A Dynamic Programming Approach to Counting Hamiltonian Cycles in Bipartite Graphs

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A Hierarchy of Problems

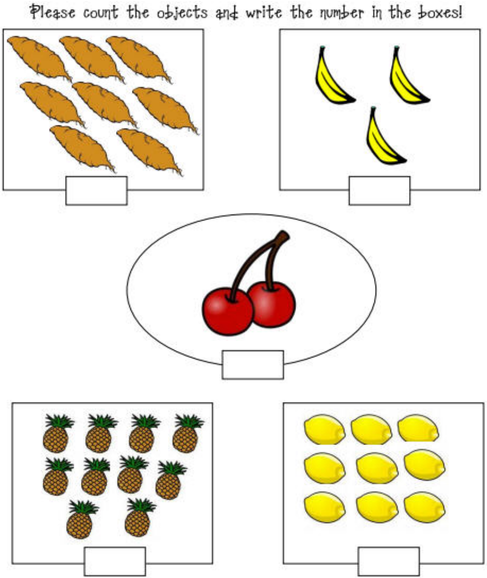
Existence problem Given a collection of properties, decide whether there exists an object realizing these properties.

Counting problem Given a collection of properties, count the number of distinct objects meeting the properties. Two versions: all, all up to isomorphism/equivalence.

Classification problem Given a collection of properties, describe, up to some criterion of isomorphism, all the objects that have the desired properties.

Characterization problem Develop a deeper understanding of classified objects.

Counting



Counting by Dynamic Programming

Counting by Dynamic Programming

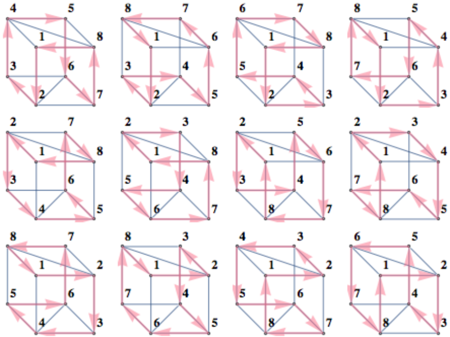
Count the matches in one box: 50

Count the boxes: 19

$50 \cdot 19 = 950$

Earlier: 1-factorizations of complete graphs, Latin squares,. . .

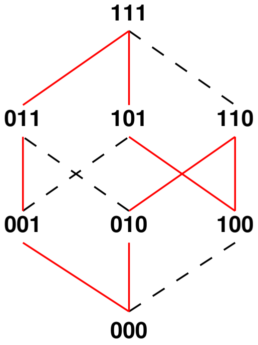
A) Directed **Hamiltonian cycles**:



B) (Undirected) **Hamiltonian cycles**: Divide #A by 2.

C) **Gray code**: 000, 001, 011, 111, 101, 100, 110, 010

Example: 3-Cube (Q_3)



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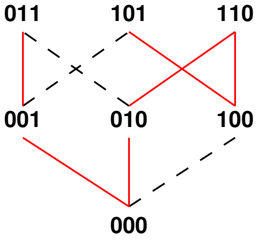
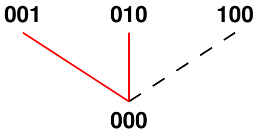


Example: 3-Cube (Q_3)



Example: 3-Cube (Q_3)

000

Outline of Algorithm

Consider a finite bipartite graph $\Gamma = (V, E)$.

1. Partition V into sets V_i such that every edge has its endpoints in consecutive sets V_k and V_{k+1} for some k.
2. For $k = 0, 1, \ldots$ build up (and count) collections of subpaths of a Hamiltonian cycle that are induced by $\cup_{i=0}^{j} V_i$ for $j = 0, 1, \ldots$.

Related old algorithm: W. Kocay, An extension of the multi-path algorithm for finding hamilton cycles, *Discrete Math.* **101** (1992), 171–188.

Utilizing Symmetries

The stabilizer of the partition $V_0, V_1, \ldots, V_M$ can be used to speed up the counting.

