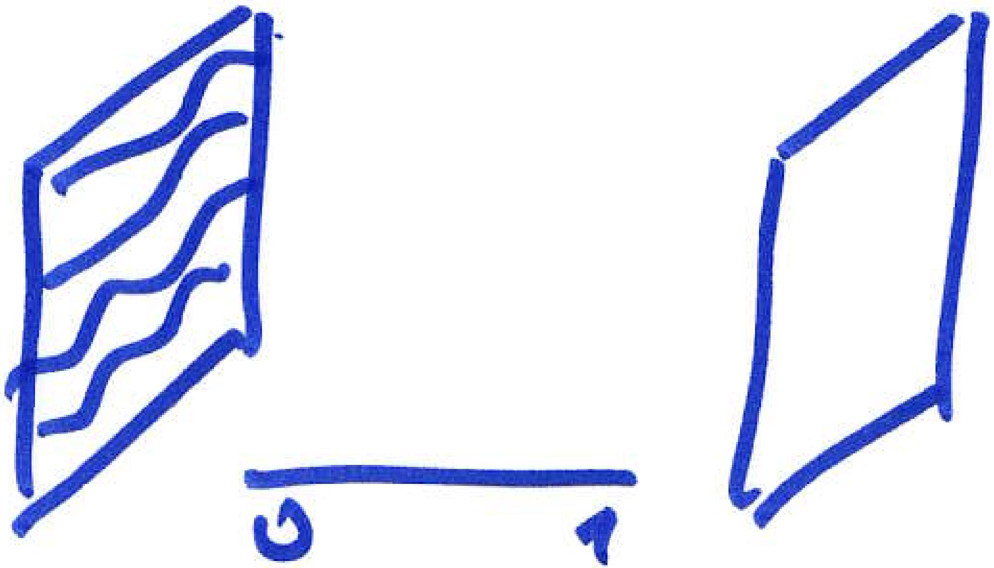
$B \in SL(2, \mathbb{Z})$

$$M = \mathbb{T}^2 \times [0,1] \Big/ (x,1) \sim (Bx, 0)$$

$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ — Nil

$B = A$ hyp. $\longrightarrow$ Solv.

Thm (Roussarie/Gabai) $\mathcal{F}$ a foliation w.o. torus leaves in M and T is incompressible torus $\Rightarrow$ T is isotopic to a torus transverse to $\mathcal{F}$.



Consequence: In $\mathbb{T}^3$ every foliation w.o. torus leaves is A.P. to a linear foliation.

$$\mathcal{F}^{cs} \xrightarrow{AP} L^{cs}$$

$$\mathcal{F}^{cu} \xrightarrow{AP} L^{cu}$$

2) Show that L^{cs} and L^{cu} are f_*-invariant.

