Orientability thresholds for random hypergraphs

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Abstract

Let $h > w > 0$ be two fixed integers. Let H be a random hypergraph whose hyperedges are all of cardinality h. To *w-orient* a hyperedge, we assign exactly w of its vertices positive signs with respect to the hyperedge, and the rest negative. A (w, k)-orientation of H consists of a w-orientation of all hyperedges of H, such that each vertex receives at most k positive signs from its incident hyperedges. When k is large enough, we determine the threshold of the existence of a (w, k)-orientation of a random hypergraph. The (w, k)-orientation of hypergraphs is strongly related to a general version of the off-line load balancing problem. The graph case, when $h = 2$ and $w = 1$, was solved recently by Cain, Sanders and Wormald and independently by Fernholz and Ramachandran, which settled a conjecture of Karp and Saks.

1 Introduction

In this paper we consider a generalisation to random hypergraphs of a commonly studied orientation problem on graphs. An *h-hypergraph* is a hypergraph whose hyperedges are all of size h. Let $h > w$ be two given positive integers. We consider $\mathcal{G}_{n,m,h}$, the probability space of the set of all h-hypergraphs on n vertices and m hyperedges with the uniform distribution. A hyperedge is said to be *w-oriented* if exactly w distinct vertices in it are marked with positive signs with respect to the hyperedge. The *indegree* of a vertex is the number of

*The research reported in this paper forms part of this author's Ph.D. thesis "Generation and properties of random graphs and analysis of randomized algorithms" submitted to the University of Waterloo in 2009.

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‡Research supported by the Canadian Research Chairs Program and NSERC

positive signs it receives. Let k be a positive integer. A (w, k)-orientation of an h-hypergraph is a w-orientation all hyperedges such that each vertex has indegree at most k. If such a (w, k)-orientation exists, we say the hypergraph is (w, k)-*orientable*; for $w = 1$ we simply say k-*orientable.* Of course, being able to determine the (w, k)-orientability of an h-hypergraph H for all k solves the optimisation problem of minimising the maximum indegree of a w-orientation of H. If a graph (i.e. the case $h = 2$) is $(1, k)$-oriented, we may orient each edge of the graph in the normal fashion towards its vertex of positive sign, and we say the graph is k-*oriented*.

Note that a sufficiently sparse hypergraph is easily (w, k)-orientable. On the other hand, a trivial requirement for (w, k)-orientability is $m \leq kn/w$, since any w-oriented h-hypergraph with m edges has average indegree mw/n. In this paper, we show the existence and determine the value of the sharp threshold (defined more precisely later) at which the random h-hypergraph $\mathcal{G}_{n,m,h}$ fails to be (w, k)-orientable, provided k is a sufficiently large constant. We show that the threshold is the same as the threshold at which a certain type of subhypergraph achieves a critical density. In the above, as elsewhere in this paper, the phrase "for k sufficiently large" means for k larger than some constant depending only on w and h.

The hypergraph orientation problem is motivated by classical load balancing problems which have appeared in various guises in computer networking. A seminal result of Azar, Broder, Karlin and Upfal [2] is as follows. Throw n balls sequentially into n bins, with each ball put into the least-full of $h \geq 2$ randomly chosen boxes. Then, with high probability, by the time all balls are allocated, no bin contains many more than $(\ln \ln n)/ \ln h$ balls. If, instead, each ball is placed in a random bin, a much larger maximum value is likely to occur, approximately $\ln n/(\ln \ln n)$. This surprisingly simple method of reducing the maximum is widely used for load balancing. It has become known as the multiple-choice paradigm, the most common version being two-choice, when $h = 2$.

One application of load balancing occurs when work is spread among a group of computers, hard drives, CPUs, or other resources. In the on-line version, the jobs arrive sequentially and are assigned to separate machines. To save time, the load balancer decides which machine a job goes to after checking the current load of only a few (say h) machines. The goal is to minimise the maximum load of a machine. Mitzenmacher, Richa and Sitaraman [22] survey the history, applications and techniques relating to this. In particular, Berenbrink, Czumaj, Steger, and Vőcking [2, 3] show an achieveable maximum load is $m/n + O(\log \log n)$ for m jobs and n machines when $h \geq 2$.

