Infinite Designs: The Interplay Between Results in the Finite and Infinite Case

Bridget S Webb
The Open University

5-9 December 2011 / 35ACCMCC Monash

The Open University

Outline

Definition

t-(v, k, λ) design

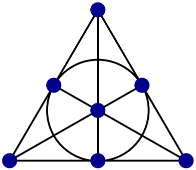
A (FINITE) t-(v, k, λ) design is a

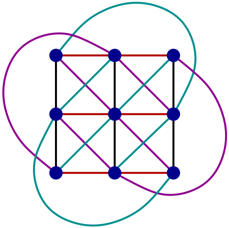
- v-set of points V
- with a collection B of k-subsets called blocks

such that

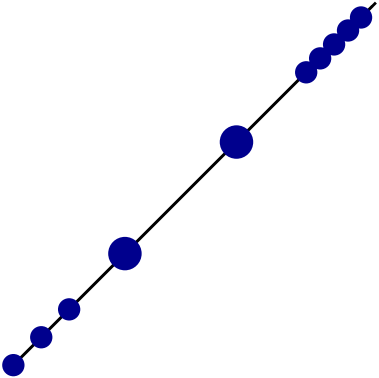
- every t-subset of points is contained in precisely λ blocks

A Steiner system is a t-$(v, k, 1)$ design
A 2-$(v, k, 1)$ design is a linear space with constant line length



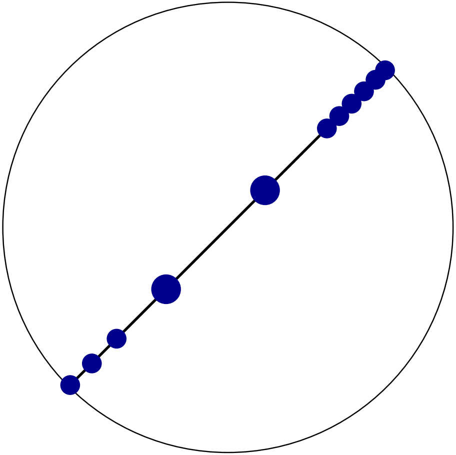


Euclidean Plane



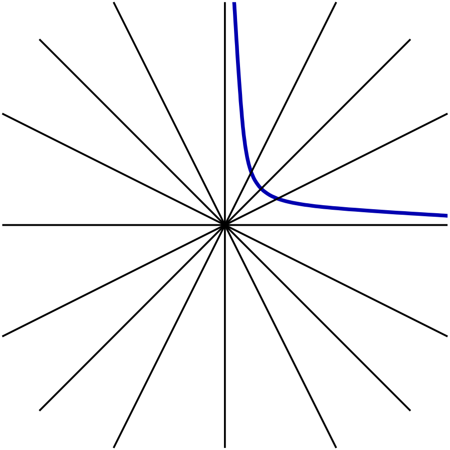
This is a $2\text{-}(2^{\aleph_0}, 2^{\aleph_0}, 1)$ design

Euclidean Disk



Another $2\text{-}(2^{\aleph_0}, 2^{\aleph_0}, 1)$ design

Strambach's Linear Space



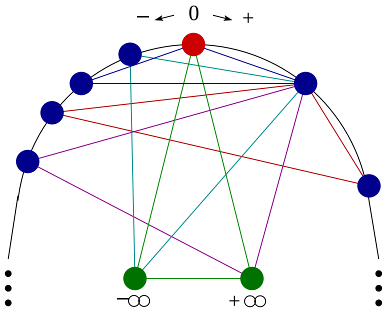
- All lines through $(0,0)$
- plus all the images of

$$y = 1/x \quad (x > 0)$$

under $\mathrm{SL}_2(\mathbf{R})$.

