Infinite Designs: The Interplay Between Results in the Finite and Infinite Case

Bridget S Webb
The Open University

5-9 December 2011 / 35ACCMCC Monash
1. **Introduction**
   - Examples
   - Definition

2. **Some Results**
   - Finite Type Concepts
   - Infinite Type Concepts
**Definition**

*t-(v, k, λ)* design

A (FINITE) *t-(v, k, λ)* design is a
- v-set of points *V*
- with a collection *B* of *k*-subsets called blocks

such that
- every *t*-subset of points is contained in precisely *λ* blocks

A Steiner system is a *t-(v, k, 1)* design
A 2-(v, k, 1) design is a linear space with constant line length
This is a $2-(2^\aleph_0, 2^\aleph_0, 1)$ design
Euclidean Disk

Another $2-(2^\aleph_0, 2^\aleph_0, 1)$ design
Strambach’s Linear Space

- All lines through \((0, 0)\)
- plus all the images of
  \[ y = \frac{1}{x} \quad (x > 0) \]
  under \(SL_2(\mathbb{R})\).

Also a \(2-(2^{\aleph_0}, 2^{\aleph_0}, 1)\) design
(Strambach 1968)
Points: $\mathbb{Q}$, $+\infty$, $-\infty$

Triples: Let

$$f : \{ r \in \mathbb{Q} : 1/2 \leq |r| < 1 \} \mapsto \{-1, 1\}$$

- $(x, y, z)$ where $x + y + z = 0$ and $x, y, z$ unequal
- $((-2)^s r, (-2)^{s+1} r, (-1)^s f(r) \infty)$
- $(0, +\infty, -\infty)$

A $2-(\aleph_0, 3, 1)$ design

(Grannell, Griggs, Phelan 1987)
Given $t$ and $k$ with $t < k$

Start with a partial Steiner system
  - $t$ points lie in at most 1 block
  - any block contains at most $k$ points

Adjoin alternatively
  - new blocks incident with those $t$-tuples of points not already in a block
  - new points so each existing block has $k$ points

After countably many steps we have a Steiner system
Triangular Lattice

A $2-(\mathbb{N}_0, 3, 2)$ design
A design with \( \nu > b \)

Points: unit circle
Blocks: indexed by
\[ S = \{ e^{2\pi ip/q} : p, q \in \mathbb{N} \} \]

For each \( s \in S \)
- \( B_{1s} \) blue block
- \( B_{2s} \) purple block

This is a \( 2-(2^{\aleph_0}, 2^{\aleph_0}, \aleph_0) \) design with \( b = r = \lambda = \aleph_0 \)

More correctly, it is a \( 2-(\aleph_0, \aleph_0, \Lambda) \) design

(Cameron, BSW 2002)
**General Definition**

**t-(v, k, Λ) design**

A ν-set $V$ of points and a collection of $k$-subsets $B$ called blocks.

- $|V \setminus B| = \bar{k}$, for all $B \in B$, where $k + \bar{k} = v$

- For $0 \leq i + j \leq t$, the cardinality $\lambda_{i,j}$ of the set of blocks containing all of $i$ points $x_1, \ldots x_i$ and none of $j$ points $y_1, \ldots y_j$, depends only on $i$ and $j$

- no block contains another block

$\Lambda = (\lambda_{i,j})$ is a $(t + 1) \times (t + 1)$ matrix

$$\lambda_{t,0} = \lambda, \quad \lambda_{1,0} = r \text{ and } \lambda_{0,0} = b$$

$0 < t \leq k \leq v$ ensures non-degeneracy

(Cameron, BSW 2002)
Finite $t$ and $\lambda$

When $t$ and $\lambda$ are both $\text{FINITE}$:

- $\lambda_{t,0} = \lambda$
- $\lambda_{i,j} = v$, for all $i < t$, $0 \leq i + j \leq t$

We can write $t-(v, k, \lambda)$, as in the finite, case without ambiguity.

