An improved upper bound on the length of the longest cycle of a supercritical random graph

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Abstract

We improve Łuczak's upper bounds on the length of the longest cycle in the random graph $\mathcal{G}(n, M)$ in the "supercritical phase" where $M = n/2 + s$ and $s = o(n)$ but $n^{2/3} = o(s)$. The new upper bound is $(6.958 + o(1))s^2/n$ with probability $1 - o(1)$ as $n \to \infty$. Letting $c = 1 + 2s/n$, the equivalence between $\mathcal{G}(n, p)$ and $\mathcal{G}(n, M)$ implies the same result for $\mathcal{G}(n, p)$ where $p = c/n$, $c \to 1$, $c - 1 = \omega(n^{-1/3})$.

1   Introduction

The probability space $\mathcal{G}(n, M)$ of all n-vertex graphs with M edges under the uniform distribution is also known as the uniform random graph model. It is one of the earliest models of random graphs, originating in a simple model introduced by Erdős [9]. We say that $\mathcal{G}(n, M)$ has a property *asymptotically almost surely* (abbreviated a.a.s.) if the probability of this event is $1 - o(1)$ as $n \to \infty$. Much of the interest in this model comes from the study of its its asymptotically almost sure (also abbreviated a.a.s.) properties as the dependence of M upon n is varied. This change from a sparse graph to a dense graph, as M increases more quickly with n, is called the evolution of the random graph. One important property is the number L of vertices in the largest component of $\mathcal{G}(n, M)$. (If there is more than one component with the maximum number of vertices, we use the lexicographically first among largest components.) When $M = cn/2$ for constant c, Erdős and Rényi [3] showed that the number of vertices in the largest component of $\mathcal{G}(n, M)$ is a.a.s. $O(\log n)$, $\Theta(n^{2/3})$, or $\Theta(n)$ according to whether $c < 1$, $c = 1$, or $c > 1$, respectively.

Because of this dramatic change in the structure of $\mathcal{G}(n, M)$, we often call $M = n/2$ a "phase transition". Further research showed that the phase transition extends throughout

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the period $M = n/2 + cn^{2/3}$ for constant c in the sense that, for this range of M, $L = c'n^{2/3}$ with a distribution over the constant c'. As a result, this range of M is known as the *critical period*. For $s = s(n)$ satisfying $n^{2/3} = o(s)$ but $s = o(n)$, the range $M = n/2 - s$ is known as the *subcritical phase* while the range $M = n/2 + s$ is known as the *supercritical phase*. For M in the supercritical phase, $\mathcal{G}(n, M)$ a.a.s. has a unique largest component on $(4 + o(1))s$ vertices and every other component has fewer than $n^{2/3}$ vertices. A "giant component" has emerged.

Another well-studied graph property is its *circumference*, the length of its longest cycle. The circumference l of $\mathcal{G}(n, M)$ also changes dramatically during the phase transition, but the way it changes is not entirely understood. Let $\omega = \omega(n) \to \infty$. When $M = cn/2$ for fixed $c < 1$, the circumference of $\mathcal{G}(n, M)$ is a.a.s. at most ω ([6], Corollary 5.8). In the subcritical phase, the circumference l a.a.s. satisfies $l/\omega < n/s < l\omega$ ([12], Section 5.4). During the critical period $M = n/2 + O(n^{2/3})$ it a.a.s. satisfies $l/\omega < n^{1/3} < l\omega$ ([12], Section 5.5). But for larger M there are not such good estimates for the circumference. (Of course, when $M = n(\log n + \log \log n + \omega)/2$ the circumference is a.a.s. equal to n as the graph is a.a.s. Hamiltonian [15].) When $M = cn/2$ for fixed $c > 1$, there are several known a.a.s. lower bounds on the circumference of the form $(f(c) + o(1))n$ [10, 7, 11]. One of the earliest and most significant breakthroughs was given by Ajtai, Komlós and Szemerédi [1], who also showed an equivalence between the problems of finding paths of length $(f(c) + o(1))n$ and finding cycles of length $(f(c) + o(1))n$. Łuczak [17] has shown that the circumference of $\mathcal{G}(n, M)$ is a.a.s. between $(16/3 + o(1))s^2/n$ and $(7.496 + o(1))s^2/n$ when $M = n/2 + s$ for $s = o(n)$ and $n^{2/3} = o(s)$. Kim and the second author [14] have improved Łuczak's lower bound to $(6 + o(1))s^2/n$.

In this paper we improve upon Łuczak's upper bound as follows.

Theorem 1 *Let $M = n/2 + s$ with $n^{2/3} = o(s)$ and $s = o(n)$. The circumference of $\mathcal{G}(n, M)$ is a.a.s. at most $(6.958 + o(1))s^2/n$.*

In proving his result, Łuczak focused on the *core* and *kernel* of $\mathcal{G}(n, M)$. The core of a graph is its maximal subgraph of minimum degree at least 2. The *prekernel* of a graph is obtained from the core by throwing away any cycle components. The *kernel* of a graph is obtained from the prekernel by replacing each maximal path of degree-2 vertices by a single edge. We say that a graph is a prekernel (respectively, a kernel) if it is the prekernel (respectively, kernel) of some graph.

Łuczak's main insight was that, for this range of M, the kernel is much like a random 3-regular graph, and the core is much like the graph formed from the kernel by randomly subdividing its edges about $(8 + o(1))s^2/n$ times. A random 3-regular graph a.a.s. contains a Hamilton cycle. This gives a cycle in $\mathcal{G}(n, M)$ containing about $(2/3) \times (8 + o(1))s^2/n = (16/3 + o(1))s^2/n$ vertices of the core. This was Łuczak's lower bound on the circumference. His upper bound came from viewing the core as constructed from the kernel together with a sequence of numbers, summing to $(8 + o(1))s^2/n$, describing how many degree-2 vertices belong on each edge of the kernel. From probability theory, the sum of the largest two-thirds of the terms of such a random sequence is at most $(7.496 + o(1))s^2/n$.

We obtain our result by a different, more detailed study of how a cycle can pass through such a structure. Our main tool is the kernel configuration model, introduced in [19] to

facilitate arguments like Łuczak's. Our method may be of independent interest because it provides a quite general framework for studying properties of the giant component for $s = O(n)$. Ding et al.[8] recently announced another model which may be used for $s = o(n)$.

Following Łuczak's example, it is helpful to put weights on the edges of the kernel; the weight of an edge tells us how many times the edge should be subdivided to recover the core. These weights form a random sequence whose asymptotic properties we investigate in Section 2. In particular, we show that any bounded number of terms in such a sequence behave like independent random variables with exponential distribution. We also show that when a function of a bounded number of these terms is summed over many sets of such terms, the result is concentrated about its expected value. These properties are needed in Section 3 where we establish an a.a.s. upper bound on the weight of the heaviest cycle in a pseudograph with random edge weights. The upper bound is expressed in terms of a family of constants, some of which we explicitly calculate in Section 4. In Section 5 we prove an a.a.s. upper bound on the circumference of a random prekernel with a degree sequence that resembles a random 3-regular graph with subdivided edges. In Section 8 we use this result to prove Theorem 1 after, in Sections 6 and 7, establishing that the degree sequence of the prekernel of $\mathcal{G}(n, M)$ indeed shows the required resemblance.

2   Random sequences

Let Ω be the probability space, equipped with the uniform distribution, of all sequences of m positive integers $(X_1, X_2, \ldots, X_m)$ summing to N. We are interested in the asymptotic value of certain functions of these random variables. Letting $\omega = \omega(N) \to \infty$, our asymptotics are in terms of $N \to \infty$, uniformly over all m satisfying $\omega < m < N/\omega$. For the rest of the paper we write $\mu = N/m$.

In this section we prove two main results. The first is a lemma that tells us the expected value of certain functions of $X_1, X_2, \ldots, X_j$ for j bounded.

Lemma 2 *Let g be a nonnegative integrable function of a bounded number j of nonnegative variables. Suppose that for some C and d, $g(x_1, \ldots, x_j) \leq C(x_1 + \cdots + x_j)^d$ for all $x_1, \ldots, x_j$. Then,*

$$\mathbf{E}\left[g\left(\frac{X_1}{\mu}, \ldots, \frac{X_j}{\mu}\right)\right] = \int_0^\infty \cdots \int_0^\infty g(x_1, \ldots, x_j) e^{-x_1 - x_2 - \cdots - x_j} dx_1 \cdots dx_j + o(1).$$

Since the X_i are identically distributed, the above theorem also holds when $(X_1, \ldots, X_j)$ is replaced by $(X_{\sigma(1)}, \ldots, X_{\sigma(j)})$ for any j distinct $\sigma(1), \ldots, \sigma(j)$ in $\{1, 2, \ldots, m\}$. Furthermore, the error represented by $o(1)$ is independent of σ.

The second result that we will prove states that when such a function is summed over σ in a sufficiently rich family, the sum is asymptotically almost surely (a.a.s.) concentrated about its expected value.