Another application of load balancing, more relevant to the topic of this paper, is mentioned by Cain, Sanders and the second author [5]. This is the disk scheduling problem, in the context where any w out of h pieces of data are needed to reconstruct a logical data block. Individual pieces can be initially stored on different disks. Such an arrangement has advantageous fault tolerance features to guard against disk failures. It is also good for load balancing: when a request for a data block arrives, the scheduler can choose any w disks among the h relevant ones. See Sanders, Egner and Korst [25] for further references.

These load balancing problems correspond to the (w, k)-orientation problem for h-uniform hypergraphs, with $w = 1$ in the case of the job scheduling problem. The machines (bins) are vertices and a job (ball) is an edge consisting of precisely the set of machines to which it can be allocated. A job is allocated to a machine by assigning a positive sign to that vertex. The maximum load is then equal to the maximum indegree of a vertex in the (w, k)-oriented hypergraph.

The work in this paper is motivated by the off-line version of this problem, in which the edges are all exposed at the start. This has obvious applications, for instance, in the disk scheduling problem, the scheduler may be able to quickly process a large number of requests together off-line, to balance the load better. This can be useful if there is a backlog of requests; of course, if backlogs do not occur, the online problem is more relevant, but this would indicate ample processing capacity, in which case there may be less need for load balancing in the first place. Trivially, the on-line and off-line versions are the same if $h = 1$, i.e. there is no choice. For $h = 2$, the off-line version experiences an even better improvement than the on-line one. If $m < cn$ items are allocated to n bins, for c constant, the expected maximum load is bounded above by some constant c' depending on c.

To our knowledge, previous theoretical results concern only the case $w = 1$ (this also applies to on-line). For $w = 1$ it is well known that an optimal off-line solution, i.e. achieving minimum possible maximum load, can be found in polynomial time ($O(m^2)$) by solving a maximum flow problem. As explained in [5], it is desirable to achieve fast algorithms that are close to optimal with respect to the maximum load. There are linear time algorithms that achieve maximum load $O(m/n)$ [11, 19, 21].

A central role in solutions of the off-line orientation problem with $(w, h) = (1, 2)$ is played by the *k-core* of a graph, being the largest subgraph with minimum degree at least k. The sharp threshold for the k-orientability of the random graph $\mathcal{G}(n, m) = \mathcal{G}_{n,m,2}$ was found in [5], and simultaneously by Fernholz and Ramachandran [14]. These were proofs of a conjecture of Karp and Saks that this threshold coincides with the threshold at which the $(k+1)$-core has average degree at most $2k$. (It is obvious that a graph cannot be k-oriented if it has a subgraph of average degree greater than $2k$.) In each case, the proof analysed a linear time algorithm that finds a k-orientation a.a.s. when the mean degree of the $(k+1)$-core is slightly less than $2k$. In this sense, the algorithms are asymptotically optimal since the threshold for the algorithms succeeding coincides with the threshold for existence of a k-orientation. The proof in [14] was significantly simpler than the other, which was made possible because a different algorithm was employed. It used a trick of "splitting vertices" to postpone decisions and thereby reduced the number of variables to be considered.

During the preparation of this paper, three preprints appeared by Frieze and Melsted [12], Fountoulakis and Panagiotou [13], and by Dietzfelbinger, Goerdt, Mitzenmacher, Montanari, Pagh and Rink [10] which independently study the threshold of $(1, 1)$-orientability of $\mathcal{G}_{n,m,h}$, i.e. the case $w = k = 1$. This has applications to cuckoo hashing. However, there seems to be no easy way to extend the proofs in [10, 12, 13] to solve for the case $k > 1$, even when $w = 1$.

In this paper, we solve the generalisation of the conjecture of Karp and Saks mentioned above, for fixed $h > w > 0$, provided k is sufficiently large. That is, we find the threshold of (w, k)-orientation of random h-hypergraphs in $\mathcal{G}_{n,m,h}$. The determination of this threshold helps to predict loads in the off-line w-out-of-h disk scheduling problem, where the randomness of the hypergraph is justified by the random intial allocation of file segments to disks. We believe furthermore that the characterisation of the threshold in terms of density of a type of core, and possibly our method of proof, will potentially help lead to fast algorithms for finding asymptotically optimal orientations.