These designs are generally well behaved:

- Fisher’s Inequality $b \geq v$ holds since $v = b$

From now on $t$ and $\lambda$ will be assumed to be $\text{FINITE}$.
In contrast to the finite case, the existence problem for INFINITE $t$-designs is incomparably simpler — basically, they exist!

Existence with $t \geq 2$

$k$ **FINITE**

- Cyclic $t-(\aleph_0, k, \lambda)$ (Köhler 1977)
- Large sets $t-(\infty, t + 1, 1)$ (Grannell, Griggs, Phelan 1991)
- Large sets $t-(\infty, k, 1)$ (Cameron 1995)
- $t$-fold transitive $t-(\aleph_0, t + 1, 1)$ (Cameron 1984)
- Uncountable family of rigid 2-$t-(\aleph_0, 3, 1)$ (Franek 1994)

$k$ **not necessarily** **FINITE**

- Any $t-(\infty, k, 1)$ can be extended (Beutelspacher, Cameron 1994)
Block's Lemma (1967)

$G$ any automorphism group of a (FINITE) $t$-$(v, k, \lambda)$ design with $m$ orbits on the $v$ points and $n$ on the $b$ blocks

$$m \leq n \leq m + b - v$$

There is no infinite analogue of Block's Lemma

Examples of linear spaces: $k$ INFINITE

$n = 1$ and $m = 2$  
(Valette 1967)

$n = 2$ and $m = 3$  
(Prazmowski 1989)
### Steiner Triple Systems

A 2-$(v, 3, 1)$ design has at least as many block orbits as point orbits ($n \geq m$) (Cameron 1994)

### 2-$(\infty, k, \lambda)$ Designs

\[
n \geq \frac{m + \binom{m}{2}}{\binom{k}{2}}
\]

so $n \geq m$ if $n \geq k^2 - k$ (BSW 1997)

### 2-$(v, 3, \lambda)$ Designs

A 2-$(v, 3, \lambda)$ design has at least as many block orbits as point orbits ($n \geq m$) (BSW 1997)
Sketch Proofs

Let $G$ be an automorphism group of a $2-(\infty, k, \lambda)$ design

Colour the $m$ point orbits with $m$ colours:

- $\lambda$ blocks between any pair of points
- colours of blocks are $G$-invariant

only finitely many blocks through $p$ and points of $Q$
but infinitely many through $p$ with points of $P \setminus p$ and $R$
so infinite orbits with $p'$ and $r$ but not $q$

so to **minimise** $n$ we can consider only infinite point orbits
Sketch Proofs

- Crude bound: \[ n \geq m + \binom{m}{2} \cdot \binom{k}{2} \]

- So \( n \geq m \) holds if \( n \geq k^2 - k \)

- \( k = 3 \)

A 2-\((v, 3, \lambda)\) design has at least as many block orbits as point orbits
Designs with more point orbits than block orbits

Model Theoretic construction of Hrushovski (1993) used to construct
- \(2-(\aleph_0, 4, 14)\) design with \(n = 1\) and \(m = 2\) (Evans 1994?)
- \(2-(\aleph_0, k, k + 1)\) designs with \(k \geq 6\), \(n = 1\) and \(m = 2\) (Camina 1999)
- \(2-(\aleph_0, k, \lambda)\) designs with \(k \geq 4\), \(n = 1\) and \(m \leq k/2\) for some \(\lambda\) (BSW 1999)
- in particular a block transitive \(2-(\aleph_0, 4, 6)\) design with two point orbits
- \(2-(\aleph_0, 4, \lambda)\) designs with \(n \leq m\) (where \(n\) is feasible) for some \(\lambda\) (BSW 1999)
- \(t-(\aleph_0, k, 1)\) designs with \(k > t \geq 2\), \(n = 1\) and \(m \leq k/t\) (Evans 2004)
- in particular a block transitive \(2-(\aleph_0, 4, 1)\) design with two point orbits
Existential Closure Property

Block Intersection Graph of a Design $\mathcal{D}$

$G_\mathcal{D}$ has vertex set the blocks of $\mathcal{D}$
- two vertices are joined if the two blocks share at least one point