Lemma 3 *Let f be a nonnegative integrable function of a bounded number k of nonnegative variables. Suppose that for some C and d, $f(x_1, \ldots, x_k) \leq C(x_1 + \cdots + x_k)^d$ for all $x_1, \ldots, x_k$. Define the constant*

$$E^* := \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_k) e^{-x_1 - x_2 - \cdots - x_k} dx_1 \cdots dx_k.$$

and assume $E^* > 0$. Let S be a set of k-tuples with entries from $\{1, 2, \ldots, m\}$, with each k-tuple having distinct components. Let $I = I(S) \in S \times S$ be the pairs of tuples which intersect; that is,

$$I = \{(\sigma, \tau) \in S \times S \mid \{\sigma(1), \ldots, \sigma(k)\} \cap \{\tau(1), \ldots, \tau(k)\} \neq \emptyset\}.$$

If $|I| = o(|S|^2)$ then

$$\sum_{\sigma \in S} f\left(\frac{X_{\sigma(1)}}{\mu}, \ldots, \frac{X_{\sigma(k)}}{\mu}\right) = (E^* + o(1))|S|$$

a.a.s.; that is, with probability $1 - o(1)$. Furthermore, the $o(1)$ terms may be bounded independently of S.

These types of concentration results are often proved using martingales or inequalities like Talagrand's; however, because we are aiming for such a coarse result, a simple application of Chebyshev's inequality will suffice for the proof.

2.1 Distribution of terms

In this section we establish some preliminary results about the distribution of the positive terms $X_1, X_2, \ldots, X_j$ for bounded j. It is an exercise in basic counting to show that the number of sequences in Ω is $\binom{N-1}{m-1}$. It immediately follows that for positive integers $t_1, t_2, \ldots, t_j$, the number of sequences in Ω with $X_1 = t_1$, $X_2 = t_2$, ..., $X_j = t_j$ is

$$B(t) := \binom{N-1-t}{m-j-1}$$

where $t = t_1 + t_2 + \cdots + t_j$.

Proposition 4 *Let x satisfy $x < \sqrt{m}/\omega$ and $x < \mu/\omega$. For positive integers $t \leq x\mu$ we have*

$$\frac{B(t)}{|\Omega|} = (1 + O(\omega^{-1}))\mu^{-j} e^{-t/\mu}.$$

Proof.

$$
\begin{aligned}
\frac{B(t)}{|\Omega|} &= \binom{N-1-t}{m-j-1}\binom{N-1}{m-1}^{-1} \\
&= \left(\prod_{i=1}^{m-1} \frac{N-t-i}{N-i}\right)\left(\prod_{i=1}^{j} \frac{m-i}{N-t-m+i}\right) \\
&= \left(\prod_{i=1}^{m-1}\left(1 - \frac{t}{N-i}\right)\right)\mu^{-j}\left(1 + O(m^{-1}) + O(t/N) + O(m/N)\right).
\end{aligned}
$$

Since $m > \omega$, $x < m/\omega$ and $m < N/\omega$, the error term is $O(1/\omega)$. Also

$$
\begin{aligned}
1 - \frac{t}{N-i} &= 1 - \frac{t}{N}\left(1 + O\left(\frac{i}{N}\right)\right) \\
&= 1 - \frac{t}{N}(1 + O(\mu^{-1})); \\
\prod_{i=1}^{m-1}\left(1 - \frac{t}{N-i}\right) &= e^{-t/\mu}\left(1 + O(tm^2/N^2 + mt^2/N^2)\right) \\
&= e^{-t/\mu}\left(1 - O(\omega^{-1})\right). \quad \blacksquare
\end{aligned}
$$

Corollary 5 *Let $x > 0$ be fixed. For any positive integers $t_1, t_2, \ldots, t_j$ summing to $t \le x\mu$ we have*

$$
\mathbf{P}[X_1 = t_1, X_2 = t_2, \ldots, X_j = t_j] = (1 + O(\omega^{-1}))\mu^{-j}e^{-t/\mu}.
$$

Next we bound the probability of larger terms.

Lemma 6 *Let $x > 0$ be fixed. For positive integers $t_1, t_2, \ldots, t_j$ summing to $t \ge x\mu$ we have*

$$
\mathbf{P}[X_1 = t_1, X_2 = t_2, \ldots, X_j = t_j] < 2\mu^{-j}e^{-x}\left(1 - \frac{1}{2\mu}\right)^{t-x\mu}
$$

when N is sufficiently large.

Proof. If $B(t) = 0$ then the required probability is zero and we are done. Otherwise, $B(i)$ is nonzero for all positive integers $i \le t$ and the probability which we must estimate is

$$
\frac{B(t)}{|\Omega|} = |\Omega|^{-1}B(\lfloor x\mu \rfloor)\prod_{i=\lfloor x\mu \rfloor+1}^{t}\frac{B(i)}{B(i-1)}.
$$

By Proposition 4, the product of the first two terms is $(1 + O(\omega^{-1}))\mu^{-j}e^{-\lfloor x\mu \rfloor/\mu}$. This is less than $2\mu^{-j}e^{-x}$ when N is sufficiently large. To bound the remaining product, we estimate the ratio

$$
\frac{B(i)}{B(i-1)} = \frac{N-i-m+j+1}{N-i} < 1 - \frac{m/2}{N}
$$

where the inequality holds for N sufficiently large. So, for N sufficiently large,

$$
\prod_{i=\lfloor x\mu \rfloor+1}^{t}\frac{B(i)}{B(i-1)} < \left(1 - \frac{1}{2\mu}\right)^{t-\lfloor x\mu \rfloor} \le \left(1 - \frac{1}{2\mu}\right)^{t-x\mu}.
$$

The result follows. $\quad \blacksquare$

2.2 Proof of Lemma 2

By the definition of expected value, we have

$$\mathbf{E}\left[g\left(\frac{X_1}{\mu},\ldots,\frac{X_j}{\mu}\right)\right] = \sum g\left(\frac{t_1}{\mu},\ldots,\frac{t_j}{\mu}\right)\mathbf{P}[X_1 = t_1, X_2 = t_2,\ldots,X_j = t_j]$$

where the sum is over all positive integer j-tuples $t_1, t_2,\ldots,t_j$.

Fix $x > 0$. Let us split the sum into two parts, $S_1(x)$ being the sum over j-tuples where each $t_i < x\mu$, and $S_2(x)$ being the remainder. We will show that, as $N \to \infty$,

$$S_1(x) \to \int_0^x \cdots \int_0^x g(x_1,\ldots,x_j)e^{-x_1-x_2-\cdots-x_j}dx_1\cdots dx_j,$$

while

$$|S_2(x)| < Ke^{-x/2}$$

for some constant K. As x grows, $|S_2(x)|$ approaches 0 and $S_1(x)$ is nonnegative and nondecreasing since g is nonnegative. So, taking $x \to \infty$ proves the lemma.

We begin by estimating $S_1(x)$. These terms have each $t_i \leq x\mu$ so by Corollary 5,

$$S_1(x) = (1 + O(\omega^{-1}))\sum_{t_1 < M}\cdots\sum_{t_j < M} g\left(t_1\frac{x}{M},\ldots,t_j\frac{x}{M}\right)e^{-(t_1+\cdots+t_j)x/M}\left(\frac{x}{M}\right)^j$$

where $M = x\mu$. As $N \to \infty$ we have $M \to \infty$ and this expression becomes the required Riemann integral.

The terms of the sum $S_2(x)$ are indexed by j-tuples $t_1, t_2,\ldots,t_j$ with at least one $t_i \geq x\mu$. Consider such a term, and let $t = t_1 + t_2 + \cdots + t_j$. For N sufficiently large, the absolute value of the term is

$$g\left(\frac{t_1}{\mu},\ldots,\frac{t_j}{\mu}\right)\mathbf{P}[X_1 = t_1, X_2 = t_2,\ldots,X_j = t_j] < C\left(\frac{t}{\mu}\right)^d 2\mu^{-j}e^{-x}\left(1 - \frac{1}{2\mu}\right)^{t-x\mu}$$

by the hypotheses about g and Lemma 6. The number of terms in $S_2(x)$ indexed by j-tuples summing to t is at most $\binom{t-1}{j-1} \leq (t+j)^{j-1} \leq (2t)^{j-1}$ for N (and hence t) sufficiently large. Thus, for N large, we have

$$
\begin{aligned}
|S_2(x)| &< \sum_{t \geq x\mu}(2t)^{j-1}C\left(\frac{t}{\mu}\right)^d 2\mu^{-j}e^{-x}\left(1 - \frac{1}{2\mu}\right)^{t-x\mu} \\
&= 2e^{-x}\frac{C2^{j-1}}{\mu^{j+d}}\left(1 - \frac{1}{2\mu}\right)^{-x\mu}\sum_{t \geq x\mu}t^{j+d-1}\left(1 - \frac{1}{2\mu}\right)^t.
\end{aligned}
$$

The factor $(1 - 1/(2\mu))^{-x\mu}$ approaches $e^{x/2}$ as $N \to \infty$. The remaining sum is

$$
\begin{aligned}
\sum_{t \geq x\mu}t^{j+d-1}\left(1 - \frac{1}{2\mu}\right)^t &\leq \sum_{t \geq 0}(t+1)(t+2)\cdots(t+j+d-1)\left(1 - \frac{1}{2\mu}\right)^t \\
&= (j+d-1)!(2\mu)^{j+d}
\end{aligned}
$$

using the Maclaurin series expansion $k!(1-x)^{-k-1} = \sum_{t \geq 0}(t+1)(t+2)\cdots(t+k)x^t$. Combining this with the previous results, we get the desired estimate. This proves the lemma. ∎

2.3 Proof of Lemma 3

For each σ in S, define the random variable $Y_\sigma := f(X_{\sigma(1)}/\mu, \ldots, X_{\sigma(k)}/\mu)$. As we remarked after Lemma 2, each of these variables has the same distribution as the random variable $Y_1 := f(X_1/\mu, \ldots, X_k/\mu)$. In particular, the expected value is the constant E^*, up to an additive error of $o(1)$. We will establish the concentration of the random variable $Z := \sum_{\sigma \in S} Y_\sigma$ by showing that the variance of Z is $o((\mathbf{E}Z)^2)$. The lemma then follows by Chebyshev's inequality.

