Lecture 1

Ex cat map

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

Anosov diffeo

$f : M \circlearrowleft$

Df-inv't splitting

$$TM = E^u \oplus E^s$$

Diffeos f, g are **top conj** if $\exists$ homeo h s.t.

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ h\downarrow & & \downarrow h \\ N & \xrightarrow[g]{} & N \end{array}$$

commutes.

An Anosov diffeo f is <u>structurally stable</u>: any g C^1-close to f is Anosov and top conj to f.

<u>Thm</u> Franks-Manning

Any Anosov diffeo on $\mathbb{T}^n$ is top conj to a linear example.

In dim 2,

$$f \text{ is robustly trans} \underset{\text{Mañé}}{\Longleftrightarrow} f \text{ is Anosov.}$$

In higher dim.

$$f \text{ is RT} \overset{?}{\Longleftarrow} f \text{ is Anosov}$$

$$f \text{ is RT} \not\Longrightarrow f \text{ is Anosov}$$

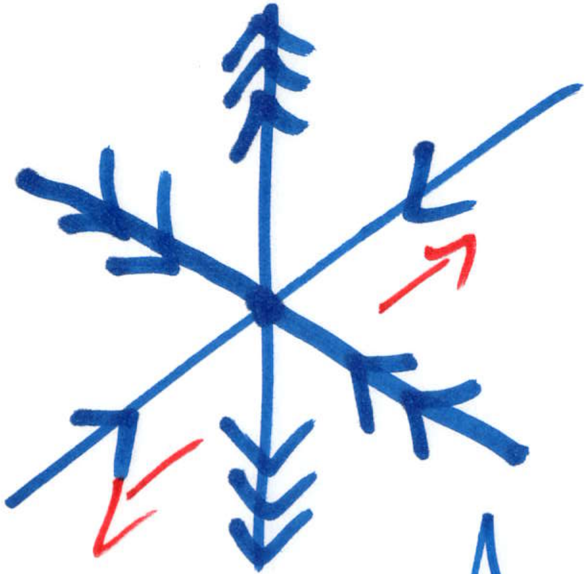
Shub $_{\mathbb{T}^4}$ Mañé $\mathbb{T}^3$

Mañé's Example

linear Anosov map $A_3 : \mathbb{T}^3 \circlearrowleft$

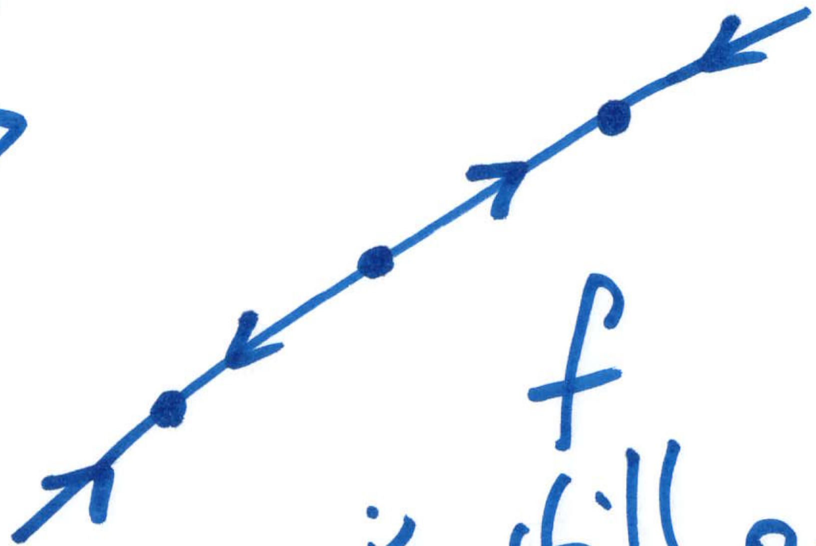
eigenvalues $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3$

$E^s \quad E^c \quad E^u$



A_3

deform



f

is still p.h.