Q_n, the n-cube: V_i contains the vertices with Hamming weight i. The order of the stabilizer is $n!$.

Example. 3-cube:

$V_0 = \{000\}$
$V_1 = \{001, 010, 100\}$
$V_2 = \{011, 101, 110\}$
$V_3 = \{111\}$

A Bidirectional Approach, or Gluing

(a) Proceed $V_o \to V_1 \to$.
(b) Proceed $V_M \to V_{M-1} \to$.

Glue the structures when they meet!
Meet in the middle.

If there is an automorphism of the original graph mapping the elements of V_i to V_{M-i}, then (b) can be omitted.

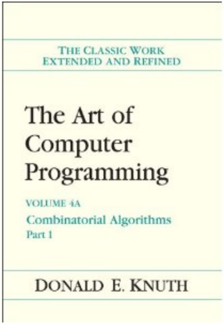
Two gluing strategies

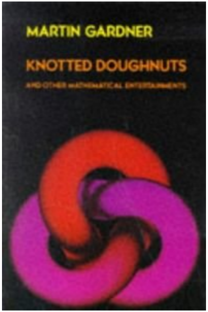
Two gluing strategies

1. Make exhaustive attempts
2. Determine what the counterpart should look like

The 6-Cube

The number of Hamiltonian cycles of the n-cube is 0, 1, 6, 1344, 906 545 760 for $i = 1$, 2, 3, 4, and 5, respectively (A066037 in the OEIS). The case $n = 6$ has attracted a lot of interest along the years:





Solutions up to Level 3, up to Equivalence

Level 0: 1 solution (counter value 1)
Level 1: 1 solution (counter value 15)
Level 2: 3446 solutions

Paths	#
1	13 495
2	263 305
3	2 782 510
4	17 003 576
5	61 154 671
6	127 360 225
7	142 398 993
8	65 084 556
9	7 887 199
10	139 098
Total	424 087 628

Glued Hamiltonian Cycles of the 6-cube

Paths	#
1	269 635 088 041 094 880
2	19 221 791 375 622 767 040
3	361 924 641 407 769 994 080
4	2 623 087 675 470 868 439 040
5	8 443 693 910 745 312 544 800
6	12 696 602 985 718 261 583 040
7	8 812 957 118 756 042 697 120
8	2 606 036 710 760 600 434 560
9	268 829 026 417 644 883 200
10	5 590 226 830 719 432 960
Total	35 838 213 722 570 883 870 720

Note. Deza and Shklyar make incorrect claims for $n = 6$ in arXiv:10043.4291v1

arXiv:1004.4391v1

Enumeration of Hamiltonian Cycles in 6-cube

March 24, 2010

Michel Deza[1] and Roman Shklyar[2]

Abstract

Finding the number $2H_6$ of directed Hamiltonian cycles in 6-cube is problem 43 in Section 7.2.1.1 of Knuth's *The Art of Computer Programming* ([Kn10]); various proposed estimates are surveyed below. We computed exact value:

H_6=14,754,666,508,334,433,250,560=6!*2^4*217,199*1,085,989*5,429,923.

Also the number Aut_6 of those cycles up to automorphisms of 6-cube was computed as 147,365,405,634,413,085

Key Words: hypercube, Hamiltonian cycle, computation.

A *Hamiltonian cycle* in a graph is a cycle that visits each vertex exactly once. Let H_n denote the number of Hamiltonian cycles in *n-cube*(the graph of n-dimensional hypercube). An *automorphism* of a graph is a permutation of its vertex-set preserving its edge-set. Let Aut_n denote the number of Hamiltonian cycles in n-cube up to the group of automorphisms of n-cube.