We begin by estimating

$$(\mathbf{E}Z)^2 = \sum_{(\sigma,\tau) \in S \times S} \mathbf{E}Y_\sigma \mathbf{E}Y_\tau = \sum_{(\sigma,\tau) \in S \times S} (E^* + o(1))(E^* + o(1)) = \Theta(|S|^2)$$

(using the lower bound assumed on E^* in the lemma). We can write the variance in terms of two sums

$$\mathbf{E}[Z^2] - (\mathbf{E}Z)^2 = \sum_{(\sigma,\tau) \in I} \left(\mathbf{E}[Y_\sigma Y_\tau] - \mathbf{E}Y_\sigma \mathbf{E}Y_\tau\right) + \sum_{(\sigma,\tau) \in (S \times S) \setminus I} \left(\mathbf{E}[Y_\sigma Y_\tau] - \mathbf{E}Y_\sigma \mathbf{E}Y_\tau\right).$$

To study the terms of the second sum, let $(\sigma, \tau) \in (S \times S) \setminus I$. By Lemma 2, we have

$$
\begin{aligned}
&\mathbf{E}[Y_\sigma Y_\tau] \\
=\ & \mathbf{E}\left[f\left(\frac{X_{\sigma(1)}}{\mu}, \ldots, \frac{X_{\sigma(k)}}{\mu} \right) f\left(\frac{X_{\tau(1)}}{\mu}, \ldots, \frac{X_{\tau(k)}}{\mu} \right) \right] \\
=\ & \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_k) f(x_{k+1}, \ldots, x_{2k}) e^{-x_1 - \cdots - x_{2k}} dx_1 \cdots dx_{2k} + o(1) \\
=\ & \left(\int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_k) e^{-x_1 - x_2 - \cdots - x_k} dx_1 \cdots dx_k \right)^2 + o(1) \\
=\ & \mathbf{E}Y_\sigma \mathbf{E}Y_\tau + o(1)
\end{aligned}
$$

where $o(1)$ is independent of σ and τ. So the second sum is $o(|S|^2)$. To study the terms of the first sum, we can be more crude. By Lemma 2 and the remark following it, we know that each $\mathbf{E}[Y_\sigma Y_\tau]$ and $\mathbf{E}Y_\sigma \mathbf{E}Y_\tau$ depends only on the tuple positions where σ and τ intersect, and each value is $O(1)$. So the first sum is $O(|I|)$, which is $o(|S|^2)$ by hypothesis. Combining the two sums, we see that the variance of Z is $o(|S|^2)$, which is $o((\mathbf{E}Z)^2)$, as required. ∎

3 Heavy cycles in a weighted pseudograph

In the introduction we saw that the problem of bounding the circumference of $\mathcal{G}(n, M)$ is connected to the problem of bounding the weight of the heaviest cycle in a certain edge-weighted graph. In this section we study a graph, technically a pseudograph since it may have loops and/or multiple edges, whose m edges are randomly weighted by positive integers summing to N. The sequence of weights is chosen uniformly at random from among all

such sequences. Equivalently, we can think of the weights as being generated by the following random process applied to make a sequence of pseudographs, beginning with the given one. At each step, choose an edge uniformly at random from the current pseudograph and subdivide the edge into two edges. Repeat the procedure until the resulting pseudograph has exactly N edges. For each edge in the original pseudograph, define its weight to be the number of edges into which it has been subdivided. These weights form a sequence of m positive integers summing to N. There are exactly $(N - m)!$ ways that the process can form a given sequence, so the sequence is chosen uniformly at random from among all such sequences. Another random process for generating the weights initially gives a weight of 1 to each edge, then selects an edge at random with probability proportional to the weight of the edge and increments the weight of the selected edge by 1. The selection and incrementing is repeated until the total weight is N. It is easy to see that this process is equivalent to the previous one.

Given a subgraph of an edge-weighted pseudograph, we define the weight of the subgraph to be the sum of the weights on its edges. To establish an upper bound for the weight of a cycle in a large pseudograph, we will consider the intersection of the cycle with small trees in the pseudograph. The intersection of the cycle and the small tree will form a set of vertex-disjoint paths which begin and end at leaf vertices of the tree. We will use the maximum-weight set of such vertex-disjoint paths to bound the weight of the intersection. This motivates the following definitions.

Fix an integer $k \geq 2$. A *biased tree* T on k edges is a tree on k edges with each non-leaf vertex having degree 3 and each edge e_i having a nonnegative number b_i called its bias. We may assume that the sum of the biases $\mathbf{b} = (b_1, b_2, \ldots, b_k)$ is 1.

Let $\mathcal{P}$ be the set of all maximal subgraphs of T which are a union of vertex-disjoint paths which begin and end at leaf vertices. Define the function

$$f_T(x_1, x_2, \ldots, x_k) = \max_{P \in \mathcal{P}} \sum_{i : e_i \in E(P)} b_i x_i$$

and the constant

$$E_T = \int_0^\infty \int_0^\infty \cdots \int_0^\infty f_T(x_1, x_2, \ldots, x_k) e^{-x_1 - x_2 - \cdots - x_k} dx_1 dx_2 \cdots dx_k. \tag{3.1}$$

If $x_1, x_2, \ldots, x_k$ are weights on the edges of T, we can think of f as the maximum "biased weight" of any graph in $\mathcal{P}$.

We say that the positive constant c^* is *k-admissible* if $E_T < c^*$ for some biased tree T on k edges.

Lemma 7 *Fix an integer $k \geq 2$. Let the positive number c^* be k-admissible. Let $G = G(n)$ be a pseudograph on $v = v(n) \to \infty$ (as $n \to \infty$) vertices and $m = m(n)$ edges with minimum degree at least 3. Suppose the subgraph B of G induced by cycles of length at most k (including loops and parallel edges) and edges incident to vertices of degree greater than 3 satisfies $|E(B)| = o(v)$. Let $N = N(n)$ be a positive integer satisfying $m = o(N)$. On the edges of G put weights, a sequence chosen uniformly at random from among all sequences of m positive integers summing to N. Then, the heaviest cycle in G has weight a.a.s. at most $c^* N$.*

Proof. Denote the edges of G by $w_1, w_2, \ldots, w_m$ and their random weights by $X_1, X_2, \ldots, X_m$. We estimate m by recalling that in any graph the sum of the vertex degrees equals twice the number of edges. Since G has minimum degree at least 3, we have $2m \geq 3v$. Since G has only $o(v)$ edges incident to vertices of degree greater than 3, we have $2m \leq 3v + o(v)$. Thus $m \sim 3v/2$.

For a subgraph S of G, define its k-*neighbourhood* $\Gamma(S)$ to be the subgraph of G reachable from S by paths of length at most k. Recalling that the subgraph B of G contains all edges incident with vertices of degree greater than 3, its k-neighbourhood satisfies $|E(\Gamma(B))| \leq 2^k |E(B)| = o(v)$.

Let C be a cycle in G. Its weight $wt(C)$ is

$$
\begin{aligned}
wt(C) \quad &:= \quad \sum_{j=1}^{m} X_j I(w_j \in E(C)) \\
&= \quad \sum_{j:w_j \in E(\Gamma(B))} X_j I(w_j \in E(C)) + \sum_{j:w_j \notin E(\Gamma(B))} X_j I(w_j \in E(C))
\end{aligned}
$$

where $I(\alpha)$ is the indicator function equal to 1 if α is true and 0 otherwise. The expected value of each X_j is $\mu = N/m$, so the first sum has expected value at most $|E(\Gamma(B))| N/m = o(vN/m) = o(N)$ since $m \sim 3v/2$. It follows by Markov's inequality that the first sum is a.a.s. $o(N)$. Thus a.a.s.,

$$
wt(C) = o(N) + \sum_{j:w_j \notin E(\Gamma(B))} X_j I(w_j \in E(C)). \tag{3.2}
$$

Since c^* is k-admissible, there is a biased tree T on k edges with $E_T < c^*$. We will study the copies of T in the graph $G \setminus B$. Let S be the set of 1-1 homomorphisms σ mapping T to $G \setminus B$. Since $k \geq 2$, each σ is uniquely defined by the mapping it induces between the edge sets. We write $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(k))$ and interpret $\sigma(i) = j$ to mean that σ maps edge e_i of T to edge w_j of G.