Also a $2\text{-}(2^{\aleph_0}, 2^{\aleph_0}, 1)$ design (Strambach 1968)

Countably Infinite Steiner Triple System



Points: $\mathbf{Q}$, $+\infty$, $-\infty$
Triples: Let

$$f : \{r \in \mathbf{Q} : 1/2 \le |r| < 1\} \mapsto \{-1, 1\}$$

- (x, y, z) where $x + y + z = 0$ and x, y, z unequal
- $((-2)^s r, (-2)^{s+1} r, (-1)^s f(r)\infty)$
- $(0, +\infty, -\infty)$

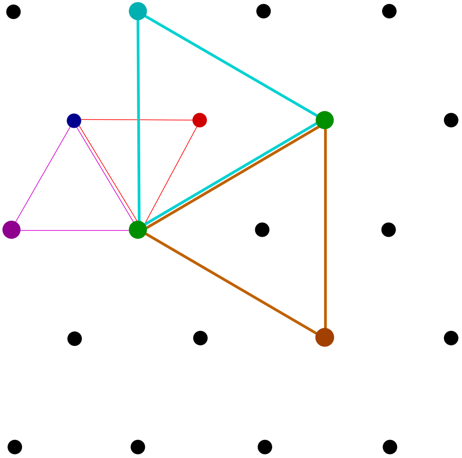
A $2\text{-}(\aleph_0, 3, 1)$ design

(Grannell, Griggs, Phelan 1987)

Free Construction of Countably Infinite Steiner Systems

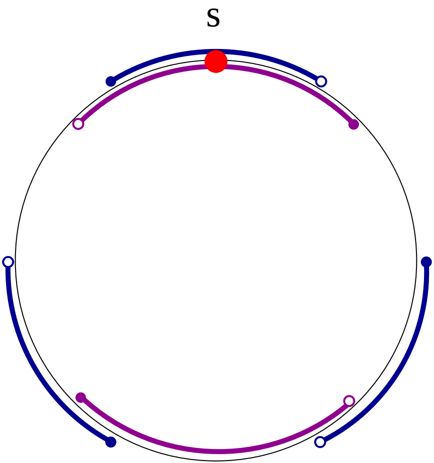
- Given t and k with $t < k$
- Start with a partial Steiner system
  - t points lie in at most 1 block
  - any block contains at most k points
- Adjoin alternatively
  - new blocks incident with those t-tuples of points not already in a block
  - new points so each existing block has k points
- After countably many steps we have a Steiner system

Triangular Lattice



A $2\text{-}(\aleph_0, 3, 2)$ design

A design with $v > b$



Points: unit circle
Blocks: indexed by
$S = \{e^{2\pi i p/q} : p, q \in \mathbf{N}\}$
For each $s \in S$

- B_{1s} blue block
- B_{2s} purple block

This is a $2\text{-}(2^{\aleph_0}, 2^{\aleph_0}, \aleph_0)$ design with $b = r = \lambda = \aleph_0$

More correctly, it is a $2\text{-}(2^{\aleph_0}, 2^{\aleph_0}, \Lambda)$ design

(Cameron, BSW 2002)

General Definition

t-(v, k, Λ) design

A v-set V of points and a collection of k-subsets $\mathcal{B}$ called blocks.

- $|V \backslash B| = \bar{k}$, for all $B \in \mathcal{B}$, where $k + \bar{k} = v$
- For $0 \leq i + j \leq t$, the cardinality $\lambda_{i,j}$ of the set of blocks containing all of i points $x_1, \ldots x_i$ and none of j points $y_1, \ldots y_j$, depends only on i and j
- no block contains another block

$\Lambda = (\lambda_{i,j})$ is a $(t+1) \times (t+1)$ matrix

$$\lambda_{t,0} = \lambda,\ \lambda_{1,0} = r \text{ and } \lambda_{0,0} = b$$

$0 < t \leq k \leq v$ ensures non-degeneracy

(Cameron, BSW 2002)

Finite t and λ

When t and λ are both FINITE:

- $\lambda_{t,0} = \lambda$
- $\lambda_{i,j} = v, \quad$ for all $i < t, \ 0 \leq i + j \leq t$

We can write t-(v, k, λ), as in the finite, case without ambiguity

These designs are generally well behaved:

- Fisher's Inequality $b \geq v$ holds since $v = b$

From now on t and λ will be assumed to be FINITE

Existence and Large Sets

In contrast to the finite case, the existence problem for INFINITE t-designs is incomparably simpler — basically, they exist!

