Connectivity for Wireless Agents Moving on a Cycle or Grid*

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Abstract. We present a mathematical model to analyse the establishment and maintenance of communication between mobile agents. We assume that the agents move through a fixed environment modelled by a motion graph, and are able to communicate if they are at distance at most d. As the agents move randomly, we analyse the evolution in time of the connectivity between a set of w agents, asymptotically for a large number N of nodes, when w also grows large, and for different values of d. The particular topologies of the environment we study in this paper are the cycle and the toroidal grid.

1 Introduction

Consider a setting in which a large number of mobile agents can perform concurrent basic movements: ahead/behind/left/right, determining a grid pattern, or left/right, describing a line. Each agent can communicate directly with any other agents which are within a given distance d. This enables communications with agents at a further distance using several intermediate agents. At each step in time there is an ad-hoc network defined by the dynamic graph whose vertex set consists of the agents, with an edge between any two agents iff they are within the distance d of each other.

In this paper, we study the static and dynamic connectivity characteristics of communicating agents, in a framework called the *walkers model*, which we define as follows. A connected graph $G = (V, E)$ with $|V| = N$ is given, and a number w of *walkers* (agents). Also given is a "distance" d. A set W of walkers, with $|W| = w$, are placed randomly and independently on the vertices of G (a vertex may contain more than one walker). Each walker has a range d for communication; that is, two walkers w_1 and w_2 can communicate in one hop if the distance, in G, between the position of the walkers is at most d. Two walkers can communicate if they can reach each other by a sequence of such hops. In

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addition, each walker takes an independent standard random walk on G, i.e. moves at each time step to a eighbouring vertex, each neighbour chosen with equal probability.

The interesting features of the walkers model are encapsulated by the *graph of walkers*, $G_f[W]$. Here f is an random assignment $f : W \to V$ of walkers into the vertices of G. The vertices of $G_f[W]$ are the vertices in G that contain at least one walker, two vertices in $G_f[W]$ being joined by an edge iff they are at distance at most d in G. We refer to *components* of $G_f[W]$ in the usual sense, and call a component *simple* if it is formed by only one isolated vertex. We are interested in the probability of $G_f[W]$ being connected, or in the number of components and their sizes, with mild asymptotic restrictions on w and d.

Our primary goal with the walkers model is to characterise the dynamics of the connectivity of the network. To obtain enough information to do this, we first examine a variation of the model called the *static* model. This is a snapshot of the model at one point in time: there is merely the random function f, and we are interested in the distribution of the number of components, as well as other information which helps to answer the dynamic questions.

In the dynamic situation, there is an initial placement of walkers as in the static case, and at each time step, every walker simultaneously moves one step to a randomly selected neighbour vertex in G. This gives rise to a random graph process, where $G_{f_t}[W]$ denotes the graph of walkers at time $t = 0, 1, \ldots$. We are interested in studying the birth and death of components, and the sudden connection and disconnection of $G_{f_t}[W]$ in a dynamic setting.

We consider a sequence of graphs G with increasing numbers of vertices N, for N tending to infinity. The parameters w and d are functions of N. We restrict to the case $w \to \infty$ in order to avoid considering small-case effects. Of course we take $d \geq 1$. We make further restrictions on w and d in order to rule out non interesting cases, such as values of the parameters in which the network is asymptotically almost surely (a.a.s.) disconnected or a.a.s. connected. In this paper, we study the walkers model for two particular sequences of graphs G: the cycle of length N and the $n \times n$ toroidal grid. (In the case of the grid, we use the ℓ^1 distance, modelling the distance along roads in a city grid, but our approach is useful for other metrics.) Amongst other things, we determine the limiting probability of connectedness of the graph for the appropriate range of w and d, and also the expected time the graph spends in the connected state after it undergoes a transition from disconnected to connected (and similarly, for the disconnected state).