A diffeo $f: M \circlearrowleft$ is *partially hyperbolic* (p.h.) if there is a Df-inv't splitting:

$$TM = E^u \oplus E^c \oplus E^s.$$

s.t. if $p \in M$ and $v^* \in E^*_p$ are *unit* vectors, then

$$1 > |Df v^s| < |Df v^c| < |Df v^u| > 1.$$

A_3 linear $\qquad$ f p.h.

$E^u_A \oplus E^c_A \oplus E^s_A \qquad E^u_f \oplus E^c_f \oplus E^s_f$

foliation W^c_A $\qquad$ fol'n W^c_f

tangent to E^c_A $\qquad$ tangent E^c_f

$\exists$ homeo h s.t.

$h(W^c_f) = W^c_A \cancel{h(E^c_f)} =$

$A : \mathbb{T}^2 \circlearrowleft$ Anosov

$f_0 = A \times id : \underbrace{\mathbb{T}^2 \times S^1}_{\mathbb{T}^3} \circlearrowleft$ p.h.

u, s c.

$P_0(v,t) = v$.

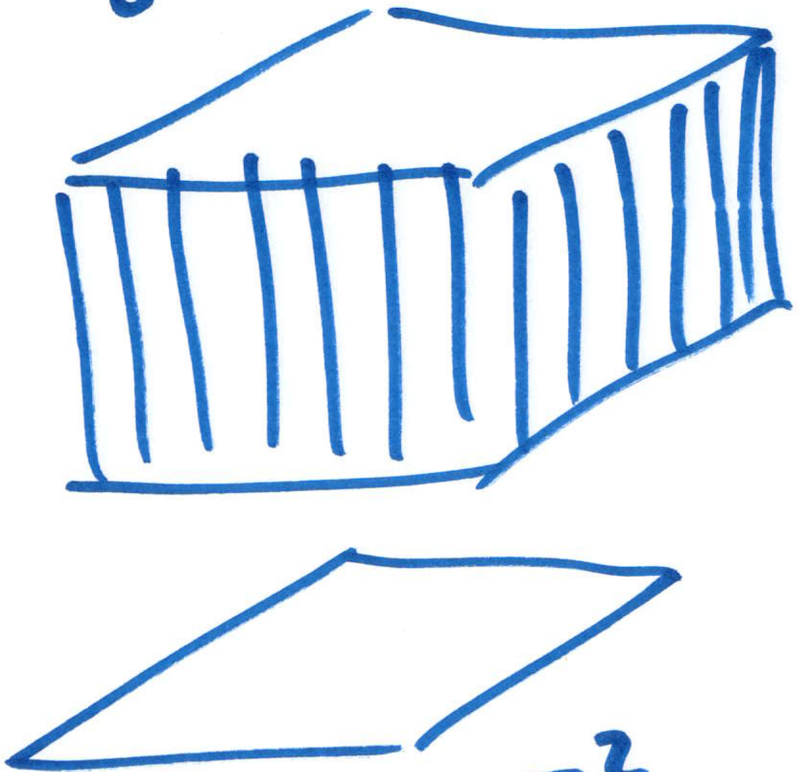
$$\begin{tikzcd} \mathbb{T}^3 \arrow[r, "f_0"] \arrow[d, "P_0"'] & \mathbb{T}^3 \arrow[d, "P_0"] \\ \mathbb{T}^2 \arrow[r, "A"'] & \mathbb{T}^2 \end{tikzcd}$$

If f is C^1-close to f_0

P is cts.

$$\begin{tikzcd} \mathbb{T}^3 \arrow[r, "f"] \arrow[d, "P"'] & \mathbb{T}^3 \arrow[d, "P"] \\ \mathbb{T}^2 \arrow[r, "A"'] & \mathbb{T}^2 \end{tikzcd}$$

$f_0 = A \times id$ every center leaf is a circle.



