Large Forbidden Trade Volumes of Random Graphs

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Abstract

Let G be a graph. A G-trade of volume m is a pair $(\mathcal{T}, \mathcal{T}')$, where each of $\mathcal{T}$ and $\mathcal{T}'$ consists of m graphs, pairwise edge-disjoint, isomorphic to G, such that $\mathcal{T} \cap \mathcal{T}' = \emptyset$ and the union of the edge sets of the graphs in $\mathcal{T}$ is identical to the union of the edge sets of the graphs in $\mathcal{T}'$. Let $X(G)$ be the set of non-negative integers m such that no G-trade of volume m exists. In this paper we prove that, for $G \in \mathcal{G}(n, \frac{1}{2})$, $\{1, 2, \ldots, \lceil cn/\log n \rceil\} \subseteq X(G)$ holds asymptotically almost surely, where $c = \log(4/3)/88$.

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Let $G = (V(G), E(G))$ be a simple graph. A *G-decomposition* of a simple graph $H = (V(H), E(H))$ is a set $\mathcal{T} = \{G_i : 1 \le i \le m\}$ of graphs such that $G_i \cong G$, $1 \le i \le m$, and $\{E(G_i) : 1 \le i \le m\}$ is a partition of $E(H)$. A *G-trade of volume* m is a pair $(\mathcal{T}, \mathcal{T}')$, where each of $\mathcal{T}$ and $\mathcal{T}'$ is a G-decomposition of the same simple graph H such that $|\mathcal{T}| = |\mathcal{T}'| = m$ and $\mathcal{T} \cap \mathcal{T}' = \emptyset$. The *trade spectrum* of G, denoted $TS(G)$, is defined to be the set of integers m such that a G-trade of volume m exists. From this definition it follows that $0 \in TS(G)$, and $1 \in TS(G)$ if and only if G contains at least one isolated vertex. Denote by $X(G)$

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the set of *forbidden trade volumes*, that is, the set of non-negative integers m such that no G-trade of volume m exists. Then

$$X(G) = \{0, 1, 2, \ldots\} \setminus TS(G).$$

The reader is referred to [2] and the references cited therein for results on trade spectra, and the connection between trades and block designs.

One can see that $TS(G)$ is *additive*, that is, if $m_1, \ldots, m_k \in TS(G)$, then $\sum_{i=1}^{k} c_i m_i \in TS(G)$ for any non-negative integers c_i. Thus, $X(G) = \emptyset$ if and only if [2, Lemma 2.1] G contains isolated vertices. Also, if $2, 3 \in TS(G)$, then $X(G) = \{1\}$ since any integer no less than 2 can be written as $2c_1 + 3c_2$ for some c_1 and c_2. In general, Billington and Hoffman [2] proved that $X(G) \subseteq \{1, 2\}$ holds for several families of graphs. Also, they show [2, Theorem 3.2] that, for any graph $G \neq K_2$, $2s, 3s \notin X(G)$ holds for any integer $s \geq \delta(G)$, where $\delta(G)$ is the minimum degree of G. As a consequence all integers large enough, say, no less than $5\delta(G)+2$, are not in $X(G)$ (see [2, Theorem 3.2] for details). That is, graphs with small minimum degree cannot have large forbidden trade volumes. On the other hand, for complete graphs K_n of order n, we have $\{1, 2, \ldots, 2n-3\} \subseteq X(K_n)$ [2, Lemma 4.1], and hence the forbidden trade volumes increase with the order. Complete graphs are the only known graphs with this property. Billington [1] asked whether there exist non-complete graphs G of order n such that the forbidden trade volumes of G increase with n. In this paper we answer this question affirmatively for random graphs.

As usual we use $\mathcal{G}(n, \frac{1}{2})$ to denote the probability space of random graphs of order n with any two vertices being adjacent with probability 1/2. For a sequence of probability spaces Ω_n, $n \geq 1$, an event A_n of Ω_n occurs *asymptotically almost surely*, abbreviated to *a.a.s.* in the following, if $\lim_{n\to\infty} \mathbf{P}(A_n) = 1$. Set

$$c = \frac{\log(4/3)}{88}.$$

Our main result is the following theorem.