Nowadays, the *random geometric graph* has became the basic network model to study communication in wireless networks. In this model, the broadcasting stations (centre of the disk) could be distributed according to a Poisson process or uniformly at random on a bounded subset of $\mathbb{R}^2$, see for example [8]. For instance, it is known that for a random geometric graphs with n vertices and radius d (where d is a function of n), a.a.s. the graph is connected if $d \geq \sqrt{\log n / n}$ [7]. The theoretical results obtained not only on connectivity but also on other graph parameters like chromatic number, have help in dealing with more technological issues as efficient broadcasting algorithms for wireless ad-hoc networks or message congestion, using as a basis the static situation. In the present paper, we obtain much sharper results on the static properties than previously obtained (albeit with a slightly different model). We give precise characterisations of connectivity for two graphs: the cycle and the toroidal grid with the Manhattan distance. In particular, given a grid with n^2 nodes, where we sprinkle uniformly at random (u.a.r.) w walkers on the nodes of grid, and given $d = o(n)$, we give a specific equation for the expected number μ of simple components in the grid, as the ratio w/n^2 tends to $0, c$ or ∞. From these expressions, we deduce the connectivity threshold for $G[W]$, when $\mu \to \infty$ (disconnected a.a.s.), when $\mu \to \Theta(1)$ (simple components except one isolated component) and when $\mu \to 0$ (connected a.a.s.).

In recent times, the big issue has been the mobility of the agents, where connections in the network are created and destroyed as the agents move further apart or closer together. There has been quite a bit of work designing efficient communication algorithms for motion agents, see [1, 6] for nice surveys. Most of the work is experimental [9]. Other interesting work deals with a data structure which is able at time t to decide quickly if two given stations are connected [5]. However, no theoretical work has been done with the global connectivity properties of dynamic wireless networks. Again we consider first the case of the cycle and the toroidal grid. In particular for the toroidal grid, we give firstly a precise estimation of the probability that, if the walkers are connected, they become disconnected in the next step (Theorem 8). Then using that result, we give precise asymptotic estimations on the expected number of steps that the grid will maintain connected (once it becomes connected) or disconnected, as the agents perform random movements on the nodes of the grid (Theorem 10). We believe that the study of the behaviour of multiple, simultaneous random walks is an important open problem which could have further applications in other fields of computer science. By lack of space, the proofs as well as the results on the grid for the l_2 norm, are left for the long version.

2 General definitions and basic results

The reader is referred to [3] for the basic definitions and theorems on probability. As usual, for any integer n, we use $[n] = \{1, 2, \dots, n\}$ and for any integers n and m, $[n]_m = n!/(n - m + 1)!$.

For our specific work, we begin with some definitions and results which are common for all G. Define K to be the random variable counting the number of connected components, in $G_f[W]$, under random assignment f of walkers. Let ϱ denote the expected number of walkers at a vertex. Then $\varrho = w/N$. For $v \in V$, define h_v to be the number of vertices in G at distance at most d from v, and define $h = \min_{v \in V} h_v$. Notice that h is the minimum number of empty vertices in G around a simple component. (We say that a vertex v is *empty* if it contains no walkers, and *occupied* if it contains at least one.)

By considering the well known coupon-collector's problem, we observe that if $w = N \log N + \omega(N)$ then $G_f[W]$ is trivially a.a.s. connected due to every vertex being occupied. Moreover, for the graphs G which we consider in this paper, if $h \in \Omega(N/\sqrt{w})$ then $G_f[W]$ is a.a.s. connected as well. This last claim will be seen in Observations 1 and 3. Thus, we consider throughout the paper $w < N \log N + O(N)$ and $h = o(N/\sqrt{w})$. In fact, our proofs will just require h to be $o(N)$.

As a key step in most of our proofs, we often need to compute the probability of having a certain configuration of walkers at a given time t. For this we apply Lemma 1 below.

Sometimes we also need the probability of certain configurations of walkers involving two consecutive time steps, in order to record the event that walkers jump to the appropriate place at time t. There is a convenient way to view this by partitioning every vertex v of G into as many *sub-vertices* as its degree, where each sub-vertex of v is associated with a different neighbour of v. Any given walker on one vertex will occupy the sub-vertex corresponding to the neighbour to which it will move at the next time step. Thus, a walker at v moving to a neighbour will occupy each of the sub-vertices of v with the same probability. In this case, we can also apply Lemma 1 with sub-vertices, since these form a static configuration (even though it encodes a dynamic transition).

Assign *size* 1 to all vertices in G. For a given sub-vertex in a vertex v with degree δ_v, its *size* will be $1/\delta_v$. Given a set A of vertices or sub-vertices, we define the *size* of A to be the sum of the sizes of its elements.

The following lemma comes by inclusion-exclusion.