Consider the random variable

$$
Z = \sum_{\sigma \in S} \sum_{i=1}^{k} b_i X_{\sigma(i)} I(w_{\sigma(i)} \in E(C)).
$$

Expressing Z in terms of the edges of G we may write

$$
\begin{aligned}
Z \quad &= \quad \sum_{j=1}^{m} \sum_{\sigma \in S} \sum_{i=1}^{k} b_i X_j I(w_j \in E(C)) I(\sigma(i) = j) \\
&= \quad \sum_{j=1}^{m} \sum_{i=1}^{k} b_i X_j I(w_j \in E(C)) |\{\sigma \in S \mid \sigma(i) = j\}|.
\end{aligned}
$$

For each edge w_j of G not in $E(\Gamma(B))$ the k-neighbourhood of w_j is the depth-k tree with internal vertices of degree 3. Thus, $|\{\sigma \in S \mid \sigma(i) = j\}|$ equals some constant independent

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of j. In fact, this constant is a number a, independent of i, because any σ in this set is determined by choosing one of the 2 ways to embed e_i onto w_j and then, moving outward from e_i, making one binary choice for each non-leaf vertex of T. On the other hand, for an edge $w_j \in E(\Gamma(B))$, $|\{\sigma \in S \mid \sigma(i) = j\}|$ is at most a (by the same argument, recalling that some choices are impossible because σ maps into $G \setminus B$), so we have

$$
\begin{aligned}
Z &= \sum_{j:w_j \in E(\Gamma(B))} O(X_j) + \sum_{j:w_j \notin E(\Gamma(B))} \sum_{i=1}^{k} ab_i X_j I(w_j \in E(C)) \\
&= \sum_{j:w_j \in E(\Gamma(B))} O(X_j) + \sum_{j:w_j \notin E(\Gamma(B))} aX_j I(w_j \in E(C))
\end{aligned}
$$

since $\sum_{i=1}^{k} b_i = 1$. The first sum has $o(v)$ terms, each having expected value $O(N/m)$, so the sum is a.a.s. $o(vN/m) = o(N)$ by Markov's inequality. We now have a.a.s.

$$
Z = o(N) + a \sum_{j \notin E(\Gamma(B))} X_j I(w_j \in E(C)).
$$

Combining this result with (3.2) we get a.a.s.

$$
wt(C) = \frac{1}{a}Z + o(N). \tag{3.3}
$$

Returning to the definition of Z, we notice that the inner sum is the "biased weight" of the edges of C passing through the copy of T given by σ. These edges must form vertex-disjoint paths beginning and ending at leaves of the copy of T given by σ, so this sum is at most $f_T(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(k)})$. So

$$
Z \leq \sum_{\sigma \in S} f_T(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(k)}).
$$

We will estimate this sum by applying Lemma 3 to

$$
\frac{1}{\mu} \sum_{\sigma \in S} f_T(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(k)}) = \sum_{\sigma \in S} f_T\left(\frac{X_{\sigma(1)}}{\mu}, \frac{X_{\sigma(2)}}{\mu}, \ldots, \frac{X_{\sigma(k)}}{\mu}\right).
$$

To verify the hypotheses of Lemma 3 we first note that $f_T(x_1, x_2, \ldots, x_k)$ is nonnegative, piecewise linear (and hence integrable), and bounded above by $x_1 + x_2 + \cdots + x_k$. We estimate $|S|$ by

$$
\begin{aligned}
|S| &= \sum_{j:w_j \in E(G)} |\{\sigma \in S \mid \sigma(1) = j\}| \\
&= \sum_{j:w_j \in E(\Gamma(B))} |\{\sigma \in S \mid \sigma(1) = j\}| + \sum_{j:w_j \notin E(\Gamma(B))} |\{\sigma \in S \mid \sigma(1) = j\}| \\
&= o(v) + \sum_{w_j \notin E(\Gamma(B))} a \\
&= o(v) + (1 + o(1))ma \\
&\sim ma
\end{aligned}
$$

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using $o(v) = o(m)$. To estimate the cardinality of the set I of pairs $(\sigma, \tau) \in S \times S$ for which σ and τ represent intersecting copies of T, consider any edge f in $G \setminus B$. As we have seen previously, there are at most a copies of T using f. So, a crude upper bound for $|I|$ is $a^2|E(G \setminus B)| \le a^2 m$, giving us $|I| = o(|S|^2)$ as required. Recalling the definition of E_T from (3.1) we may apply Lemma 3 and conclude a.a.s.

$$\frac{1}{\mu}Z \le (E_T + o(1))|S| = (E_T + o(1))am.$$

Combining this with Equation (3.3) we get a.a.s.

$$wt(C) \le \frac{1}{a}\mu(E_T + o(1))am + o(N) = (E_T + o(1))N + o(N) < c^*N$$

as required. ∎

Remark 1. A random 3-regular graph a.a.s. satisfies all of the hypotheses of Lemma 7. The lemma thus gives an upper bound which holds a.a.s. on the weight of the heaviest cycle in a randomly-weighted random 3-regular graph.

Remark 2. There are essentially two ingredients in the proof of Lemma 7. The first ingredient is a method for bounding the weight of a cycle in a large edge-weighted 3-regular subgraph. The second ingredient is the argument that the weight of the heaviest cycle does not change much when the remainder of the graph is included. This second ingredient is implicit in Łuczak's proof of his upper bound on the circumference of $\mathcal{G}(n, M)$ in the supercritical phase [17]. It is the first ingredient that is the new contribution.

Remark 3. For the task of bounding the weight of a cycle in a large edge-weighted 3-regular subgraph, one might suggest investigating the weight of the least-weight matching. Certainly the complement of a Hamilton cycle in a 3-regular graph forms a perfect matching. But, in general, the maximum-weight cycle is not necessarily Hamiltonian. Thus, its removal from the graph does not always form a perfect matching.

4 Computing E_T

Recall the definitions of $f_T(\mathbf{x}) = f_T(x_1, x_2, \ldots, x_k)$ and E_T from (3.1). In the previous section we saw that the weight of the heaviest cycle in a certain edge-weighted pseudograph can be bounded in terms of E_T for any biased tree T. In this section we compute the value of E_T for a few specific biased trees T. For some trees T we also state the biases $\mathbf{b}$ which make E_T as small as possible.

Proposition 8 *Let T be the biased tree on two degree-3 vertices and four leaf vertices with bias b on the edges incident to leaves and bias $1 - 4b$ on the remaining edge, where b is the unique zero of $105b^3 - 90b^2 + 24b - 2$ on $0 < b < 1/4$. Then*

$$E_T = \frac{4(1 - 3b)(5b^2 - 5b + 1)}{(7b - 2)^2}$$

which lies in $(0.8797, 0.8798)$ and hence $c^* = 0.8798$ is k-admissible for $k = 5$.

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It can be shown that, for this tree, no other choice of biases $\mathbf{b}$ yields a lower value of E_T. See [13] for details.

Proof. We begin by letting v and w denote the two non-leaf vertices of T. Let e_1 and e_2 denote the two edges which are each incident to v and a leaf. Denote by e_3 the edge joining v to w, and christen the other two edges as e_4 and e_5. Under this ordering the biases are $\mathbf{b} = (b, b, 1 - 4b, b, b)$.

To evaluate the integral E_T we exploit some of its symmetry. It suffices to integrate over only nonnegative x_1, x_2, x_3, x_4, x_5 satisfying $x_1 \le x_2$ and $x_4 \le x_5$ and multiply the final result by 4. For such points, only two of the $P \in \mathcal{P}$ can attain the maximum in the definition of f_T, giving us

$$f_T(\mathbf{x}) = f_T(x_1, x_2, x_3, x_4, x_5) = \max\{bx_1 + bx_2 + bx_4 + bx_5, bx_2 + (1 - 4b)x_3 + bx_5\}.$$

We split the region of integration into two parts, according to whether

$$bx_1 + bx_2 + bx_4 + bx_5 \ge bx_2 + (1 - 4b)x_3 + bx_5,$$

i.e. $b(x_1 + x_4)/(1 - 4b) \ge x_3$. The integrals are

$$\int_{x_2=0}^{\infty} \int_{x_5=0}^{\infty} \int_{x_1=0}^{x_2} \int_{x_4=0}^{x_5} \int_{x_3=0}^{\frac{b(x_1+x_4)}{1-4b}} b(x_1 + x_2 + x_4 + x_5)e^{-x_1-x_2-x_3-x_4-x_5}dx_1 dx_2 \cdots dx_5$$

and

$$\int_{x_2=0}^{\infty} \int_{x_5=0}^{\infty} \int_{x_1=0}^{x_2} \int_{x_4=0}^{x_5} \int_{x_3=\frac{b(x_1+x_4)}{1-4b}}^{\infty} (bx_2 + (1 - 4b)x_3 + bx_5)e^{-x_1-x_2-x_3-x_4-x_5}d\mathbf{x}$$

which, when evaluated, added together, and multiplied by 4, give us

$$E_T = \frac{4(1 - 3b)(5b^2 - 5b + 1)}{(7b - 2)^2}.$$

The result follows by simple computations. ∎

Proposition 9 *Let T be the biased tree on three degree-3 vertices and five leaf vertices with bias b on edges incident to leaves and bias $(1 - 5b)/2$ on the other edges, where b is the unique zero of $-3993b^4 + 2765b^3 + 1452b^5 - 804b^2 + 105b - 5$ on $0 < b < 1/5$. Then*

$$E_T = \frac{726b^4 - 601b^3 + 245b^2 - 55b + 5}{5(1 - b)(4b - 1)^2}$$

which lies in $(0.8741, 0.8742)$ and hence $c^* = 0.8742$ is k-admissible for $k = 7$.