$d_{C^0}(\tilde{f}, f_*) < K \qquad \tilde{f}(\mathcal{L}) = \mathcal{L}$

$d(f_* L^\sigma, L^\sigma)$ bounded

$f_* L^\sigma = L^\sigma \qquad \sigma = cs, cu$

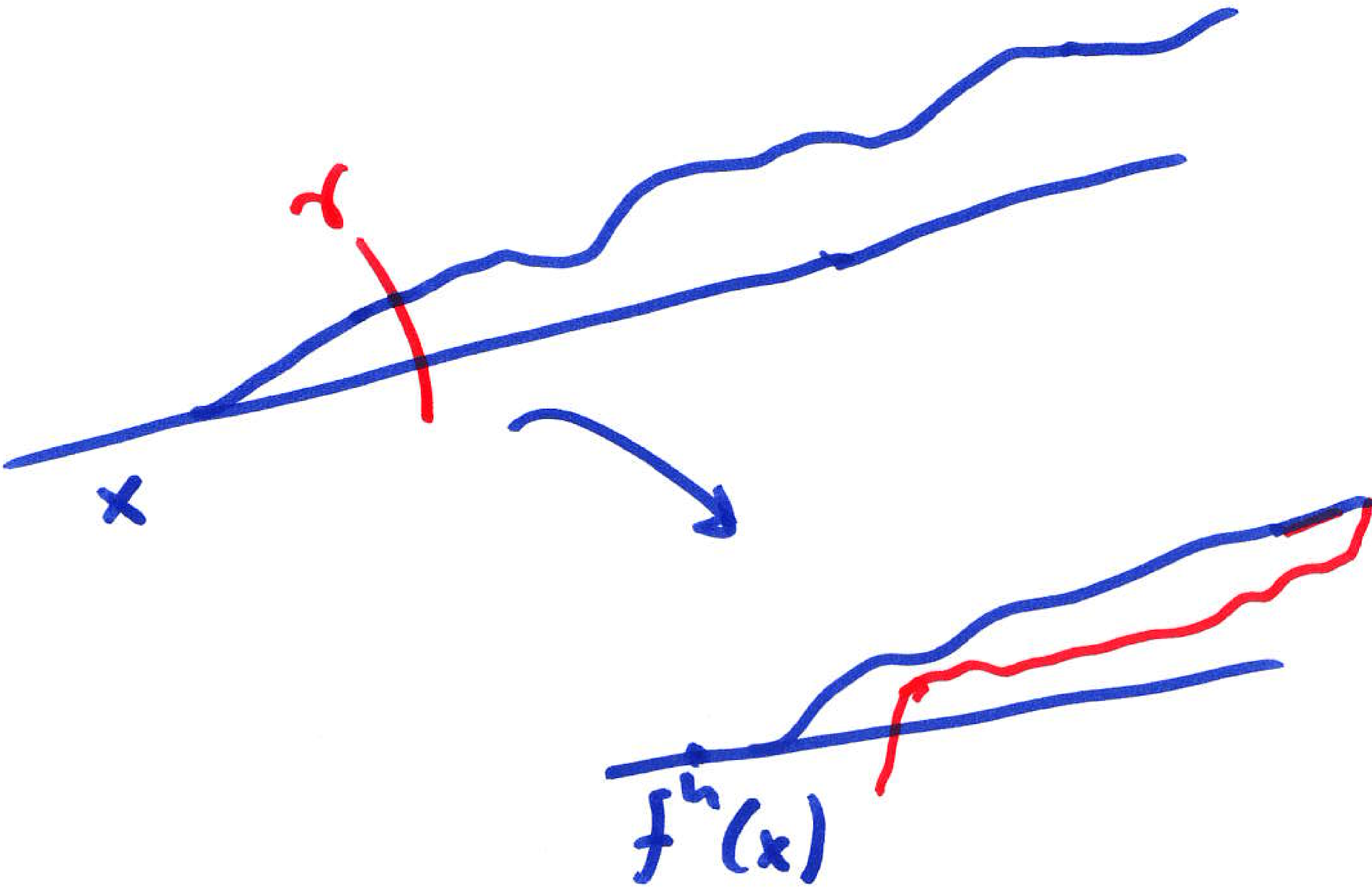
3) GPS $\Rightarrow$ dyn. coherence

$\mathcal{F}^{cs}$ and W^{u} have GPS

if $\forall L \in \tilde{\mathcal{F}}^{cs}$ and $\forall \gamma \in \tilde{W}^{u}$

we have $\# L \cap \gamma = 1$.

<u>We know</u> $\# L \cap \gamma \leq 1$.



γ
x
f^n(x)

Comments: $\rightarrow$ We have still not used that f_* is Anosov

$\rightarrow$ We DO use that $M = \mathbb{T}^3$
(At least that leafs of $\mathcal{L}^{cs}$ are parallel to each other)

$\rightarrow$ $\exists$ unique f-inv. foliation
$\mathcal{L}^{cs}$ and $\mathcal{L}^{cu}$ resist perturbations

4) Prove GPS when f_* is Anosov

Proposition: L^{cu} and L^{cs} are totally irrational.

Pf: f_* is Anosov.

Novikov / Hector Hirsch ($\dim M = 3$)

A foliation by simply connected leafs has GPS with every transverse foliation.

<u>Consequence</u>: We got GPS

$\Longrightarrow$ dyn. coherence.

$\mathcal{F}^{cs} \rightsquigarrow \mathcal{W}^{cs} — L^{cs}$

$\mathcal{F}^{cu} \rightsquigarrow \mathcal{W}^{cu} — L^{cu}$

<u>Lema</u> $L^{cs} \neq L^{cu}$

<u>Consequence</u>: $3 \neq$ real eigenvalues

$\tilde{f}$ has $\tilde{W}^{cs}, \tilde{W}^{cu}$ — L^{cs}, L^{cu}

$$
\begin{array}{lccccccc}
f_* & |\lambda_1| & < & |\lambda_2| & < 1 < & |\lambda_3| \\
 & | & & | & & | \\
 & E^1_* & & E^2_* & & E^3_*
\end{array}
$$

<u>**Goal**</u>: $L^{cs} = E^1_* \oplus E^2_* \quad L^{cu} = E^2_* \oplus E^3_*$

L^{cs} and L^{cu} resist perturbations

$\exists H : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $d(H, id) < K$

$$H \circ \tilde{f} = f_* \circ H$$

$$H(\tilde{W}^u(x)) = E_*^3 + H(x)$$

$$H(\tilde{W}^s(x)) \subseteq E_*^1 \oplus E_*^2 + H(x)$$

Implies: $L^{cu} \neq E^1_* \oplus E^2_*$

$L^{cs} = E^1_* \oplus E^2_*$.

<u>Problem</u>: $L^{cu} \overset{?}{=} E^1_* \oplus E^3_*$