Lemma 1 *Let $A_0, \ldots, A_m$ be pairwise disjoint sets of vertices (or sub-vertices) in G, with sizes $S_0, \ldots, S_m$ respectively. Let $N = |V(G)|$. If $\sum_{i=0}^{m} S_i = o(N)$, then*

$$\mathbf{P}\left(A_0 \text{ empty } \wedge \bigwedge_{i=1}^{m} A_i \text{ not empty}\right) \sim \left(1 - \frac{S_0}{N}\right)^{w} \prod_{i=1}^{m} \left(1 - e^{-S_i \varrho}\right).$$

To cover large sizes S (not necessarily $o(N)$) we need the following variation on the previous lemma.

Lemma 2 *Let A be a set of vertices in G of size S, and $v_1, \ldots, v_m$ vertices not in A, with $m \geq 1$. Assume $|V(G)| = N$ and $N - S \to \infty$. The probability that no vertex in A is occupied and $v_1, \ldots, v_m$ are all occupied is at most $p_0 p^{m-1} \alpha^w$ where $p_0 = 1 - e^{-\varrho/\alpha}$, $\alpha = 1 - S/N$ and*

$$p = \begin{cases} 1 & \text{if } \rho/\alpha \to \infty, \\ \rho/\alpha & \text{if } \rho/\alpha = O(1). \end{cases}$$

3 The cycle

Let $G = C_N$ be the cycle with N nodes.

Observation 1 *Notice that for C_N, $h = 2d$. Cover the cycle with $\lceil \frac{N}{\lceil d/2 \rceil} \rceil$ paths of $\lceil d/2 \rceil$ vertices. If $h = \Omega(N/\sqrt{w})$, then the probability that some path is empty of walkers is at most*

$$\left\lceil \frac{N}{\lceil d/2 \rceil} \right\rceil \left(1 - \frac{\lceil d/2 \rceil}{N}\right)^w \leq O(\sqrt{w})e^{-\Omega(\sqrt{w})} \to 0$$

Thus, a.a.s. each of these paths is occupied by at least one walker, and $G_f[W]$ is connected. So we assume for the rest of the section that $h = o(N/\sqrt{w})$, in fact the assumption $d = o(N)$ is all we require.

To study the connectivity of C_N, we introduce the concept of *hole*. Let us say there is a *hole* between two vertices u and v if u and v each contain at least one walker, but no vertex in the clockwise path from u to v contains a walker. We say that such a hole *follows* u, or that u is the *start vertex* of the hole. The number of internal vertices in a hole is its *size*. An *s-hole* is a hole whose size is at least s. Notice that at least two d-holes are needed to disconnect the walkers on C_N.

Let H be the random variable counting the number of d-holes, when w walkers are placed u.a.r. on C_N, and let $\mu_H = \mathbf{E}[H]$ be its expectation (just μ for short throughout this section).

Holes are closely related to components: trivially,

$$K = \begin{cases} 1 & \text{if } H = 0, 1, \\ H & \text{if } H > 1. \end{cases} \qquad \text{and thus} \quad \mathbf{E}[K] = \mathbf{P}(H = 0) + \mathbf{E}[H]. \tag{1}$$

Static properties Here, we study the connectivity of the graph of walkers $G_f[W]$ in the static situation, by analysing the behaviour of H. In view of (1), notice that if $\mu \to 0$ then $\mathbf{P}(K = 1) \to 1$, i.e. $G_f[W]$ is a.a.s. connected.

Theorem 1 *The expected number of holes satisfies*

$$\mu \sim N\left(1 - e^{-\varrho}\right)\left(1 - \frac{d}{N}\right)^w \sim \begin{cases} w\left(1 - \frac{d}{N}\right)^w & \text{if } \varrho \to 0, \\ N\left(1 - e^{-\varrho}\right)\left(1 - \frac{d}{N}\right)^w & \text{if } \varrho \to c, \\ N\left(1 - \frac{d}{N}\right)^w & \text{if } \varrho \to \infty. \end{cases}$$

Furthermore, if μ is bounded then H is asymptotically Poisson with mean μ, and if μ is bounded away from 0 then $\mu \sim N\left(1 - e^{-\varrho}\right) e^{-d\varrho}$.

The proof is done by estimating the factorial moments of H, using indicator variables.

From the second part of this theorem we can immediately obtain the probability that $G_f[W]$ is connected, when w walkers are scattered u.a.r. through the vertices of C_N.

Corollary 1 *If w walkers are placed u.a.r. on C_N, then $\mathbf{P}(K = 1) = e^{-\mu}(1 + \mu) + o(1)$.*

This implies that $G_f[W]$ is a.a.s. disconnected if $\mu \to \infty$, and a.a.s. connected if $\mu \to 0$. So we may restrict attention to $\mu = \Theta(1)$.