$\uparrow c$

Katok

$\mathbb{T}^2$ $\rightsquigarrow$ perturb $p \downarrow$

Can find $f \sim f_0$

and $X \subset \mathbb{T}^3$ so that X intersects each

full meas c-leaf in 2 points.

$$\begin{tikzcd} \mathbb{T}^3 \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{T}^3 \arrow[d, "p"] \\ \mathbb{T}^2 \arrow[r, "A"'] & \mathbb{T}^2 \end{tikzcd}$$

top. skew product

$$\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{T}^1$$

Suppose M is any circle bundle over $\mathbb{T}^2$.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "p"'] & M \arrow[d, "p"] \\ \mathbb{T}^2 \arrow[r, "A"'] & \mathbb{T}^2 \end{tikzcd}$$

skew product.

3-nilmanifolds.

Anosov flows.

A v. field X generates a flow
$\varphi : M \times \mathbb{R} \to M$

φ is an <ins>Anosov flow</ins> if there is an inv't splitting

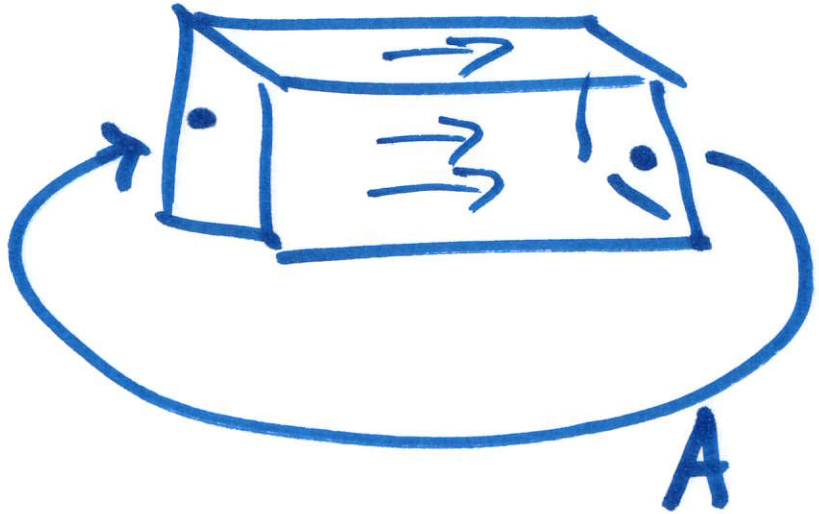
$$TM = \underset{\substack{\uparrow \\ \text{expanded}}}{E^u} \oplus \underset{\substack{\uparrow \\ E^c}}{\mathbb{R}X} \oplus \underset{\substack{\uparrow \\ \text{contracted}}}{E^s}$$

If φ is an Anosov flow, then the time t map ($t \neq 0$) is p.h.

<u>Ex</u>

$A : \mathbb{T}^2 \circlearrowleft$ Anosov diffeo.

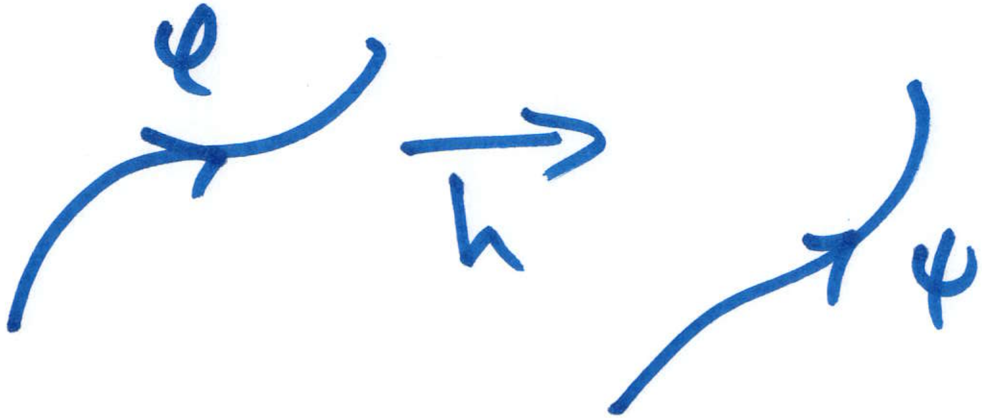
$$M_A = \mathbb{T}^2 \times [0,1] \big/ (x,1) \sim (Ax,0)$$



$$\varphi_t(x,s) = (x, s+t)$$

A <u>suspension</u> <u>Anosov</u> <u>flow</u>.