For the tree in the above proposition, it can be shown [13] that no other choice of biases $\mathbf{b}$ yields a lower value of E_T.

Proof. We may view T as the complete binary tree on six edges with one additional edge e_1 joining the root to an additional vertex. Denote by e_2 and e_3 the other edges incident to

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the root. Denote by e_4 and e_5 the edges incident with e_2. Denote by e_6 and e_7 the two edges incident with e_3. Under this ordering, the biases are

$$\mathbf{b} = \left(b, \frac{1-5b}{2}, \frac{1-5b}{2}, b, b, b, b \right).$$

By symmetry we may compute E_T by integrating over only $x_4 \le x_5$ and $x_6 \le x_7$ and multiplying the final result by 4. In this range, f_T is the maximum of four expressions,

1. $b(x_4 + x_5 + x_6 + x_7)$,

2. $bx_1 + (1 - 5b)x_2/2 + b(x_5 + x_6 + x_7)$,

3. $(1 - 5b)(x_2 + x_3)/2 + b(x_5 + x_7)$, and

4. $bx_1 + (1 - 5b)x_3/2 + b(x_4 + x_5 + x_7)$.

To compute the integral, the region of integration is divided into four parts, according to which of the above expressions gives the maximum. We present the details for the first part only.

The first expression exceeds the other three if and only if $bx_4 > bx_1 + (1 - 5b)x_2/2$ and $bx_6 > bx_1 + (1 - 5b)x_3/2$. To express the integral over this part as an iterated integral, we divide the part into two regions, according to whether $x_4 > x_6$ or not. The region on which $x_4 > x_6$ gives the integral

$$\int_{x_4=0}^{\infty} \int_{x_6=0}^{x_4} \int_{x_1=0}^{x_6} \int_{x_2=0}^{\frac{2b(x_4-x_1)}{1-5b}} \int_{x_3=0}^{\frac{2b(x_6-x_1)}{1-5b}} \int_{x_5=x_4}^{\infty} \int_{x_7=x_6}^{\infty} I \, dx_1 dx_2 dx_3 dx_4 dx_5 dx_6 dx_7$$

with integrand $I = b(x_4 + x_5 + x_6 + x_7)e^{-x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7}$. The integral evaluates to

$$\frac{1}{100} \frac{(73b - 17)b^3}{(4b - 1)^3}.$$

The region on which $x_6 > x_4$ gives

$$\frac{1}{100} \frac{(73b - 17)b^3}{(4b - 1)(16b^2 - 8b + 1)}.$$

The other three parts can be expressed and evaluated similarly, giving a final result of

$$E_T = \frac{726b^4 - 601b^3 + 245b^2 - 55b + 5}{5(1 - b)(4b - 1)^2}.$$

The result follows by simple computations. ∎

Our final computation is for a nine-edge tree. Its lengthy proof uses the same method that we used in the previous computations, so we omit it.

Proposition 10 *Let T be the biased tree on four degree-3 vertices and six leaf vertices with bias b on edges incident to leaves and bias $(1-6b)/4$ on the other edges, where b is the unique zero of*

$$2372895b^6 - 3013200b^5 + 1501416b^4 - 389232b^3 + 56016b^2 - 4224b + 128$$

on $0 < b < 1/6$. Then

$$E_T = \frac{-2(-128 + 2448b - 17856b^2 + 60372b^3 - 88938b^4 + 37665b^5)}{9(32 - 600b + 4212b^2 - 13122b^3 + 15309b^4)}$$

which lies in $(0.8696, 0.8697)$ and hence $c^* = 0.8697$ is k-admissible for $k = 9$.

Computer simulations suggest that the value of c^* decreases only slightly as k is increased further so we do not pursue this here.

5 Circumference of a random prekernel with given degree sequence

In the previous sections we have established Lemma 7, an a.a.s. upper bound on the weight of the heaviest cycle in certain randomly-edge-weighted pseudographs. In this section we use that lemma to establish an upper bound on the circumference of a random prekernel whose degree sequence satisfies certain conditions. In later sections we will see that the degree sequence of the prekernel of $\mathcal{G}(n, M)$ a.a.s. satisfies these conditions, allowing us to use this result to establish an a.a.s. upper bound on the circumference of the prekernel of $\mathcal{G}(n, M)$.

One of the challenges in this section arises because Lemma 7 is a statement about non-random pseudographs with random edge weightings, while we are proving a statement about random prekernels. The kernel configuration model of Pittel and Wormald, described below, allows us to rigorously make this transition. It combines a pairing model, for generating the kernel, with a random sequence of weights on the kernel edges.

Another challenge in this section is to show that the conditions on the degree sequence imply that the hypotheses of Lemma 7 are satisfied. One hypothesis requires that there are few edges incident with vertices whose degree exceeds 3. Another hypothesis requires that the number of short cycles in the kernel be small. In Łuczak's proof of his upper bound for the circumference of $\mathcal{G}(n, M)$ in the supercritical phase, he established the first hypothesis by direct enumeration over degree sequences. (See the proof of Theorem 10 in [17].) However, Łuczak does not require the second hypothesis, so we will need to prove it here. We will see that, without much extra effort, our proof of the second hypothesis gives an alternative derivation of the first hypothesis. In [5] and [20] there are results about short cycles arising in this pairing model. However, these results apply only when the maximum degree is bounded, so they cannot be used for our application.

We are interested in studying prekernels with a given degree sequence $\mathbf{d} = (d_i)$. We say that $\mathbf{d}$ is a *prekernel degree sequence* if its number of terms $v = v(\mathbf{d})$ is finite, each term is a positive integer at least 2, and $r = r(\mathbf{d}) = \sum_i (d_i - 2)$ is even. For $j = 2, 3, \ldots$ we define

$$D_j = D_j(\mathbf{d}) = |\{i : d_i = j\}|. \tag{5.1}$$

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The *kernel configuration model* $\mathcal{H}(\mathbf{d})$ is used to generate prekernels with degree sequence $\mathbf{d}$. It has been used successfully to calculate improved estimates for the size of the core, excess, and tree mantle [19]. We describe the model next.

For each i with $d_i \geq 3$ create a set S_i of d_i points. Let $\mathcal{P}$ be the set of perfect matchings on the union of these sets of points and choose $P \in \mathcal{P}$ uniformly at random. Then, assign the remaining numbers $\{i : d_i = 2\}$ to the edges of the perfect matching and, for each edge, choose a linear order for these numbers. The assignments and the linear ordering, denoted by f, are chosen uniformly at random. The pair (P, f) defines a random configuration in the model $\mathcal{H}(\mathbf{d})$.

Each configuration (P, f) corresponds to a prekernel $G(P, f)$ by collapsing each set S_i to a vertex (producing a kernel $K(P)$) and placing the degree-2 vertices on the edges of the kernel according to the assignment and linear orderings.

Lemma 11 *Let* $\mathbf{d} = \mathbf{d}(n)$ *be a prekernel degree sequence satisfying* $v = v(\mathbf{d}) \to \infty$, $r = r(\mathbf{d}) \to \infty$, $r = o(v)$, $D_3 = D_3(\mathbf{d}) \sim r$, *and*

$$\sum_{i:d_i \geq 3} \binom{d_i}{2} < 4r.$$

Fix a positive integer $k \geq 2$ *and suppose that the positive constant* c^* *is k-admissible. For a random configuration* (P, f) *in* $\mathcal{H}(\mathbf{d})$, *the longest cycle in* $G(P, f)$ *has length a.a.s. at most* c^*v *as* $n \to \infty$.

Proof. Define $\mathcal{P}^*$ to be the set of $P \in \mathcal{P}$ for which $K(P)$ has at most $\sqrt{r}$ edges in cycles of length at most k. We will show that a random configuration (P, f) a.a.s. has $P \in \mathcal{P}^*$. Recall that P is a random perfect matching on the points in the union of the S_i. For $j \in \{1, 2, \ldots, k\}$, the number of ways of choosing j pairs of points to form a cycle is at most

$$\frac{1}{2j} \left(\sum_{i:d_i \geq 3} 2\binom{d_i}{2} \right)^j = O\left(r^j\right).$$

The probability that j given pairs of points appear in the pairing P is asymptotic to

$$\left(\sum_{i:d_i \geq 3} d_i \right)^{-j}$$

since j is bounded. Now

$$\sum_{i:d_i \geq 3} d_i \; > \; \sum_{i:d_i \geq 3} (d_i - 2)$$
$$= \; \sum_i (d_i - 2)$$
$$= \; r$$

so the expected number of cycles of length j is $O(r^j r^{-j}) = O(1)$. Since k is fixed, the expected number of edges in such cycles is also $O(1)$. By Markov's inequality, the number of edges

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in cycles of length j is a.a.s. bounded above by any function $\omega = \omega(n) \to \infty$, in particular $\sqrt{r}/k$. Thus, a.a.s. $P \in \mathcal{P}^*$.