Our approach has a significant difference from that used in the graph case when $(w, h) = (1, 2)$. The algorithm used in [14] does not seem to apply in the hypergraph case, at least, splitting vertices cannot be done without creating hyperedges of larger and larger size. The algorithm used by [5], on the other hand, generalises in an obvious way, but it is already very complicated to analyse in the graph case, and the extension of the analysis to the hypergraph case seems formidable. However, in common with those two approaches, we first find what we call the $(w, k+1)$-core in the hypergraph, which is an analogue of the $(k+1)$-core in graphs. We determine the size and density of this core when the random hypergraph's density is significantly larger than what is required for the core to form. This result may be of independent interest, and uses the differential equation method in a setting which contains a twist not encountered when it is applied to the graph case: some of the functions involved have singularities at the starting point. (See Section 3 for details.)

Although we gain information on the threshold of appearance of this core, we do not, and do not need to, determine it precisely. From here, we use the natural representation of the orientation problem in terms of flows. It is quite easy to generalise the network flow formulation from the case $h = 2$, $w = 1$ to the arbitrary case, giving a problem that can be solved in time $O(m^2)$ for $m = \Theta(n)$. Unlike the approaches for the graph case, we do not study an algorithm that solves the load balancing problem. Instead, we use the minimum cut characterisation of the maximum flow to show that a.a.s. the hypergraph can be (w, k)-oriented if and only if the density of its $(w, k+1)$-core is below a certain threshold. When the density of the $(w, k+1)$-core is above this threshold, it is trivially too dense to be (w, k)-oriented. Even the case $w = 1$ of our result gives a significant generalisation of the known results. We prove that the threshold of the orientability coincides with the threshold at which certain type of density (in the case $w = 1$, this refers to the average degree divided by h) of the $(w, k+1)$-core is at most k, and also the threshold at which certain type of induced subgraph (in the case $w = 1$, this refers to the standard induced subgraph) does not appear. For the graph case, our method provides a new proof (for sufficiently large k) of the Karp-Saks conjecture that we believe is simpler than the proofs of [5] and [14].

We give precise statements of our results, including definition of the $(w, k+1)$-core, in Section 2. In Section 3 we study the properties of the $(w, k+1)$-core. In Section 4, we formulate the appropriate network flow problem, determine a canonical minimum cut for a network corresponding to a non-(w, k)-orientable hypergraph, and give conditions under which such a minimum cut can exist. Finally, in Section 5, we show that for k is sufficiently large, such a cut a.a.s. does not exist when the density of the core is below a certain threshold.

An extended abstract for this paper, omitting most proofs, will appear in STOC 2010 [16].

2 Main results

Let $h > w > 0$ and $k \geq 2$ be fixed. For any h-hypergraph H, we examine whether a (w, k)-orientation exists. We call a vertex *light* if the degree of the vertex is at most k. For any light vertex v, we can give v the positive sign respect to any hyperedge x that is incident to v (we call this *partially orienting* x towards to v), without violating the condition that each vertex has indegree at most k. Remove v from H, and for each hyperedge x incident to v, simply update x by removing v. Then the size of x decreases by 1, and it has one less vertex that needs to be given the positive sign. If the size of a hyperedge falls to $h-w$, we can simply remove that hyperedge from the hypergraph. Repeating this until no light vertex exists, we call the remaining hypergraph $\widehat{H}$ the $(w,k+1)$-*core* of H. Every vertex in $\widehat{H}$ has degree at least $k+1$, and every hyperedge in $\widehat{H}$ of size $h-j$ requires a $(w-j)$-orientation in order to obtain a w-orientation of the original hyperedge in H.

In order to simplify the notation, we use $\bar{n}$, $\bar{m}$ and $\bar{\mu}$ to denote the numbers of vertices and of hyperedges, and the average degree, of $H \in \mathcal{G}_{\bar{n},\bar{m},h}$, reserving n, m_{h-j} and μ to denote the numbers of vertices and of hyperedges of size $h-j$, and the average degree, of $\widehat{H}$.

Instead of considering the probability space $\mathcal{G}_{\bar{n},\bar{m},h}$, we may consider $\mathcal{M}_{\bar{n},\bar{m},h}$, the probability space of random multihypergraphs with $\bar{n}$ vertices and $\bar{m}$ hyperedges, such that each hyperedge x is of size h, and each vertex in x is chosen independently, uniformly at random from $[\bar{n}]$. Actually $\mathcal{M}_{\bar{n},\bar{m},h}$ may be a more accurate model for the off-line load balancing problem in some applications, and as we shall see, results for the non-multiple edge case can be deduced from it. For a nonnegative integer vector $\mathbf{m} = (m_2, \dots, m_h)$, we also define the probability space $\mathcal{M}_{n,\mathbf{m}}$, being the obvious generalisation of $\mathcal{M}_{n,m,h}$ to non-uniform multihypergraphs in which m_i is the number of hyperedges of size i.