Theorem 1 **[thm:main]** *For $G \in \mathcal{G}(n, \frac{1}{2})$, a.a.s.*

$$[\textbf{excl}] \quad \{1, 2, \ldots, \lceil cn/\log n \rceil\} \subseteq X(G). \tag{1}$$

In order to prove this we introduce the following two concepts. A graph $G = (V(G), E(G))$ of order n is called *j-non-meshing*, for some integer j with $2 \leq j \leq n$, if every way of identifying j vertices of one copy of G with j vertices of another copy of G gives a graph with multiple edges. In other words, G is j-non-meshing if, for any two graphs G_1 and G_2 isomorphic to G and having j vertices in common, there exist $u, v \in V(G_1) \cap V(G_2)$ such that u and v are adjacent in both G_1 and G_2. For example, K_n is j-non-meshing for $2 \le j \le n$. For a graph G, a subset K of $V(G)$ is G-*defining* if there exists no non-identity permutation σ of $V(G)$ such that, for all $u \in K$ and $v \in V(G)$, $uv \in E(G)$ if and only if $\sigma^{-1}(u)\sigma^{-1}(v) \in E(G)$. Denote

$$j_0(n) = \frac{8 \log n}{\log(4/3)}.$$

Lemma 1 **[lem:non-meshing]** *Asymptotically almost surely,* $G \in \mathcal{G}(n, \frac{1}{2})$ *is* j*-non-meshing for all* j *with* $j_0(n) < j \le n$.

Proof. Let J be a subset of $V(G)$ with $|J| = j$. Let $A(J)$ be the event that there exists an injection σ from J to $V(G)$ such that for all pairs $\{u, v\}$ of distinct vertices u, v in J,

$$[\mathbf{eq:cond}] \quad \text{either } uv \notin E(G) \text{ or } \sigma(u)\sigma(v) \notin E(G). \tag{2}$$

For a fixed pair $\{u, v\}$, the probability that (2) holds is $1/2$ when $\{\sigma(u), \sigma(v)\} = \{u, v\}$ (as this can only happen if $uv \notin E(G)$) and $3/4$ otherwise. However, these events are not independent for different pairs $\{u, v\}$. If $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}$ is a set of distinct pairs of vertices in J such that $\sigma(u_i) = u_{i+1}$ and $\sigma(v_i) = v_{i+1}$ for $1 \le i < k$ ($k \ge 2$), we say that these pairs are *associated* by σ. For all of these pairs to satisfy (2), it is necessary that no two consecutive pairs in the sequence $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}$ are edges of G. (The extra condition on the image of $\{u_k, v_k\}$ under σ gives no improvement, as it turns out, since it may happen that $\{u_1, v_1\} = \{u_k, v_k\}$, and $k = 2$ is the value of k which determines the final result.) The probability that this happens is $3/4$ for $k = 2$, whilst for $k \ge 3$ it is

$$2^{-k} \sum_{i=0}^{k/2} \binom{k-i+1}{i} \le 2^{-k} \lfloor k/2 + 1 \rfloor 2^{0.724k} < \left(\frac{3}{4}\right)^{k/2}.$$

The middle step here follows on noting that the binomial is increasing in i for $i \le \frac{1}{2} - \frac{1}{2\sqrt{5}}$, and the last step follows by calculus and checking the small values of k. The pairs of vertices in J can be partitioned into maximal associated sets, and the event considered above is, for a maximal associated set, independent of all other pairs of vertices in J. Thus, for a given injection σ from J to $V(G)$, the probability that σ satisfies (2) for all $\binom{j}{2}$ pairs of vertices in J is at most $(3/4)^{j(j-1)/4}$. Thus, $\mathbf{P}(A(J)) \le [n]_j (3/4)^{j(j-1)/4}$, where $[n]_j = n(n-1)\cdots(n-j+1)$. Consequently, if X_j is the number of sets J with $|J| = j$ such that $A(J)$ holds,

$$\mathbf{E}(X_j) \le \binom{n}{j} [n]_j \left(\frac{3}{4}\right)^{j(j-1)/4} \le \frac{n^{2j}}{j!} \left(\frac{3}{4}\right)^{j(j-1)/4} = \frac{e^{(2\log n + (j-1)\log(3/4)/4)j}}{j!} \tag{3}$$

which is $O(1/j!)$ since $j > j_0(n)$. Thus $\mathbf{E}\sum_{j\geq j_0} X_j = o(1)$ using linearity of expectation. So by the first moment principle, $\mathbf{P}\left(\sum_{j\geq j_0} X_j \geq 1\right) = o(1)$, and the result follows. ∎