Let (P, f) be a random configuration from $\mathcal{H}(\mathbf{d})$. Define $G'(P, f)$ to be the edge-weighted pseudograph whose underlying pseudograph is $K(P)$ and whose edge-weight on e, for each edge e, is one more than the number of vertices assigned to e by f. Let A be the event that the heaviest cycle in $G'(P, f)$ has weight at most c^*v. Let P_0 be the P^* minimizing $\mathbf{P}[A \mid P = P^*]$ over $P^* \in \mathcal{P}^*$. The minimum exists because $\mathcal{P}^*$ is finite. Next we verify that, conditioned on $P = P_0$, $G'(P, f)$ satisfies the hypotheses of Lemma 7. The number of vertices v' of $G'(P, f)$ is at least $D_3 \sim r \to \infty$. The minimum degree is at least 3 because it is a kernel. The number of edges incident to cycles of length at most k (including loops and parallel edges) is at most $\sqrt{r} = o(r) = o(v')$ since $P_0 \in \mathcal{P}^*$. The number of edges incident to vertices of degree greater than 3 is at most

$$
\begin{aligned}
\sum_{j:d_j \geq 4} d_j &\leq 2 \sum_{j:d_j \geq 4} (d_j - 2) \\
&= 2 \sum_j (d_j - 2) - 2D_3 \\
&= 2r - 2D_3 \\
&= o(r)
\end{aligned}
$$

which is $o(v')$. The number of edges m' satisfies

$$
\begin{aligned}
2m' &= \sum_{j:d_j \geq 3} d_j \\
&= 3D_3 + o(r)
\end{aligned}
$$

by the previous calculation, so $m' = O(D_3) = O(r) = o(v)$. To see that m' is little-oh of the sum N of the edge-weights, observe that N is the number of edges of $G(P, f)$, which is $\sum_j d_j/2 \geq v$. Next observe that the edge weights form a sequence of positive integers that is determined by the assignment f in the random configuration. There are exactly $|\{i : d_i = 2\}|!$ choices for f that produce any given sequence, so the sequence is chosen uniformly at random. We have shown that the hypotheses of Lemma 7 hold for $G'(P, f)$ conditioned on $P = P_0$, so we have $\mathbf{P}[A \mid P = P_0] = 1 - o(1)$. Now

$$
\begin{aligned}
\mathbf{P}[A] &\geq \sum_{P^* \in \mathcal{P}^*} \mathbf{P}[A \mid P = P^*]\mathbf{P}[P = P^*] \\
&\geq \mathbf{P}[A \mid P = P_0] \sum_{P^* \in \mathcal{P}^*} \mathbf{P}[P = P^*]
\end{aligned}
$$

by the choice of P_0. Since we showed $P \in \mathcal{P}^*$ a.a.s. we get $\mathbf{P}[A] = 1 - o(1)$; that is, the heaviest cycle in $G'(P, f)$ a.a.s. has weight at most c^*v. But if C is a cycle in $G(P, f)$ of some length l, C corresponds naturally to a cycle in $G'(P, f)$ of weight l. So the longest cycle in $G(P, f)$ a.a.s. has length at most c^*v. ∎

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Corollary 12 *Let* $\mathbf{d} = \mathbf{d}(n)$ *be a prekernel degree sequence satisfying* $v = v(\mathbf{d}) \to \infty$, $r = r(\mathbf{d}) \to \infty$, $r = o(v)$, $D_3 = D_3(\mathbf{d}) \sim r$, *and*

$$\sum_{i:d_i \geq 3} \binom{d_i}{2} < 4r.$$

Fix $k \geq 2$ *and suppose that the positive constant* c^* *is* k-*admissible. Let* G *be chosen uniformly at random from all prekernels with degree sequence* $\mathbf{d}$. *The longest cycle in* G *has length a.a.s. at most* $c^* v$ *as* $n \to \infty$.

Proof. The probability space $\mathcal{H}(\mathbf{d})$, conditioned on the event that $G(P, f)$ is a simple graph, is a uniform probability space on the prekernels with degree sequence $\mathbf{d}$ ([19], Lemma 3). By Lemma 5 in [19], $G(P, f)$ is a.a.s. a simple graph. (In fact, Lemma 5 in [19] is stated with an additional hypothesis on $\max d_i$, but this hypothesis is not used in the proof.) The result now follows from Lemma 11. ∎

6 Truncated multinomial distribution

In order to apply Corollary 12 to the prekernel of $\mathcal{G}(n, M)$, we must verify the hypotheses about properties of the degree sequence. We give a new derivation of these properties, which will require some facts about the following distribution.

Let v and t be positive integers. The probability space $\mathbf{Multi}(v, t)$ consists of vectors $(d_1, d_2, \ldots, d_v)$ with distribution

$$\mathbf{P}[d_1 = j_1, d_2 = j_2, \ldots, d_v = j_v] = \frac{t!}{v^t j_1! j_2! \cdots j_v!}$$

for any vector $(j_1, j_2, \ldots, j_v)$ of nonnegative integers summing to t. This is the well-known multinomial distribution, modelling the number of balls in each bin when each of t balls is tossed into one of v bins, independently and uniformly at random. The space $\mathbf{Multi}(v, t)|_{\geq 2}$ is obtained from $\mathbf{Multi}(v, t)$ by conditioning on the event that each $d_i \geq 2$.

Lemma 13 *Let* $v = v(n)$ *and* $r = r(n)$ *satisfy* $v \to \infty$, $r \to \infty$ *and* $r = o(v)$. *If the random vector* $\mathbf{d}$ *is distributed as* $\mathbf{Multi}(v, 2v + r)|_{\geq 2}$ *then a.a.s.* $D_3(\mathbf{d}) \sim r$ *and*

$$\sum_{i:d_i \geq 3} \binom{d_i}{2} < 4r.$$

Proof. Define the positive number λ by

$$\frac{\lambda(e^\lambda - 1)}{e^\lambda - 1 - \lambda} = 2 + \frac{r}{v}.$$

In [4], the authors show that λ exists and they use it to define a vector of independent truncated Poisson random variables which approximate $\mathbf{Multi}(v, 2v + r)|_{\geq 2}$ as follows. Define the random variable Y taking values $j = 2, 3, \ldots$ according to the distribution

$$\mathbf{P}[Y = j] = p_j = \frac{\lambda^j}{j!(e^\lambda - 1 - \lambda)}.$$

Consider the probability space formed by vectors $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_v)$ of v independent copies of Y and let Σ be the event that their sum satisfies $\sum_i Y_i = 2v + r$. For nonnegative integers $j_1, j_2, \ldots, j_v$ summing to $2v + r$ with each $j_i \geq 2$ we have

$$\mathbf{P}[Y_1 = j_1, \ldots, Y_v = j_v] = \frac{\lambda^{2v+r}}{(e^\lambda - 1 - \lambda)^v} \prod_{i=1}^{v} \frac{1}{j_i!}$$

so this probability space, conditioned on Σ, is identical to $\mathbf{Multi}(v, 2v+r)|_{\geq 2}$. Equation (5.7) in [19] says that

$$\mathbf{P}\left[\max_i Y_i < \log v \text{ or } \sum_{i:Y_i \geq 3} \binom{Y_i}{2} \geq 4r \mid \Sigma\right] = O(r^{-1} + rv^{-1}).$$

The second claim in the lemma follows. Theorem 4(a) in [18] states that for $r \to \infty$,

$$\mathbf{P}[\Sigma] = \frac{1 + O(r^{-1})}{\sqrt{2\pi vc(1 + \bar{\eta} - c)}}$$

where $c(1 + \bar{\eta} - c) \sim c - k = r/v$ by Equation (20) in [18]. It follows that

$$\mathbf{P}[\Sigma]^{-1} = O(\sqrt{r}). \tag{6.1}$$

To establish the first claim in the lemma, we observe that $D_3(\mathbf{Y})$ is distributed as a binomial random variable with v trials and p_3 probability of success. By Chernoff's bound,

$$\mathbf{P}\left[|D_3(\mathbf{Y}) - vp_3| > a\right] < 2\exp(-a^2/(3vp_3))$$

for $0 < a \leq vp_3$. Recalling (6.1), in $\mathbf{Multi}(v, 2v+r)|_{\geq 2}$ we have

$$\mathbf{P}\left[|D_3(\mathbf{d}) - vp_3| > a\right] = O(\sqrt{r})\exp(-a^2/(3vp_3)).$$

Setting $a = \sqrt{vp_3} \log r$ (which satisfies $a \leq vp_3$, as we will see shortly) we get $D_3(\mathbf{d}) = vp_3 + O(\sqrt{vp_3} \log r)$ with probability $1 - O(\exp(-(\log r)^2/3))$. Now $\lambda \sim 3rv^{-1}$ by Theorem 1(a) in [18], so

$$vp_3 = v\frac{\lambda^3}{3!(e^\lambda - 1 - \lambda)} = v\frac{\lambda^3}{3!(\lambda^2/2 + O(\lambda^3))} = \frac{1}{3}v\lambda(1 + O(\lambda)) = r(1 + O(rv^{-1}))$$

giving us $D_3(\mathbf{d}) \sim r$ a.a.s. as required. ∎

7 Properties of vertex degrees in $\mathcal{G}(n, M)$

Now we proceed to establish the properties of the degree sequence of the prekernel of $\mathcal{G}(n, M)$ that are required to apply Corollary 12. Recall that we are assuming $M = M(n) = n/2 + s$ for some $s = s(n)$ satisfying $s = o(n)$ and $n^{2/3} = o(s)$. For this range of M, it is well-known that $\mathcal{G}(n, M)$ a.a.s. has a unique component with maximum number of vertices [2], which we call the largest component.