It is a simple matter determine from this and the first part of Theorem 1 the threshold value of ρ (or of d) at which the walkers graph becomes connected, and the probability of connectedness when around the threshold.

Dynamic properties Assume that from an initial random placement f of the walkers, at each step, every walker moves from its current position to one of its neighbours, with probability 1/2 of going either way. This is a standard random walk on the cycle for each walker. To study the connectivity properties of the dynamic graph of walkers we need to introduce some notation.

A *configuration* is an arrangement of the w walkers on the vertices of C_N. Consider the graph of configurations, where the vertices are the N^w different configurations. Any configuration has 2^w neighbours, and the dynamic process can be viewed as a random walk on the graph of configurations, in particular, a Markov chain $\mathcal{M}(N, w)$. If N is odd, then $\mathcal{M}(N, w)$ is ergodic. For the purposes of this extended abstract, we will treat in detail only the case of N odd, if non-ergodicity causes extra complications.

Observation 2 *For any fixed t, we can regard $G_{f_t}[W]$ as $G_f[W]$ in the static case. Hence, by Corollary 1, if $\mu \to 0$ or ∞ then, for t in any fixed bounded time interval, $G_{f_t}[W]$ is either a.a.s. connected or a.a.s. disconnected. So we assume $\mu = \Theta(1)$ for the remaining of the section, since we wish to study only the nontrivial dynamic situations.*

We define $H = H(t)$ to be the random variable that counts the number of d-holes at time t. Under the assumptions in Observation 2, for t in any fixed bounded time interval, $H(t)$ is asymptotically Poisson with expectation $\mu = \Theta(1)$ as studied in C_N.

For the dynamic properties of $G_{f_t}[W]$, we are interested in the probability that a new d-hole appears at a given time. Moreover, we require knowledge of this probability conditional upon the number of d-holes already existing.

If there is a d-hole from u to v at time t and all walkers at u and v move in the same direction on the next step, a new d-hole may appear following one of the neighbours of u (provided no new walkers move in to destroy this). These two d-holes, though being different, are related, and we prefer to think of them as the same thing. A similar comment applies when the exact size of a d-hole following u changes in one step. Define a *d-hole line* to be a maximal sequence of pairs $(h_1, t_1), \ldots, (h_l, t_l)$ where h_i is a d-hole existing at time t_i for $1 \le i \le l$, and such that $t_i = t_{i-1} + 1$ and the start vertex of h_i is adjacent to, or equal to, the start vertex of h_{i-1}, for $2 \le i \le l$. Fix two consecutive time steps t and $t + 1$. If $t_1 = t + 1$, we say that the line is *born* between t and $t + 1$, if $t_l = t$ the line *dies* between t and $t + 1$, and if $t = t_i$, $i \in \{1, \ldots, l - 1\}$ we say that the line *survives* during the interval $[t, t + 1]$. Note that the time-reversal of the process has a d-hole line born at vertex u between $t + 1$ and t iff the d-hole line dies at u between t and $t + 1$. Define $S = S(t)$, $B = B(t)$ and $D = (t)$ to be the number of d-hole lines surviving, being born and dying between t and $t+1$. We obviously have $D(t) + S(t) = H(t)$ and $B(t) + S(t) = H(t+1)$.

Theorem 2 *For t in any fixed bounded time interval, the random variables $S(t)$, $B(t)$ and $D(t)$ are asymptotically jointly independent Poisson, with the expectations*

$$\mathbf{E}\left[S\right] \sim \begin{cases} \mu & \text{if } \varrho \to 0, \\ \mu - \lambda & \text{if } \varrho \to c, \\ 3\mu e^{-\varrho} & \text{if } \varrho \to \infty, \end{cases} \quad \text{and} \quad \mathbf{E}\left[B(t)\right] = \mathbf{E}\left[D(t)\right] \sim \begin{cases} \frac{1}{2}\mu\varrho & \text{if } \varrho \to 0, \\ \lambda & \text{if } \varrho \to c, \\ \mu & \text{if } \varrho \to \infty, \end{cases}$$

where $\lambda = \left(1 - \dfrac{3e^{-\varrho} - e^{-\frac{3}{2}\varrho}}{1 + e^{-\frac{1}{2}\varrho}}\right)\mu$. *Here $0 < \lambda < \mu$ for $\varrho \to c$.*

Under the assumptions in Observation 2 and using this result, we can obtain several important consequences. The first gives the probability of having no holes at time t and at least one at time $t+1$. Note that more than one hole is required in C_N to disconnect $G_{f_t}[W]$. As $H(t) = S(t) + D(t)$, it follows from the theorem that $H(t)$ and $B(t)$ are asymptotically independent. We can write $\mathbf{P}\left(H(t+1) \geq 1 \wedge H(t) = 0\right) = \mathbf{P}\left(H(t) = 0 \wedge B(t) \geq 1\right)$, and immediately obtain the following.