Lemma 2 *Let $G \in \mathcal{G}(n, \frac{1}{2})$. Then a.a.s. all subsets $K \subseteq V(G)$ with $|K| \geq 10n/11$ are G-defining.*

Proof. Let $K \subseteq V(G)$ with $|K| = k \geq 10n/11$. Suppose that σ is a non-identity permutation on $V(G)$ with support R, i.e., $R = \{v \in V(G) : \sigma(v) \neq v\}$, and let $r = |R|$. Then σ induces a permutation σ^* on the set of unordered pairs $\{u, v\}$ of distinct vertices in $V(G)$, defined by $\sigma^*(\{u, v\}) = \{\sigma(u), \sigma(v)\}$. Let S be the set of unordered pairs $\{u, v\}$ not fixed (as an unordered pair) by σ^* and with at least one of u, v in K. That is,

$$S = \{\{u, v\} : u, v \in V(G), \{u, v\} \cap K \neq \emptyset, \{\sigma(u), \sigma(v)\} \neq \{u, v\}\}.$$

Let $i = |K \cap R|$. The number of unordered pairs $\{u, v\}$ with one of u, v in K and the other in R is $i(k - i) + k(r - i) + \binom{i}{2}$. All these unordered pairs are in S, except for at most $r/2$ which correspond to transpositions in σ. So we have

$$[\mathbf{S}] \quad |S| \geq kr - \frac{i(i+1)}{2} - \frac{r}{2} \geq \frac{(k-2)r}{2} \tag{4}$$

using $i \leq r$ and $i + 1 \leq k + 1$.

The permutation σ^* induces a digraph on the set of unordered pairs of distinct vertices of G, in which there is an arc from $\{u, v\}$ to $\{u', v'\}$ if and only if $\sigma^*(\{u, v\}) = \{u', v'\}$. The sub-digraph D of this digraph induced by S consists of directed paths, and directed cycles of length at least 2. Let d be the number of such cycles, so that $d \leq |S|/2$. Suppose that for all $\{u, v\} \in S$ we have

$$[\mathbf{cond}] \quad uv \in E(G) \text{ if and only if } \sigma^{-1}(u)\sigma^{-1}(v) \in E(G). \tag{5}$$

Suppose all edges uv of G with $\{u, v\} \notin S$ are given. Then the number of possibilities for assigning edges of G to these paths and cycles is 2^d, because the edges in paths of D are determined by (5) and for each cycle of D there are two possibilities. The probability that $G \in \mathcal{G}(n, \frac{1}{2})$ satisfies (5) is thus at most $2^{d-|S|} \leq 2^{-|S|/2} \leq 2^{-(k-2)r/4}$ by (4).

There are $\binom{n}{k}$ subsets $K \subseteq V(G)$ with $|K| = k$ and at most $\binom{n}{r} r! < n^r$ permutations σ as above (note that $r \geq 2$ by its definition). Since $k \geq 10n/11$ we have by Stirling's formula that for sufficiently large n

$$\binom{n}{k} \leq \binom{n}{\lceil 10n/11 \rceil} \leq (11/10^{10/11})^n < 1.36^n.$$

So the probability that $G \in \mathcal{G}(n, \frac{1}{2})$ satisifies (5) for some K and σ is at most

$$1.36^n n^r 2^{-(k-2)r/4} = 1.36^n 2^{-(k-2-4\log n)r/4} \leq 1.36^n 2^{-(5/11-\varepsilon)n}$$

for all $\varepsilon > 0$. Since $2^{5/11} > 1.37$, the sum of this expression over all $k \geq 10n/11$ and $r \geq 2$ goes to zero, and the lemma is proved. ∎

We will use the two lemmas above in the proof of Theorem 1. We will also use the following known results, see e.g. [3, Lemma 2.1]. For a graph G and $v \in V(G)$, denote by $N_G(v)$ the set of neighbours of v in G, and $d(v) = |N_G(v)|$ the degree of v.