We begin by showing a.a.s. there are few vertices in the core that lie outside the largest component. The next result is part of the proof of Theorem 4 of [17]. Here we present a slightly more thorough proof.

Lemma 14 *Let $M = M(n) = n/2 + s$ for some $s = s(n)$ satisfying $s = o(n)$ and $n^{2/3} = o(s)$. The number of vertices in cycles of $\mathcal{G}(n, M)$ not in the largest component is a.a.s. at most $\omega n/s$ for any $\omega = \omega(n) \to \infty$.*

Proof. Let $\bar{G}$ be the graph formed from $\mathcal{G}(n, M)$ by removing its (lexicographically first) largest component. Let $n(\bar{G})$ and $M(\bar{G})$ represent its number of vertices and edges, respectively. Let $\epsilon > 0$ and define S to be the set of ordered pairs $(\bar{n}, \bar{M})$ satisfying

1. $(1 - \epsilon)4s \leq n - \bar{n} \leq (1 + \epsilon)4s$,

2. $(1 - \epsilon)4s \leq M - \bar{M} \leq (1 + \epsilon)4s$, and

3. $\mathbf{P}[n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}] > 0$.

It is known that the largest component of $\mathcal{G}(n, M)$ has a.a.s. $4s(1 + o(1))$ vertices and $4s(1 + o(1))$ edges [2, 16]. So a.a.s. $(n(\bar{G}), M(\bar{G})) \in S$. For $(\bar{n}, \bar{M}) \in S$ we have $\bar{M} \leq M - 4s(1 - \epsilon) = n/2 + s - 4s(1 - \epsilon)$ and $n \leq \bar{n} + 4s(1 + \epsilon)$, giving us

$$\bar{M} \leq \bar{n}/2 - s(1 - 6\epsilon). \tag{7.1}$$

To estimate the number X of vertices in cycles in $\bar{G}$ we let $(\bar{n}, \bar{M}) \in S$ and condition on the non-empty event $n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}$. In the conditioned space, $\bar{G}$ is equally likely to be any graph on $\bar{n}$ vertices and $\bar{M}$ edges. For $3 \leq k \leq \bar{n}$ the number of such graphs having a cycle of length k is at most

$$\binom{\bar{n}}{k} \frac{k!}{2k} \binom{\binom{\bar{n}}{2}}{\bar{M} - k}$$

so the expected value of X in this conditioned space is

$$\mathbf{E}[X \mid n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}] \leq \sum_{k=3}^{\bar{n}} k \binom{\bar{n}}{k} \frac{k!}{2k} \binom{\binom{\bar{n}}{2}}{\bar{M} - k} \binom{\binom{\bar{n}}{2}}{\bar{M}}^{-1} < \frac{1}{2} \sum_{k=3}^{\bar{n}} \frac{\bar{n}^k \bar{M}^k}{\left(\binom{\bar{n}}{2} - \bar{n}\right)^k}.$$

Using (7.1) this becomes

$$
\begin{aligned}
\mathbf{E}[X \mid n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}] \; &< \; \frac{1}{2} \sum_{k=3}^{\bar{n}} \left(\frac{\frac{\bar{n}}{2} - s(1 - 6\epsilon)}{\frac{\bar{n} - 1}{2} - 1} \right)^k \\
&< \; \frac{1}{2} \sum_{k=3}^{\bar{n}} \left(1 - \frac{s(1 - 6\epsilon)}{\bar{n} - 3} \right)^k \\
&< \; \frac{1}{2} \sum_{k=3}^{\infty} \left(1 - \frac{s(1 - 6\epsilon)}{\bar{n} - 3} \right)^k \\
&= \; \frac{1}{2} \times \frac{\bar{n} - 3}{s(1 - 6\epsilon)}
\end{aligned}
$$

which is at most n/s for n sufficiently large. By Markov's inequality,

$$\mathbf{P}[X \geq \omega n/s \mid n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}] \leq \frac{\mathbf{E}[X \mid n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}]}{\omega n/s} < \frac{1}{\omega}.$$

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So

$$
\begin{aligned}
\mathbf{P}[X < \omega n/s] \ & \geq \ \mathbf{P}[X < \omega n/s, (n(\bar{G}), M(\bar{G})) \in S] \\
& = \ \sum_{(\bar{n}, \bar{M}) \in S} \big(\mathbf{P}[X < \omega n/s \mid n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}] \\
& \quad\quad \times \mathbf{P}[n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}] \big) \\
& \geq \ \left(1 - \frac{1}{\omega} \right) \mathbf{P}[(n(\bar{G}), M(\bar{G})) \in S] \\
& = \ \left(1 - \frac{1}{\omega} \right) (1 - o(1))
\end{aligned}
$$

since a.a.s. $(n(\bar{G}), M(\bar{G})) \in S$. Therefore a.a.s. $X < \omega n/s$. ∎

Instead of proving results about the degree sequence of the prekernel of $\mathcal{G}(n, M)$ directly, we will actually prove results about the degree sequence of the core. The next result will allow us to transfer results about the core to the prekernel.

Lemma 15 *Let $M = M(n) = n/2 + s$ for some $s = s(n)$ satisfying $s = o(n)$ and $n^{2/3} = o(s)$. The core of the largest component of $\mathcal{G}(n, M)$ is a.a.s. formed from the core of $\mathcal{G}(n, M)$ by removing $o(s^2/n)$ vertices of degree 2. Also, the prekernel of $\mathcal{G}(n, M)$ is a.a.s. formed from the core of $\mathcal{G}(n, M)$ by removing $o(s^2/n)$ vertices of degree 2.*

Proof. It is well-known that the largest component of $\mathcal{G}(n, M)$ is a.a.s. the only component that has more than one cycle. (See Theorem 5.12 in [12].) So, the core of $\mathcal{G}(n, M)$ is a.a.s. composed of the core of the largest component together with some cycle components. By Lemma 14 the number of vertices in the cycle components is a.a.s. at most $\omega n/s = (s/n^{2/3})(n/s) = n^{1/3} = (n^{2/3})^2/n = o(s^2/n)$, since we may take $\omega = s/n^{2/3}$. Because the largest component of $\mathcal{G}(n, M)$ a.a.s. contains more than one cycle, it follows that these cycle components are a.a.s. all of the cycle components in the core of $\mathcal{G}(n, M)$, making the prekernel a.a.s. equal to the core of the largest component. ∎

Now we establish the required properties of the degree sequence of the prekernel of $\mathcal{G}(n, M)$. Recall the definitions of $D_j(\mathbf{d})$, $v(\mathbf{d})$, $r(\mathbf{d})$ from (5.1). The results about $r(\mathbf{d})$, $v(\mathbf{d})$, and $D_3(\mathbf{d})$ in the following lemma were used by Łuczak [17]. He used the estimate of $r(\mathbf{d})$ from [16] to establish estimates for $D_j(\mathbf{d})$ and $v(\mathbf{d})$ by direct enumeration over degree sequences. Instead of studying the prekernel directly, he studied the core of the largest component. Our proof method is different, using the known estimates of $v(\mathbf{d})$ and $r(\mathbf{d})$ to establish results about the degree sequence of the core.

Lemma 16 *Let $M = M(n) = n/2 + s$ for some $s = s(n)$ satisfying $s = o(n)$ and $n^{2/3} = o(s)$. Let $\mathbf{d}$ be the degree sequence of the prekernel of $\mathcal{G}(n, M)$. Then, a.a.s. $v(\mathbf{d}) \sim 8s^2/n$, $r(\mathbf{d}) \sim D_3(\mathbf{d}) \sim 32s^3/(3n^2)$, and*

$$
\sum_{i: d_i \geq 3} 2 \binom{d_i}{2} < 4r(\mathbf{d}).
$$

Proof. By Lemma 15 the prekernel differs from the core a.a.s. by $o(s^2/n)$ vertices of degree 2. Thus, it suffices to prove the lemma for the degree sequence $\mathbf{d}$ of the core. Appealing to Lemma 15 again, the core differs from the core of the largest component a.a.s. by $o(s^2/n)$ vertices of degree 2. It is known [19] that the degree sequence $\mathbf{d}'$ of the core of the largest component a.a.s. has $v(\mathbf{d}') \sim 8s^2/n$ and $r(\mathbf{d}') \sim 32s^3/(3n^2)$, so we must have a.a.s. $v(\mathbf{d}) \sim 8s^2/n$ and $r(\mathbf{d}) \sim 32s^3/(3n^2)$ also. Letting $\epsilon > 0$, this means $\mathbf{d} \in S$ a.a.s. where S is the set of ordered pairs $(\bar{v}, \bar{r})$ satisfying

1. $(1 - \epsilon)8s^2/n \leq \bar{v} \leq (1 + \epsilon)8s^2/n$,

2. $(1 - \epsilon)32s^3/(3n^2) \leq \bar{r} \leq (1 + \epsilon)32s^3/(3n^2)$, and

3. $\mathbf{P}[v(\mathbf{d}) = \bar{v}, r(\mathbf{d}) = \bar{r}] > 0$.