Flows φ and ψ are topogically equivalent if there is a homeo h which takes orbits of φ to orbits of ψ preserving orientation.



Anosov flows are structurally stable: φ Anosov then all ψ C^1-close to φ are Anosov and top equiv to ψ.

Thm (Plante-Verjorsky)

If φ is an Anosov flow on a 3-mfld with solvable fundamental group, it is top equiv to a suspension flow.

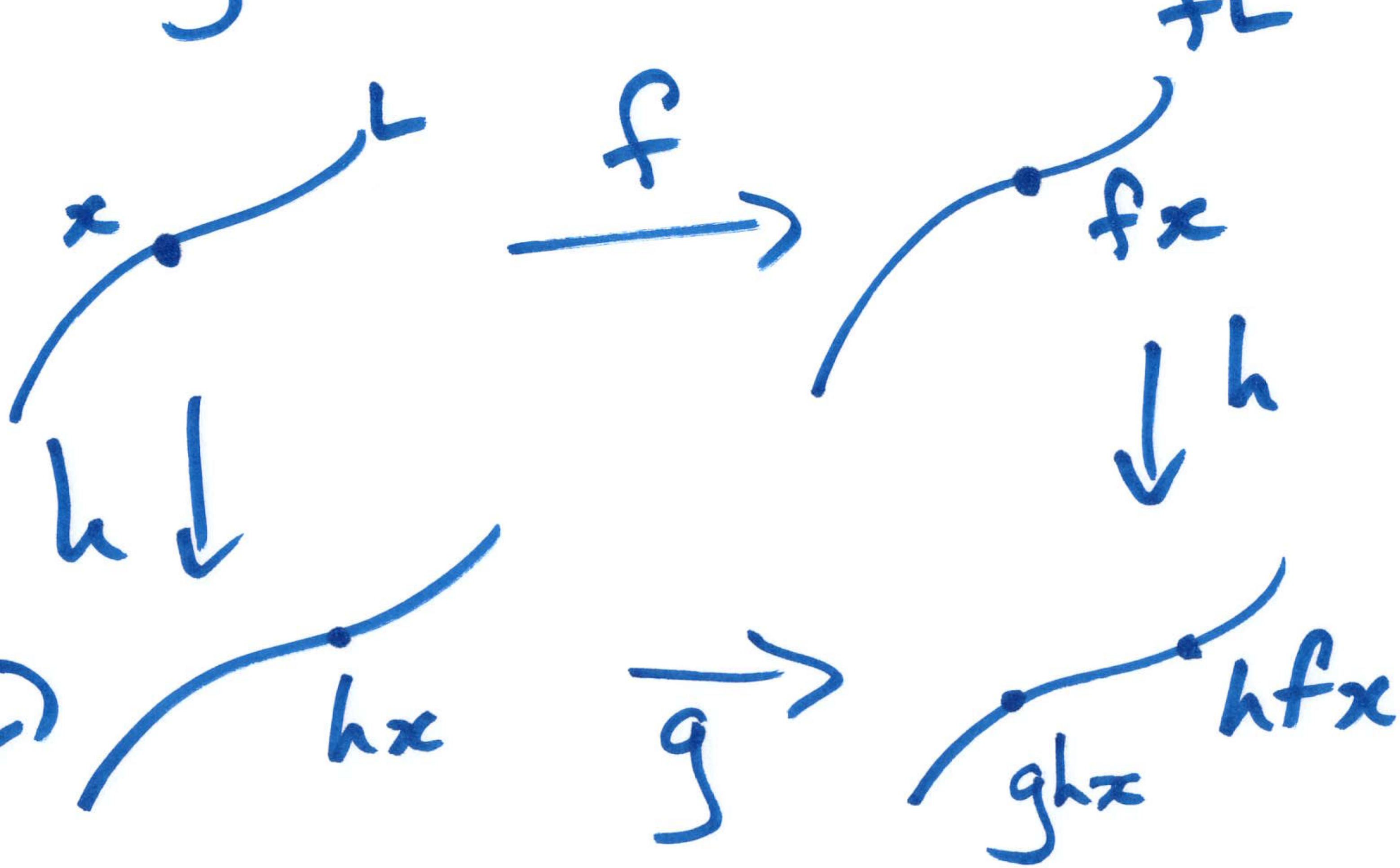
Anosov.