Corollary 2 *Let λ be defined as in Theorem 2. The probability of having no holes at time t and at least one at time $t+1$ is given by*

$$\mathbf{P}\Big(H(t+1) \geq 1 \wedge H(t) = 0\Big) \sim \begin{cases} \frac{1}{2}\mu e^{-\mu}\varrho & \text{if } \varrho \to 0, \\ e^{-\mu}\left(1 - e^{-\lambda}\right) & \text{if } \varrho \to c, \\ e^{-\mu}(1 - e^{-\mu}) & \text{if } \varrho \to \infty, \end{cases}$$

We define the *lifespan* of a d-hole line as the number of time steps for which the line is alive. For any vertex v and time t, the random variable $L_{v,t}$ counts the lifespan of the d-hole line born at vertex v between times t and $t+1$. If this birth does not take place $L_{v,t}$ is defined to be 0. Note that the random variables $L_{v,t}$ are identically distributed for all v and t. Notice that the expected lifespan of any given d-hole line is bounded (this bound depending on N).

Theorem 3 *Let λ be defined as in Theorem 2. For any vertex v and time t,*

$$\mathbf{E}\left[L_{v,t}\right] \sim \begin{cases} 2\varrho^{-1} & \text{if } \varrho \to 0, \\ \frac{\mu}{\lambda} & \text{if } \varrho \to c, \\ 1 & \text{if } \varrho \to \infty, \end{cases}$$

The next theorem gives us the probability that there is one component at time t, but at least two at time $t+1$.

Theorem 4 *Let λ be defined as in Theorem 2. The probability that $G_{f_t}[W]$ is connected and that $G_{f_{t+1}}[W]$ is disconnected is given by*

$$\mathbf{P}\Big(H(t+1) \geq 2 \wedge H(t) < 2\Big) \sim \begin{cases} \frac{1}{2}\mu^2 e^{-\mu}\varrho & \text{if } \varrho \to 0, \\ e^{-\mu}\left(1+\mu-(1+\mu+\lambda+\lambda^2)e^{-\lambda}\right) & \text{if } \varrho \to c, \\ (1+\mu)e^{-\mu}(1-(1+\mu)e^{-\mu}) & \text{if } \varrho \to \infty, \end{cases}$$

For any time t, let us condition upon $G_{f_t}[W]$ being disconnected at time t and becoming connected at $t+1$. Let T_C be a random variable measuring the time $G_{f_t}[W]$ remains connected. Similarly, let us condition upon $G_{f_t}[W]$ being connected at time t and becoming disconnected at $t+1$. Let T_D be a random variable measuring the time $G_{f_t}[W]$ remains disconnected. Their expectations do not depend on the chosen time t and are given in the following theorem.

Theorem 5 *Let λ be defined as in Theorem 2. The expected time that the graph of walkers $G_{f_t}[W]$ will be connected or disconnected (once it becomes so) is*

$$\mathbf{E}[T_C] \sim \begin{cases} 2\frac{1+\mu}{\mu^2}\varrho^{-1} & \text{if } \varrho \to 0, \\ \frac{1+\mu}{1+\mu-(1+\mu+\lambda+\lambda^2)e^{-\lambda}} & \text{if } \varrho \to c, \\ \frac{e^{\mu}}{e^{\mu}-(1+\mu)} & \text{if } \varrho \to \infty. \end{cases}$$

$$\text{and } \mathbf{E}[T_D] \sim \begin{cases} 2\frac{e^{\mu}-1-\mu}{\mu^2}\varrho^{-1} & \text{if } \varrho \to 0, \\ \frac{e^{\mu}-1-\mu}{1+\mu-(1+\mu+\lambda+\lambda^2)e^{-\lambda}} & \text{if } \varrho \to c, \\ \frac{e^{\mu}}{1+\mu} & \text{if } \varrho \to \infty, \end{cases}$$

4 The Grid

Let $G = T_N$ be the toroidal grid with $N = n^2$ nodes. We can refer to vertices by using coordinates in $\mathbb{Z}_n \times \mathbb{Z}_n$. For the grid we encounter significant new obstacles as compared to the cycle; see for instance the Geometric Lemma below. In T_N, we shall express the distance between two vertices as the minimal ℓ_1 distance of their coordinates. (In the long version, similar results are obtained for ℓ_2 norm).