Lemma 3 **[lem:deg-con]** *Let $G \in \mathcal{G}(n, \frac{1}{2})$ and $0 < \varepsilon < 1/10$. Then the following hold a.a.s.*

(a) $|d(v) - n/2| < \varepsilon n$ *for all* $v \in V(G)$*;*

(b) *for all* $u, v \in V(G)$, $\Big||N_G(u) \cap N_G(v)| - n/4\Big| < \varepsilon n$.

Proof of Theorem 1. Select a graph G on n vertices satisfying all of the properties in Lemmas 1 to 3 which are asserted to hold a.a.s. We prove that (1) holds for such a graph G. It then follows by Lemmas 1 to 3 that a random graph $G \in \mathcal{G}(n, \frac{1}{2})$ satisfies (1) a.a.s. Let $m \leq n/(11j_0(n)) = cn/\log n$. To prove that there is no G-trade of volume m, it suffices to show that, for any two G-decompositions

$$\mathcal{T} = \{G_i : 1 \leq i \leq m\}, \ \ \mathcal{T}' = \{G'_i : 1 \leq i \leq m\}$$

of a simple graph H, we have $G_1 = G'_i$ for some i.

Since H is simple and, by Lemma 1, G is j-non-meshing for any $j > j_0(n)$, G_1 has at most $j_0(n)$ vertices in common with each of G_i, for $i = 2, \ldots, m$. Hence there are at most $mj_0(n) \leq n/11$ vertices in $V(G_1) \cap (\bigcup_{i=2}^m V(G_i))$. Denote by K the set of all other vertices of G_1, that is, $K = V(G_1) \setminus (\bigcup_{i=2}^m V(G_i))$. Then $|K| \geq 10n/11$. Note that, by the definition of K, any edge of H incident with a vertex in K must be in G_1. Hence $d_H(v) = d_{G_1}(v)$ for all $v \in K$, and in particular $d_H(v)$ is close to $n/2$ by Lemma 3(a). For distinct vertices $u \in K$ and $v \in K$, let G'_i and G'_j be the graphs in $\mathcal{T}'$ containing u and v, respectively. Then i is unique since otherwise $d_H(u)$ would close to n by Lemma 3(a), a contradiction. Similarly, j is unique. Also, $N_H(u) \cap N_H(v) = N_{G_1}(u) \cap N_{G_1}(v)$ and so by Lemma 3(b) it follows that $|N_H(u) \cap N_H(v)|$ is close to $n/4$. On the other hand, $N_H(u) \cap N_H(v) \subseteq V(G'_i) \cap V(G'_j)$, so if $i \neq j$, we have by Lemma 1 that $|V(G'_i) \cap V(G'_j)| \leq j_0(n) << n/4$. Thus, we must have $i = j$. Since this is true for all $u, v \in K$, we conclude that $K \subseteq V(G'_i)$ for some i. Moreover, since all vertices of $G_1 - K$ have degree at least $(1/2-\varepsilon)n$ in G_1 by the statement in Lemma 3(a), a vertex v of $H-K$ has neighbours in K if and only if v is in G_1. The same statement holds for G_i'. Hence $V(G_1)=V(G_i')$.

Since $G_1 \cong G \cong G_i'$, there exists a permutation σ of $V(G_1)$ which induces an isomorphism from G_1 to G_1'. Thus, $uv \in E(G)$ if and only if $\sigma^{-1}(u)\sigma^{-1}(v) \in E(G)$ for $u, v \in K$. However, as $|K| \geq 10n/11$, K is G-defining by the statement in Lemma 2. So σ must be the identity permutation. Hence $G_1 = G_i'$ and we are done. ∎

Concluding remarks

It would be interesting to know how much the interval of values in Theorem 1 can be increased without making the theorem false. Clearly the upper end of the interval can be increased, since we made no attempt to obtain the best possible constant in Lemma 1; the difficulties with cycles in σ of length 2 will not be typical. On the other hand, the upper end must be less than n, by Lemma 3 and the above-mentioned result that $2\delta(G) \notin X(G)$. Moreover, this upper bound can be decreased a little since the minimum degree of a random graph is $n/2-\Theta(\sqrt{n \log n})$.

References

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