Existence with $t \geq 2$

k FINITE

- Cyclic t-$(\aleph_0, k, \lambda)$ (Köhler 1977)
- Large sets t-$(\infty, t+1, 1)$ (Grannell, Griggs, Phelan 1991)
- Large sets t-$(\infty, k, 1)$ (Cameron 1995)
- t-fold transitive t-$(\aleph_0, t+1, 1)$ (Cameron 1984)
- Uncountable family of rigid 2-$(\aleph_0, 3, 1)$ (Franek 1994)

k not necessarily FINITE

- Any t-$(\infty, k, 1)$ can be extended (Beutelspacher, Cameron 1994)

Block's Lemma

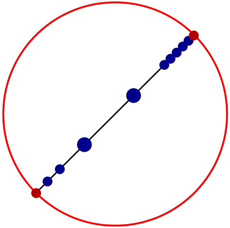
Block's Lemma (1967)

G any automorphism group of a (FINITE) t-(v, k, λ) design with m orbits on the v points and n on the b blocks

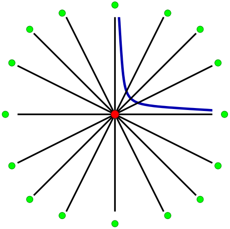
$$m \leq n \leq m + b - v$$

There is no infinite analogue of Block's Lemma

Examples of linear spaces: k INFINITE



$n = 1$ and $m = 2$
(Valette 1967)



$n = 2$ and $m = 3$
(Prazmowski 1989)

Orbit Theorems

k FINITE

Steiner Triple Systems

A 2-$(v, 3, 1)$ design has at least as many block orbits as point orbits ($n \geq m$) (Cameron 1994)

2-(∞, k, λ) Designs

$$n \geq \frac{m + \binom{m}{2}}{\binom{k}{2}} \quad \text{so} \quad n \geq m \quad \text{if} \quad n \geq k^2 - k$$

(BSW 1997)

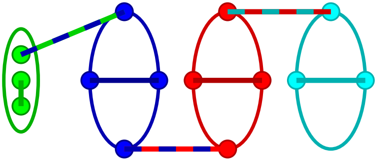
2-$(v, 3, \lambda)$ Designs

A 2-$(v, 3, \lambda)$ design has at least as many block orbits as point orbits ($n \geq m$) (BSW 1997)

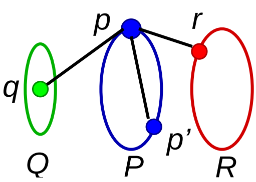
Sketch Proofs

Let G be an automorphism group of a 2-(∞, k, λ) design

Colour the m point orbits with m colours:



- λ blocks between any pair of points
- colours of blocks are G-invariant



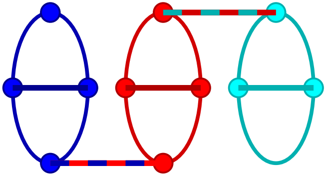
only finitely many blocks through p and points of Q

but infinitely many through p with points of $P \backslash p$ and R

so infinite orbits with p' and r but not q

- so to minimise n we can consider only infinite point orbits

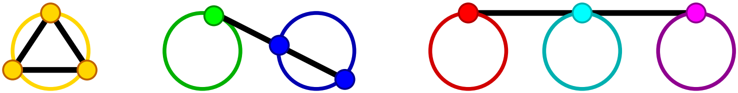
Sketch Proofs



- Crude bound: $n \geq \dfrac{m + \binom{m}{2}}{\binom{k}{2}}$

  So $n \geq m$ holds if $n \geq k^2 - k$

- $k = 3$



A 2-$(v, 3, \lambda)$ design has at least as many block orbits as point orbits

Designs with more point orbits than block orbits

Model Theoretic construction of Hrushovski (1993) used to construct

- $2\text{-}(\aleph_0, 4, 14)$ design with $n = 1$ and $m = 2$ (Evans 1994?)
- $2\text{-}(\aleph_0, k, k+1)$ designs with $k \geq 6$, $n = 1$ and $m = 2$ (Camina 1999)
- $2\text{-}(\aleph_0, k, \lambda)$ designs with $k \geq 4$, $n = 1$ and $m \leq k/2$ for some λ (BSW 1999)
- in particular a block transitive $2\text{-}(\aleph_0, 4, 6)$ design with two point orbits
- $2\text{-}(\aleph_0, 4, \lambda)$ designs with $n \leq m$ (where n is *feasible*) for some λ (BSW 1999)
- $t\text{-}(\aleph_0, k, 1)$ designs with $k > t \geq 2$, $n = 1$ and $m \leq k/t$ (Evans 2004)
- in particular a block transitive $2\text{-}(\aleph_0, 4, 1)$ design with two point orbits