Counting Equivalence Classes of Hamiltonian Cycles

N Total number of Hamiltonian cycles

N_i The number of equivalence classes of Hamiltonian cycles with an automorphism group of order i

By the Orbit-Stabilizer Theorem,

$$N = \sum_i \frac{|G|N_i}{i} = \sum_i \frac{2^n n! N_i}{i} \tag{1}$$

1. Determine $N_2, \ldots$
2. Solve N_1 from (1)
3. Determine $\sum_i N_i$

Hamiltonian Cycles with Prescribed Automorphisms

When considering automorphisms of Hamiltonian cycles in a graph $\Gamma = (V, E)$, it is convenient that these can be considered both as

- subgroups of $\mathrm{Aut}(\Gamma)$ and
- subgroups of $\mathrm{Aut}(C_{|V|})$.

Lemma 1. A Hamiltonian cycle in the n-cube cannot have an automorphism of prime order greater than 2.

A Hamiltonian cycle consists of the union of two perfect matchings.

Lemma 2. Let $n \geq 3$. The automorphisms of a Hamiltonian cycle in the n-cube stabilizes the two perfect matchings formed by taking every second edge of the cycle.

Hamiltonian Cycles with Prescribed Automorphisms

Classification is carried out via perfect matchings.

$\|\text{Aut}\| \setminus$ Type	All	Reflected
2	7 001 923 981	4 369 328 232
4	220 165	195 606
8	568	494
16	20	20
Total	7 002 144 734	4 369 524 352

It now follows that there are 777 739 016 577 752 714 inequivalent Hamiltonian cycles in the 6-cube.

Details Regarding Computations

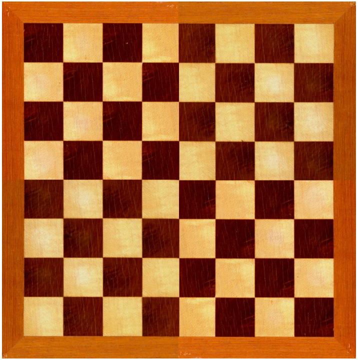
CPU-time: Gluing for the 6-cube took just under 10 core-years.

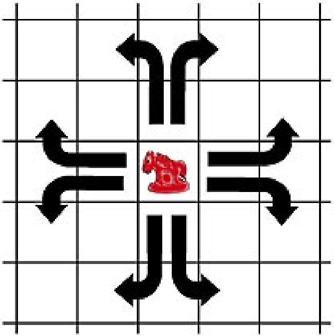
Memory: Up to 8GB.

Validation: Double counting and independent implementations, etc. $\Rightarrow r \times 10$ core-years...

Implementation: Some subproblems and many technical details omitted here.

Knight's Tours of a Chessboard





Partitions of Vertices

0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
2	3	2	3	2	3	2	3
3	2	3	2	3	2	3	2
4	3	4	3	4	3	4	3
3	4	3	4	3	4	3	4

8–16–16–16–8

0	1	2	1	2	3	2	3
1	2	1	2	3	2	3	4
2	1	2	3	2	3	4	3
1	2	3	2	3	4	3	4
2	3	2	3	4	3	4	5
3	2	3	4	3	4	5	4
2	3	4	3	4	5	4	5
3	4	3	4	5	4	5	6

1–6–15–20–15–6–1

Results

Old results:

M. Loebbing and I. Wegener, The number of knight's tours equals 33,439,123,484,294—Counting with binary decision diagrams, *Electron. J. Combin.* **3**(1) (1996), Research Paper 5 and Comment 1.

B. D. McKay, Knight's tours of an 8×8 chessboard, Technical Report TR-CS-97-03, Computer Science Department, Australian National University, Canberra, 1997.

Our result:

The number of partial solutions is 1, 143 379, and 95 345 608 on the levels 0, 1, and 2, respectively. Gluing $\Rightarrow$ 13 267 364 410 532 Hamiltonian cycles (=McKay).

Final Comment and Generalizations

The main contribution here is not the *numbers* but the *algorithm*.

Note! The problem of determining the number of Hamiltonian cycles in a graph is #P-complete and determining whether it is > 0 is NP-complete.

Possible variants and generalizations:

- Consider nonbipartite graphs
- Consider directed graphs
- Count Hamiltonian *paths*
- Count (perfect) matchings
- Snake-in-the-box (longest induced path)

The End

THANK YOU!!! QUESTIONS?