Observation 3 *For T_N, and for $d < n/2$, the number of vertices at distance at most d from any given vertex is $h = 2d(d+1)$. For each $i, j < 8n/d$, let v_{ij} denote the point with coordinates $(\lfloor id/8 \rfloor, \lfloor jd/8 \rfloor)$. Let S_{ij} denote the set of grid points closer to v_{ij} than any of the other $v_{i'j'}$. Then there are $\Theta(N/d^2)$ disjoint sets S_{ij} each containing $\Theta(d^2)$ points. The probability that at least one of these S_{ij} is empty is at most $\Theta(N/d^2)(1-\Theta(d2/N))^w = O(\sqrt{w})e^{-\Omega(\sqrt{w})} \to 0$ if $h = \Omega(N/\sqrt{w})$. Thus, a.a.s. each of these pieces is occupied by at least one walker, and $G_f[W]$ is connected. So we assume for the rest of the section $h = o(N/\sqrt{w})$, or merely $h = o(N)$, i.e. $d = o(n)$.*

We wish to study the connection and disconnection of $G_f[W]$ in a similar way to the cycle. For the grid, the notion of hole does not help, and we deal directly with components. A major role is played by simple components, and we shall prove that, for the interesting values of the parameters, a.a.s. there only exist simple components and one giant one.

Let $\mathcal{C}$ be any given component. The *edges* of $\mathcal{C}$ are the straight edges joining occupied vertices in $\mathcal{C}$ at distance at most d. The associated *empty area* $A_{\mathcal{C}}$ is the set of vertices not in $\mathcal{C}$, but at distance at most d from some vertex in $\mathcal{C}$ (i.e. those vertices which must be free of walkers for $\mathcal{C}$ to exist as a component). The *exterior* $\mathcal{E}_{\mathcal{C}}$ of $\mathcal{C}$ is all those vertices not in $\mathcal{C} \cup A_{\mathcal{C}}$. We partition $\mathcal{E}_{\mathcal{C}}$ into *external regions* as follows: two vertices belong to the same external region when they can be joined by a continuous arc not intersecting any edge of $\mathcal{C}$.

Recall that, in the terminology of planar maps, the *bounding cycle* of a face is a walk around the boundary of the face. Given an external region ${\mathcal{E}_{\mathcal{C}}}^i$, let $\mathcal{C}'$ be any connected subgraph of $\mathcal{C}$ that has no edges crossing and such that no vertices of $\mathcal{C}$ are contained in the face F of $\mathcal{C}'$ which contains ${\mathcal{E}_{\mathcal{C}}}^i$. Such graphs exist: for instance, take the spanning tree of $\mathcal{C}$ whose length (sum of lengths of edges) has been minimised in ℓ_1, and, subject to this, has the shortest Euclidean length. The bounding cycle of this face F is defined to be a *boundary walk* β in $\mathcal{C}$ with respect to ${\mathcal{E}_{\mathcal{C}}}^i$. Such a walk is *maximal* if the face F does not properly contain a face of some other subgraph of $\mathcal{C}$. We call a (directed) closed walk in $\mathcal{C}$ *regular* if, for each edge entering a vertex v, the next edge in the walk is the next edge in the clockwise direction around v.

For $i < n$, let us call a *column* of width i any subset of T_N defined by $\{a, \dots, a+i-1\} \times \mathbb{Z}_n$. We define a *row* of height j similarly. Define a *rectangle* of width i and height j to be the intersection of a column of width i and a row of height j. Notice in a rectangle we can compare vertices inside according to their coordinates, and we shall use statments as v_1 *is more to the left than* v_2 or v_3 *is the uppermost vertex*.

We say that a component $\mathcal{C}$ with at least 2 vertices is a *rectangular component* (*r-component*) if all of its vertices, edges and empty area are contained in a rectangle in the torus of height and width at most $n-1$. In particular, this implies that $\mathcal{C}$ contains no nonseparating cycle of the torus. Otherwise, it is an *nr-component*. For a given r-component $\mathcal{C}$, we define its *origin* as the leftmost of the lower-most vertices of $\mathcal{C}$, with a canonical defintion of left and lower over a containing rectangle. The *outside region* of an r-component is the only external region of the component having points outside any containing rectagle.