$n$-Existential Property of Graphs

A graph $G$ is said to be $n$-existentially closed, or $n$-e.c., if
- for each pair $(X, Y)$ of disjoint subsets of the vertex set $V(G)$ with $|X| + |Y| \leq n$
- there exists a vertex in $V(G) \setminus (X \cup Y)$ which is adjacent to each vertex in $X$ but to no vertex in $Y$

(Erdős, Rényi 1963)
Existential closure number $\Xi(G)$, is the largest $n$ for which $G$ is $n$-e.c. (if it exists)

**FINITE Steiner Triple Systems**

- a $2-(v, 3, 1)$ design is 2-e.c. iff $v \geq 13$
- if a $2-(v, 3, 1)$ design is 3-e.c. then $v = 19$ or 21
  (Forbes, Grannell, Griggs 2005)

In fact, only 2 of the STS(19) are 3-e.c. and ‘probably’ none of the STS(21)

**FINITE 2-$(v, k, \lambda)$ Designs**

- $\Xi(G_D) \leq k$, if $\lambda = 1$
- $\Xi(G_D) \leq \left\lfloor \frac{k + 1}{2} \right\rfloor$, if $\lambda \geq 2$
  (McKay, Pike 2007)
Existential Closure: INFINITE Designs

**k FINITE**

- \( \Xi(\mathcal{G}_D) = \min \{ t, \left\lfloor \frac{k-1}{t-1} \right\rfloor + 1 \} \) if \( \lambda = 1 \) and \( 2 \leq t \leq k \)
- \( 2 \leq \Xi(\mathcal{G}_D) \leq \min \{ t, \left\lceil \frac{k}{t} \right\rceil \} \) if \( \lambda \geq 2 \) and \( 2 \leq t \leq k - 1 \)

(Pike, Sanaei 2011)

**k INFINITE, \( k < v \)**

- \( \Xi(\mathcal{G}_D) = t \), if \( t = 1 \) or \( \lambda = 1 \), but \( (t, \lambda) \neq (1, 1) \)
- \( 2 \leq \Xi(\mathcal{G}_D) \leq t \), if \( t \geq 2 \) and \( \lambda \geq 2 \)  

(Horsley, Pike, Sanaei 2011)

**k INFINITE, \( k = v \)**

- \( t \) and \( \lambda \) positive integers such that \( (t, \lambda) \neq (1, 1) \)
- there exists a \( t-(\infty, \infty, \lambda) \) design with \( \Xi(\mathcal{G}_D) = n \)
- there exists a \( t-(\infty, \infty, \lambda) \) design which is \( n\)-e.c.

for each non-neg integer \( n \)  

(Horsley, Pike, Sanaei 2011)
A resolution class (parallel class) in a design is a set of blocks that partition the point set.

A design is resolvable if the block set can be partitioned into resolution classes.

The Euclidean Plane: $2-(2^\aleph_0, 2^\aleph_0, 1)$ is resolvable.

The Projective Plane: $2-(2^\aleph_0, 2^\aleph_0, 1)$ is NOT resolvable.

The Triangular Lattice: $2-(\aleph_0, 3, 2)$ design is resolvable.
Existence of Resolvable INFINITE Designs

$k < v$
- any $t-(\infty, k, \lambda)$ design is resolvable with $v$ resolution classes of size $v$  
  (Danziger, Horsley, BSW 2012?)

$k = v$
- There exists a $2-(\infty, \infty, 1)$ design with $\Xi(G_D) = 0$ iff there exists a resolvable $2-(\infty, \infty, 1)$ design
- A resolvable $t-(\infty, \infty, 1)$ design has $v$ resolution classes of $v$ blocks  
  (Horsley, Pike, Sanaei 2011)
- A resolvable $t-(\infty, \infty, \lambda)$ design has $v$ resolution classes of $v$ blocks and up to $\lambda - 1$ short resolution classes with less than $v$ blocks  
  (Danziger, Horsley, BSW 2012?)