Is there a form of structural stability for p.h. systems?

Sort of.

If f and g are p.h. and if there are (invt) foli'n's W^c_f, W^c_g tangent to E^c_f, E^c_g. A <u>leaf conjugacy</u> ~~iff~~ is a homeo h s.t. if $L \in W^c_f$ then $h(L) \in W^c_g$ and $hf(L) = gh(L)$

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "h"'] & fx \arrow[d, "h"] \\ hx \arrow[r, "g"'] & ghx = hfx \end{tikzcd}$$



Thm [Hirsch-Pugh-Shub]

If f is p.h. with an invt center foliation W^c_f and is plaque expansive,

then any g C^1-close to f is p.h. and leaf conj to f.

W^c_g.

Conjecture (E. Pujals 2001) $\exists W^c_f$

If f is p.h in dimension 3

f is transitive, then f is

~~ta~~ (up to finite covers/iterate).

leaf conjugate to

- linear Anosov map on $\mathbb{T}^3$
- skew products ($\mathbb{T}^3$ or more generally)
- Anosov flows.

Thm [H, Potrie]

If f is p.h. on a 3-mfld with solvable fundamental group, then f is leaf conj (up to finite cover/iterate) to

- a linear Anosov on $\mathbb{T}^3$
- a skew product
- suspension Anosov flow

OR

there is a repelling or attracting periodic 2-torus tangent to either

$$E^c \oplus E^u$$

or

$$E^c \oplus E^s.$$

There is no such 2-torus when f is

- dynamically coherent.
- transitive.
- meas pres
- $NW(f) = M$
- absolutely p.h.

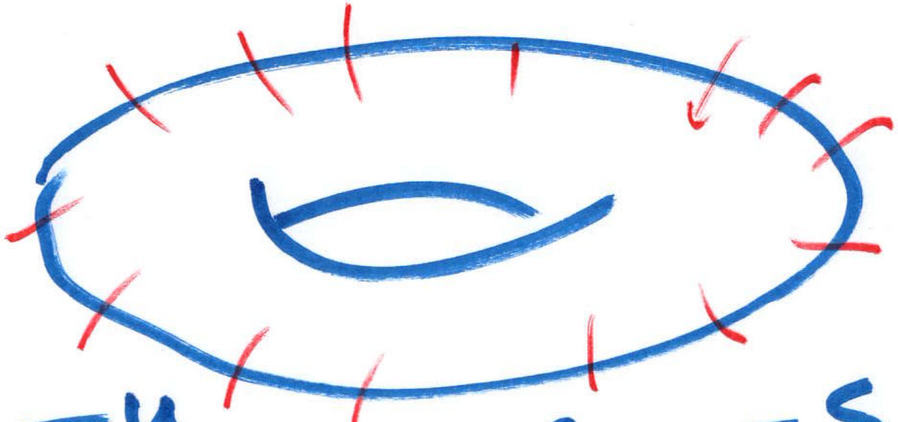
p.h. $TM = E^u \oplus E^c \oplus E^s$.