We note that for $(\bar{v}, \bar{r}) \in S$ we have $\bar{r} = o(\bar{v})$ since $(s^3/n^2)/(s^2/n) = s/n = o(1)$ and both $\bar{v} \to \infty$ and $\bar{r} \to \infty$ since $n^{2/3} = o(s)$.

To establish the remaining properties of the degree sequence of the core of $\mathcal{G}(n, M)$ we use Theorem 2 of [4], which proves the existence of a probability space of ordered pairs (G, I) in which

1. G, conditioned on the event $I = 1$, is distributed as the core of $\mathcal{G}(n, M)$,

2. $\mathbf{P}[I = 1] = \Omega(1)$, and

3. the degree sequence $\mathbf{d}(G)$ of G, conditioned on $v(\mathbf{d}(G)) = \bar{v}$ and $r(\mathbf{d}(G)) = \bar{r}$, is distributed as $\mathbf{Multi}(\bar{v}, 2\bar{v} + \bar{r})|_{\geq 2}$.

(The statement of Theorem 2 of [4] actually includes the hypothesis $M \geq n$ which is not satisfied here; however, that hypothesis is not needed for their proof.)

Write $v = v(\mathbf{d}(G))$, $r = r(\mathbf{d}(G))$ and let A be the event that

1. $(1 - \epsilon)32s^3/(3n^2) \leq D_3(\mathbf{d}(G)) \leq (1 + \epsilon)32s^3/(3n^2)$, and

2. $\sum_{i:d_i(G) \geq 3} 2\binom{d_i(G)}{2} < 4r$.

To prove the lemma, we must show $\mathbf{P}[A \mid I = 1] = 1 + o(1)$ or equivalently $\mathbf{P}[A^C \mid I = 1] = o(1)$, where X^C denotes the complement of event X. We begin by writing

$$\mathbf{P}[A^C \mid I = 1] = \mathbf{P}[A^C, (v, r) \in S \mid I = 1] + \mathbf{P}[A^C, (v, r) \notin S \mid I = 1].$$

The second term is at most $\mathbf{P}[(v, r) \notin S \mid I = 1]$ which is $o(1)$ because $\mathbf{d}(G)$, conditioned on $I = 1$, is distributed like the degree sequence of the core of $\mathcal{G}(n, M)$. We estimate the first term by

$$
\begin{aligned}
\mathbf{P}[A^C, (v, r) \in S \mid I = 1] &\leq \mathbf{P}[I = 1]^{-1}\mathbf{P}[A^C, (v, r) \in S] \\
&\leq \mathbf{P}[I = 1]^{-1}\mathbf{P}[A^C \mid v = \hat{v}, r = \hat{r}]\mathbf{P}[(v, r) \in S]
\end{aligned}
$$

where $(\hat{v}, \hat{r})$ is the ordered pair maximizing $\mathbf{P}[A^C \mid v = \bar{v}, r = \bar{r}]$ over all $(\bar{v}, \bar{r}) \in S$. (The maximum exists because S is finite.) We now use the properties of the distribution of (G, I)

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to estimate each of $\mathbf{P}[I = 1]^{-1}$, $\mathbf{P}[A^C \mid v = \hat{v}, r = \hat{r}]$, and $\mathbf{P}[(v,r) \in S]$. We have already noted that $\mathbf{P}[I = 1] = \Omega(1)$, so we have $\mathbf{P}[I = 1]^{-1} = O(1)$. Since $(\hat{v}, \hat{r}) \in S$ we have $\hat{v} \to \infty$, $\hat{r} \to \infty$, and $\hat{r} = o(\hat{v})$. We know that, conditioned on $v = \hat{v}$ and $r = \hat{r}$, the degree sequence of G is distributed as $\mathbf{Multi}(\hat{v}, 2\hat{v} + \hat{r})|_{\geq 2}$. Lemma 13 tells us that event A occurs a.a.s. in this model so we have $\mathbf{P}[A^C \mid v = \hat{v}, r = \hat{r}] = o(1)$. Finally, we may crudely estimate $\mathbf{P}[(v,r) \in S] = O(1)$. Combining these estimates we get $\mathbf{P}[A^C \mid I = 1] = o(1)$ as required.

∎

8   Circumference of $\mathcal{G}(n, M)$

Lemma 17 *Let $M = M(n) = n/2 + s$ for some $s = s(n)$ satisfying $s = o(n)$ and $n^{2/3} = o(s)$. Fix k and suppose that the positive constant c^* is k-admissible. The circumference of $\mathcal{G}(n, M)$ is a.a.s. at most $(8c^* + o(1))s^2/n$.*

Proof. Every cycle in a graph lies in the graph's core. By Lemma 15 the prekernel G of $\mathcal{G}(n, M)$ is formed from the core of $\mathcal{G}(n, M)$ by removing $o(s^2/n)$ vertices of degree 2. So, to prove the lemma, it suffices to show that the circumference of G is a.a.s. at most $(8c^* + o(1))s^2/n$.

By Lemma 16 there exists $\omega = \omega(n) \to \infty$ such that the degree sequence $\mathbf{d}(G)$ of G a.a.s. lies in the set D of prekernel degree sequences $\mathbf{d}$ satisfying

1.
$$\sum_{i : d_i \geq 3} 2\binom{d_i}{2} < 4r(\mathbf{d}),$$

2. $(1 - \omega^{-1})8s^2/n \leq v(\mathbf{d}) \leq (1 + \omega^{-1})8s^2/n$,

3. $(1 - \omega^{-1})32s^3/(3n^2) \leq r(\mathbf{d}) \leq (1 + \omega^{-1})32s^3/(3n^2)$,

4. $(1 - \omega^{-1})32s^3/(3n^2) \leq D_3(\mathbf{d}) \leq (1 + \omega^{-1})32s^3/(3n^2)$, and

5. $\mathbf{P}[\mathbf{d}(G) = \mathbf{d}] > 0$.

Define A to be the event that the circumference of G is at most $c^* v(\mathbf{d}(G))$. We have

$$\mathbf{P}[A] \geq \sum_{\mathbf{d} \in D} \mathbf{P}[A \mid \mathbf{d}(G) = \mathbf{d}]\mathbf{P}[\mathbf{d}(G) = \mathbf{d}].$$

Suppose that $\mathbf{P}[A \mid \mathbf{d}(G) = \mathbf{d}]$ is minimized over $\mathbf{d} \in D$ by $\mathbf{d} = \mathbf{d}^*$. (The minimum exists since D is finite.) Then

$$
\begin{aligned}
\mathbf{P}[A] &\geq \mathbf{P}[A \mid \mathbf{d}(G) = \mathbf{d}^*] \sum_{\mathbf{d} \in D} \mathbf{P}[\mathbf{d}(G) = \mathbf{d}] \\
&= \mathbf{P}[A \mid \mathbf{d}(G) = \mathbf{d}^*](1 + o(1))
\end{aligned}
$$

since a.a.s. $\mathbf{d} \in D$.

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In general, the number of graphs on n vertices and M edges that have a given graph as their prekernel depends only on the number of vertices and edges of the given prekernel. So, conditioning on the event $\mathbf{d}(G) = \mathbf{d}^*$, G is equally likely to be each prekernel with degree sequence $\mathbf{d}^*$. The probability $P[A \mid \mathbf{d}(G) = \mathbf{d}^*]$ is thus the probability that a graph, chosen uniformly at random from all prekernels of degree sequence $\mathbf{d}^*$, has circumference a.a.s. at most $c^* v$. Since $\mathbf{d}^* \in D$, we may apply Corollary 12 to conclude that this probability is $1 + o(1)$. Thus $\mathbf{P}[A] = 1 + o(1)$. In other words, the circumference of G is a.a.s. at most $c^* v(\mathbf{d}(G))$. But we have seen $v(\mathbf{d}(G)) \sim 8s^2/n$ a.a.s. so the circumference is a.a.s. at most $(8c^* + o(1))s^2/n$, as required. ∎

Proof of Theorem 1. Proposition 10 tells us that $c^* = 0.8697$ is k-admissible for $k = 9$. By Lemma 17 the circumference of $\mathcal{G}(n, M)$ is a.a.s. at most $(8c^* + o(1))s^2/n < (6.958 + o(1))s^2/n$. ∎

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