There exists a $2-(\aleph_0, \aleph_0, 2)$ design with $\aleph_0$ resolution classes of size $\aleph_0$ and one resolution class of 4 blocks
Sparse, Uniform and Perfect Triple Systems

An $r$-sparse STS contains no $(n, n + 2)$-configurations for $4 \leq n \leq r$

A uniform STS has all its cycle graphs $G_{a,b}$ isomorphic

A perfect STS has each cycle graph $G_{a,b}$ a single cycle of length $v - 3$

**FINITE Steiner Triple Systems**

- Infinitely many 4, 5 and 6-sparse systems but no non-trivial $r$-sparse systems known for $r \geq 7$
- Only finitely many uniform systems known, apart from the Affine, Projective, Hall and Netto triple systems
- Only finitely many perfect systems known

**Countably INFINITE Steiner triple Systems**

$2^{\aleph_0}$ nonisomorphic CISTs that are

- $r$-sparse for all $r \geq 4$
- uniform

(Chicot, Grannell, Griggs, BSW 2009)
Universality and Homogeneity

A countable structure $M$ is

- **universal** with respect to a class of structures $C$ if $M$ embeds every member of $C$
- **homogeneous** if every isomorphism between finite substructures can be extended to an automorphism of $M$

There is no universal countable Steiner Triple System (Franek 1994)

There is a unique (up to isomorphism) universal homogeneous **locally finite** Steiner Triple System, $U$ (Cameron 2007?)

NOTE: In work on linear spaces, homogeneous as defined here is called ultrahomogeneous

The classification of ultrahomogeneous linear spaces (Devillers, Doyen 1998) does *not* extend to Steiner Systems
The Fraïssé Limit

Fraïssé’s Theorem

Suppose $C$ is a class of finitely generated structures such that
- $C$ is closed under isomorphisms
- $C$ contains only countably many members up to isomorphism
- $C$ has the Hereditary Property, HP
- $C$ has the Joint Embedding Property, JEP
- $C$ has the Amalgamation Property, AP

Then there is a countable homogeneous structure $S$
- which is universal for $C$
- unique up to isomorphisms

We call $S$ the Fraïssé limit of $C$ (Fraïssé 1954, Jónsson 1956)

Such a class is called an amalgamation class
Steiner Triple Systems

Regard an STS as a Steiner quasigroup

\[ a \circ b = c \text{ iff } \{a, b, c\} \text{ is a block } \quad (\text{and } x \circ x = x) \]

Then substructures (in the sense of model theory) are subsystems

The class of all finite STS is an amalgamation class — the Fraïssé limit is the universal homogeneous locally finite STS, \( U \)

The class of all finitely generated STS is NOT an amalgamation class

The class of all affine triple systems is an amalgamation class — the Fraïssé limit is the countably infinite affine triple system, \( A \)

The class of all projective triple systems is an amalgamation class — the Fraïssé limit is the countably infinite projective triple system, \( P \)
A structure is $\aleph_0$-categorical if its automorphism group is oligomorphic. That is, it has finitely many orbits on $n$-tuples for each positive integer $n$.

- $U$ is not $\aleph_0$-categorical.
- $\Lambda$ and $\mathcal{P}$ are both $\aleph_0$-categorical.

Let $S$ and $T$ be two $\aleph_0$-categorical STSs:

- The direct product $S \times T$ is $\aleph_0$-categorical — the direct product of oligomorphic groups is oligomorphic (Cameron, Gerwurz, Merola 2008).
- $d[S]$, the result of applying the doubling construction to $S$, is $\aleph_0$-categorical (Barbina, Chicot, BSW 201?).
In general countably infinite Steiner systems are quite well behaved

In general infinite designs exist

Other \textsc{finite} type concepts can be investigated for \textsc{infinite} designs

Work on \textsc{infinite} designs can lead to interesting new problems in the \textsc{finite} world

Keep $t$ and $\lambda$ \textsc{finite} to preserve your sanity!