Existential Closure Property

Block Intersection Graph of a Design $\mathcal{D}$

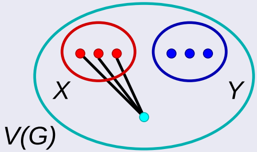
$G_{\mathcal{D}}$ has vertex set the blocks of $\mathcal{D}$

- two vertices are joined if the two blocks share at least one point

n-Existential Property of Graphs

A graph G is said to be n-existentially closed, or n-e.c., if

- for each pair (X, Y) if disjoint subsets of the vertex set $V(G)$ with $|X| + |Y| \leq n$
- there exists a vertex in $V(G) \backslash (X \cup Y)$ which is adjacent to each vertex in X but to no vertex in Y



(Erdős, Rényi 1963)

Existential Closure of Block Intersection Graphs

Existential closure number $\Xi(G)$, is the largest n for which G is n-$e.c.$ (if it exists)

Finite Steiner Triple Systems

- a 2-$(v, 3, 1)$ design is 2-$e.c.$ iff $v \geq 13$
- if a 2-$(v, 3, 1)$ design is 3-$e.c.$ then $v = 19$ or 21

(Forbes, Grannell, Griggs 2005)

In fact, only 2 of the STS(19) are 3-$e.c.$ and 'probably' none of the STS(21)

Finite 2-(v, k, λ) Designs

- $\Xi(G_{\mathcal{D}}) \leq k$, if $\lambda = 1$
- $\Xi(G_{\mathcal{D}}) \leq \left\lfloor \dfrac{k+1}{2} \right\rfloor$, if $\lambda \geq 2$

(McKay, Pike 2007)

Existential Closure: INFINITE Designs

k FINITE

- $\Xi(G_{\mathcal{D}}) = \min\{t, \left\lfloor \frac{k-1}{t-1} \right\rfloor + 1\}$ if $\lambda = 1$ and $2 \leq t \leq k$
- $2 \leq \Xi(G_{\mathcal{D}}) \leq \min\{t, \lceil \frac{k}{t} \rceil\}$ if $\lambda \geq 2$ and $2 \leq t \leq k-1$

(Pike, Sanaei 2011)

k INFINITE, $k < v$

- $\Xi(G_{\mathcal{D}}) = t$, if $t = 1$ or $\lambda = 1$, but $(t, \lambda) \neq (1, 1)$
- $2 \leq \Xi(G_{\mathcal{D}}) \leq t$, if $t \geq 2$ and $\lambda \geq 2$ (Horsley, Pike, Sanaei 2011)

k INFINITE, $k = v$

t and λ positive integers such that $(t, \lambda) \neq (1, 1)$

- there exists a t-$(\infty, \infty, \lambda)$ design with $\Xi(G_{\mathcal{D}}) = n$
- there exists a t-$(\infty, \infty, \lambda)$ design which is n-e.c.

for each non-neg integer n (Horsley, Pike, Sanaei 2011)

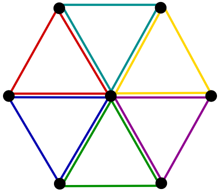
Resolvability

A resolution class (parallel class) in a design is a set of blocks that partition the point set
A design is resolvable if the block set can be partitioned into resolution classes

The Euclidean Plane: $2\text{-}(2^{\aleph_0}, 2^{\aleph_0}, 1)$ is resolvable

The Projective Plane: $2\text{-}(2^{\aleph_0}, 2^{\aleph_0}, 1)$ is NOT resolvable

The Triangular Lattice: $2\text{-}(\aleph_0, 3, 2)$ design is resolvable



Existence of Resolvable Infinite Designs

$k < v$

- any t-(∞, k, λ) design is resolvable with v resolution classes of size v (Danziger, Horsley, BSW 201?)