Let X, Y and Z be the number of simple components, r-components and nr-components respectively, and put $K = X + Y + Z$. Let $Z = Z_1 + Z_2$, where Z_1 is the number of nr-components which cannot coexist with another nr-component and Z_2 counts those ones which can. Then $\mathbf{E}[Z] = \mathbf{P}(Z_1 = 1) + \mathbf{E}[Z_2]$. Set $\mu = \mu_X = \mathbf{E}[X]$, the expected number of simple components.

Static properties Let μ denote the expected number of simple components in the grid. The next theorem gives its value asymptotically.

Theorem 6 *The expected number of simple components satisfies*

$$\mu \sim N\left(1-e^{-\varrho}\right)\left(1-\frac{h}{N}\right)^{w} \sim \begin{cases} w\left(1-\frac{h}{N}\right)^{w} & \text{if } \varrho \to 0, \\ N(1-e^{-\varrho})\left(1-\frac{h}{N}\right)^{w} & \text{if } \varrho \to c, \\ N\left(1-\frac{h}{N}\right)^{w} & \text{if } \varrho \to \infty. \end{cases}$$

Furthermore, if μ is bounded then X is asymptotically Poisson with mean μ, whilst if μ is bounded away from 0 then $\left(1-\frac{h}{N}\right)^{w} \sim e^{-h\varrho}$ and we have $\mu \sim N\left(1-e^{-\varrho}\right)e^{-h\varrho}$.

The proof, analogous to the proof of Theorem 1, follows from Lemma 1.

From the previous theorem we can immediately obtain the probability of having no simple components, when w walkers are scattered u.a.r. throughout the vertices of T_N.

Corollary 3 *The probability of having no simple components is $\mathbf{P}(X=0) = e^{-\mu}+o(1)$. Furthermore, if $h\varrho = O(1)$ then $\mu \to \infty$ and $G_f[W]$ is disconnected a.a.s.*

We may now restrict to the condition $h\varrho \to \infty$ in the study of r-components and nr-components.

The next lemma relates the empty area outside a boundary cycle of component with its length, and will play a key role in proving the main results.

Lemma 3 (Geometric Lemma) *Let $\mathcal{C}$ be a component in T_N with β one of its maximal boundary walks, and $l = length(\beta)$ its length. Assume that $\mathcal{C}$ has at least two occupied sites. Then the size of the empty area A_β outside β is bounded below by $|A_\beta| \geq dl/R$, for some big enough constant R. Moreover, if $\mathcal{C}$ is rectangular, and β is a maximal boundary walk with respect to the outside region, we have $|A_\beta| \geq h + dl/R$.*

Lemma 4 *Let $\mathcal{C}$ be an nr-component which can coexist with other nr-components. Then it has a boundary cycle β with $length(\beta) \geq n - o(n)$.*

This lemma's proof (omitted) is effected by quantifying the intuitive idea that such a component must "wrap around" the torus.

The next technical result shows that simple components are predominant a.a.s. in T_N. The proof uses the Geometric Lemma.

Lemma 5 *If $h\varrho \to \infty$, then $\mathbf{E}[Y] = o(\mathbf{E}[X])$ and $\mathbf{E}[Z_2] = o(\mathbf{E}[X])$.*

The following theorem gives the connectivity of $G_f[W]$ in the static case, under various assumptions. The proof follows from Lemma 5, Theorem 6 and Chebyshev inequality

Theorem 7 • *For $\mu \to \infty$, $G_f[W]$ is disconnected a.a.s.*
• *For $\mu = \Theta(1)$, then $K = 1 + X$ a.a.s., and X is asymptotically Poisson.*
• *For $\mu \to 0$, $G_f[W]$ is connected a.a.s.*

From the previous theorem we immediately obtain that the probability that $G_f[W]$ is connected is $e^{-\mu} + o(1)$. Since $G_f[W]$ is a.a.s. disconnected if $\mu \to \infty$, and a.a.s. connected if $\mu \to 0$, we may restrict attention to $\mu = \Theta(1)$. In this case, we only have a.a.s. simple components and the giant one found in the above proof.

Dynamic properties With the static results under our belt, the analysis of the dynamic case is quite similar to that of the cycle (though differing in details and some of the justifications) so we just state the major results.