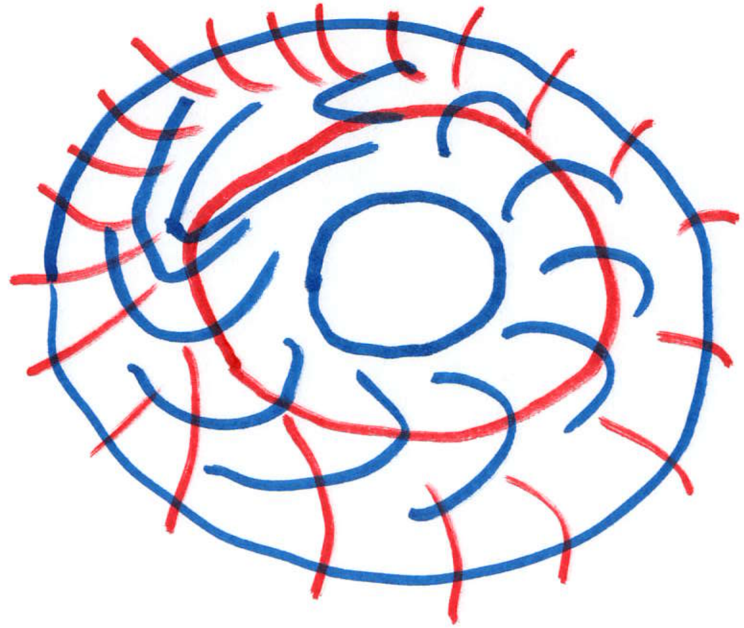
there's always unique foln W^u, W^s tangent E^u, E^s.

Defn f is dynamically coherent if there are inv't folns W^{cu} and W^{cs} tangent to $E^{cu} = E^c \oplus E^u$ and $E^{cs} = E^c \oplus E^s$.

<u>Thm</u> Brin-Burago Ivanov
There are no p.h. system on S^3.

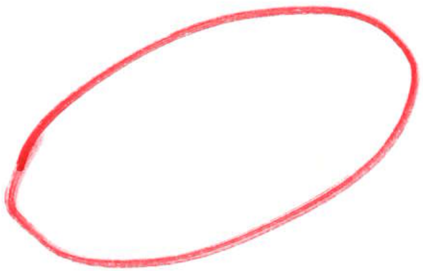
<u>Novikov's Thm</u>
Any codim 1 folia on S^3 has a compact leaf.
This compact leaf is a 2-torus bounding Reeb component.





$$E^u \oplus \underbrace{E^c \oplus E^s}_{W^{cs}}$$

W^u

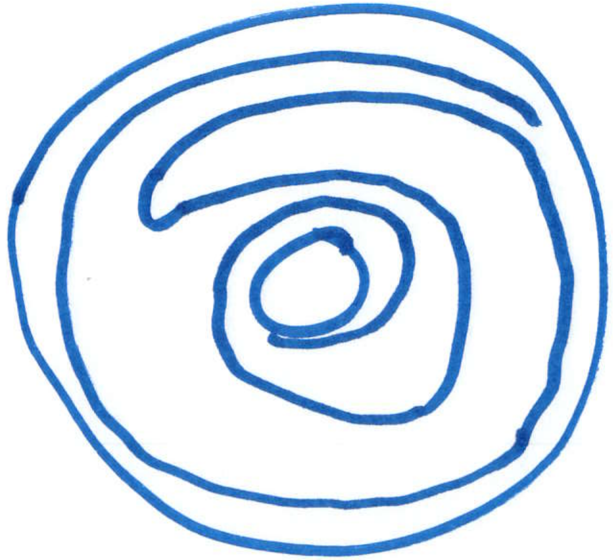


circle tangent to E^u

$\downarrow f^{-n}$

o

↯



Solodov C.