$k = v$

- There exists a 2-$(\infty, \infty, 1)$ design with $\Xi(G_{\mathcal{D}}) = 0$ iff there exists a resolvable 2-$(\infty, \infty, 1)$ design
- A resolvable t-$(\infty, \infty, 1)$ design has v resolution classes of v blocks (Horsley, Pike, Sanaei 2011)
- A resolvable t-$(\infty, \infty, \lambda)$ design has v resolution classes of v blocks and up to $\lambda - 1$ short resolution classes with less than v blocks (Danziger, Horsley, BSW 201?)

There exists a 2-$(\aleph_0, \aleph_0, 2)$ design with $\aleph_0$ resolution classes of size $\aleph_0$ and one resolution class of 4 blocks

Sparse, Uniform and Perfect Triple Systems

An *r*-sparse STS contains no $(n, n+2)$-configurations for $4 \leq n \leq r$

A uniform STS has all its cycle graphs $G_{a,b}$ isomorphic

A perfect STS has each cycle graph $G_{a,b}$ a single cycle of length $v - 3$

FINITE Steiner Triple Systems

- Infinitely many 4, 5 and 6-sparse systems but no non-trivial r-sparse systems known for $r \geq 7$
- Only finitely many uniform systems known, apart from the Affine, Projective, Hall and Netto triple systems
- Only finitely many perfect systems known

Countably INFINITE Steiner triple Systems

$2^{\aleph_0}$ nonisomorphic CISTs that are

- r-sparse for all $r \geq 4$
- uniform (Chicot, Grannell, Griggs, BSW 2009)

Universality and Homogeneity

A countable structure M is

- universal with respect to a class of structures C if M embeds every member of C
- homogeneous if every isomorphism between finite substructures can be extended to an automorphism of M

There is no universal countable Steiner Triple System (Franek 1994)

There is a unique (up to isomorphism) universal homogeneous locally finite Steiner Triple System, $\mathcal{U}$ (Cameron 2007?)

NOTE: In work on linear spaces, homogeneous as defined here is called ultrahomogeneous

The classification of ultrahomogeneous linear spaces (Devillers, Doyen 1998)

does *not* extend to Steiner Systems

The Fraïssé Limit

Fraïssé's Theorem

Suppose C is a class of finitely generated structures such that

- C is closed under isomorphisms
- C contains only countably many members up to isomorphism
- C has the Hereditary Property, HP
- C has the Joint Embedding Property, JEP
- C has the Amalgamation Property, AP

Then there is a countable homogeneous structure $\mathcal{S}$

- which is universal for C
- unique up to isomorphisms

We call $\mathcal{S}$ the Fraïssé limit of C (Fraïssé 1954, Jónsson 1956)

Such a class is called an amalgamation class

Steiner Triple Systems

Regard an STS as a Steiner quasigroup

$$a \circ b = c \quad \text{iff} \quad \{a, b, c\} \text{ is a block} \quad (\text{and } x \circ x = x)$$

Then substructures (in the sense of model theory) are subsystems

The class of all finite STS is an amalgamation class — the Fraïssé limit is the universal homogeneous locally finite STS, $\mathcal{U}$

The class of all finitely generated STS is NOT an amalgamation class

The class of all affine triple systems is an amalgamation class — the Fraïssé limit is the countably infinite affine triple system, $\mathcal{A}$

The class of all projective triple systems is an amalgamation class — the Fraïssé limit is the countably infinite projective triple sysytem, $\mathcal{P}$

$\aleph_0$-categoricity

A structure is $\aleph_0$-categorical if its automorphism group is oligomorphic
That is, it has finitely many orbits on n-tuples for each positive integer n

- $\mathcal{U}$ is not $\aleph_0$-categorical
- $\mathcal{A}$ and $\mathcal{P}$ are both $\aleph_0$-categorical

Let S and T be two $\aleph_0$-categorical STS

- the direct product $S \times T$ is $\aleph_0$-categorical — the direct product of oligomorphic groups is oligomorphic
  (Cameron, Gerwurz, Merola 2008)
- $d[S]$, the result of applying the doubling construction to S, is $\aleph_0$-categorical (Barbina, Chicot, BSW 201?)

Summary

- In general countably infinite Steiner systems are quite well behaved
- In general infinite designs exist
- Other FINITE type concepts can be investigated for INFINITE designs
- Work on INFINITE designs can lead to interesting new problems in the FINITE world

- Keep t and λ FINITE to preserve your sanity!