By analogy with d-hole lines, we define a *simple component line* to be a maximal sequence of pairs $(v_1, t_1), \ldots, (v_l, t_l)$ where v_i is a simple component existing at time t_i for $1 \le i \le l$, and such that $t_i = t_{i-1} + 1$ and the vertex v_i is adjacent to v_{i-1}, for $2 \le i \le l$. Birth, death, survival and random variables S, B, D are the defined analogously to the cycle case.

Theorem 8 *For t in any fixed bounded time interval, the random variables $S(t)$, $B(t)$ and $D(t)$ are asymptotically jointly independent Poisson, with the expectations*

$$\mathbf{E}\left[S(t)\right] \sim \begin{cases} \mu & \text{if } d\varrho \to 0, \\ \mu - \lambda & \text{if } d\varrho \to c, \\ 4\frac{1-e^{-\varrho/4}}{1-e^{-\varrho}} e^{-(2d+5/4)\varrho}\mu & \text{if } d\varrho \to \infty, \end{cases} \quad \text{and}$$

$$\mathbf{E}\left[B(t)\right] = \mathbf{E}\left[D(t)\right] \sim \begin{cases} 2d\varrho\mu & \text{if } d\varrho \to 0, \\ \lambda & \text{if } d\varrho \to c, \\ \mu & \text{if } d\varrho \to \infty, \end{cases}$$

where $\lambda = \left(1 - e^{-2d\varrho}\right)\mu$. Here $0 < \lambda < \mu$ for $d\varrho \to c$.

Theorem 9 *Let $\lambda = \lambda(\varrho) = \mu\left(1 - e^{-2d\varrho}\right)$. The probability that $G_{f_t}[W]$ is connected and that $G_{f_{t+1}}[W]$ is disconnected is asymptotic to $2\mu e^{-\mu} d\varrho$ if $d\varrho \to 0$, to $e^{-\mu}(1 - e^{-\lambda})$ if $d\varrho \to c$, and to $e^{-\mu}(1 - e^{-\mu})$ if $d\varrho \to \infty$.*

A sequence of results similar to the case of the cycle yields the following analogue of Theorem 3.

Theorem 10 *The expected life of a simple component line is asymptotic to $\frac{1}{2d\varrho}$ if $\varrho \to 0$, to $\frac{\mu}{\lambda}$ if $\varrho \to c$, and to 1 if $\varrho \to \infty$. where λ is defined as in Theorem 9.*

As in the case of C_N, let T_C be a random variable measuring the time $G_{f_t}[W]$ remains connected from a moment at which it is so, and let T_D be a random variable measuring the time $G_{f_t}[W]$ remains disconnected, from the moment at which it is so. The next theorem gives the expected time that the graph of walkers $G_{f_t}[W]$ will remain connected or disconnected.

Theorem 11 *Let λ be defined as in Theorem 9. Then,*

$$\mathbf{E}\left[T_C\right] \sim \begin{cases} \frac{1}{2\mu d\varrho} & \text{if } \varrho \to 0, \\ \frac{1}{1-e^{-\lambda}} & \text{if } \varrho \to c, \\ \frac{1}{1-e^{-\mu}} & \text{if } \varrho \to \infty, \end{cases} \quad \text{and } \mathbf{E}\left[T_D\right] \sim \begin{cases} \frac{e^{\mu}-1}{2\mu d\varrho} & \text{if } \varrho \to 0, \\ \frac{e^{\mu}-1}{1-e^{-\lambda}} & \text{if } \varrho \to c, \\ e^{\mu} & \text{if } \varrho \to \infty, \end{cases}$$

5 Conclusions and open problems

In this work we have characterised the dynamic connectivity of a very large set of agents which **move** through a prescribed real or virtual graph. We believe it is the first time that this kind of characterisation has been obtained, and could open an interesting line of research. We gave characterisations for the cycle and the grid for the ℓ_1 norm, which can be extended without problems to the ℓ_2 norm, as applies for instance to robots with movement restricted to orthogonal N-E-S-W directions but with omni-directional radio-frequency communication.

Currently under way is the extension of the results presented here to the hypercube, which is interesting from the mathematical point of view, as the number of neighbours is not constant. Moreover, from the point of view of ad-hoc networks, an interesting case is the random geometric graph (with the ℓ_2 norm), where the walkers can move randomly in any direction taking a step of some random size. We think the present work constitutes a step in this direction.

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