All our asymptotic notation refers to $n \to \infty$. For clarity, we consider $H \in \mathcal{M}_{\bar{n},\bar{m},h}$. We use n, m_{h-j} and μ for the number of vertices, the number of hyperedges of size $h-j$ and the average degree of $\widehat{H}$. We parametrise the number $\bar{m}$ of edges in the hypergraphs under study by letting $\bar{\mu} = \bar{\mu}(n)$ denote $h\bar{m}/\bar{n}$, the average degree of $H \in \mathcal{M}_{\bar{n},\bar{m},h}$ (or of $H \in \mathcal{G}_{\bar{n},\bar{m},h}$).

Our first observation concerns the distribution of $\widehat{H}$ and its vertex degrees. Let $\mathrm{Multi}(n,m,k+1)$ denote the multinomial distribution of n integers summing to m, restricted to each of the integers being at least $k+1$. We call this the *truncated multinomial* distribution.

Proposition 2.1 *Let $h > w \geq 1$ be two fixed integers. Let $H \in \mathcal{M}_{\bar{n},\bar{m},h}$ and let $\widehat{H}$ be its $(w,k+1)$-core. Conditional on its number n of vertices and numbers m_{h-j} of hyperedges of size $h-j$ for $j = 0,\dots,w-1$, the random hypergraph $\widehat{H}$ is distributed uniformly at random. Furthermore, the distribution of the degree sequence of $\widehat{H}$ is the truncated multinomial distribution* $\mathrm{Multi}(n,m,k+1)$ *where $m = \sum_{j=0}^{w-1}(h-j)m_{h-j}$.*

The following theorem shows that the size and the number of hyperedges of $\widehat{H}$ are highly concentrated around the solution of a system of differential equations. The theorem covers the cases for any arbitrary $h > w \geq 2$ and holds for all sufficiently large k. The special case $w=1$ has been studied by various authors and the concentration results can be found in [5, Theorem 3] which hold for all $k \geq 0$. Since w and k are fixed, we often omit them from the notation.

Theorem 2.2 *Let $h > w \geq 2$ be two fixed integers. Assume that for some constant $c > 1$ we have $ck \leq \bar{\mu}$ where $\bar{\mu} = h\bar{m}/\bar{n}$. Let $H \in \mathcal{M}_{\bar{n},\bar{m},h}$ and let $\widehat{H}$ be its $(w,k+1)$-core. Let n be the number of vertices and m_{h-j} the number of hyperedges of size $h-j$ of $\widehat{H}$. Then, provided k is sufficiently large, there are constants $\alpha > 0$ and $\beta_{h-j} > 0$, defined in (3.3) below, depending* *only on $\bar{\mu}$, k, w and h, for which a.a.s. $n \sim \alpha\bar{n}$ and $m_{h-j} \sim \beta_{h-j}\bar{n}$ for $0 \le i \le w-1$. The same conclusion (with the same constants) holds for $H \in \mathcal{G}_{\bar{n},\bar{m},h}$.*

Note. The full definition of α and β_{h-j} in the theorem is rather complicated, involving the solution of a differential equation system given below in (3.4–3.14).

Let $\mathcal{P}$ be a hypergraph property and let $\mathcal{M}_{n,m,h} \in \mathcal{P}$ denote the event that a random hypergraph from $\mathcal{M}_{n,m,h}$ has the property $\mathcal{P}$. Following [1, Section 10.1, Definition 4], we say that $\mathcal{P}$ has a *sharp threshold function* $f(n)$ if for any constant $\epsilon > 0$, $\mathbf{P}(\mathcal{M}_{n,m,h} \in \mathcal{P}) \to 1$ when $m \le (1-\epsilon)f(n)$, and $\mathbf{P}(\mathcal{M}_{n,m,h} \in \mathcal{P}) \to 0$ when $m \ge (1+\epsilon)f(n)$.

Let $\kappa(\widehat{H})$ denote $\sum_{j=0}^{w-1}(w-j)m_{h-j}/n$, which we call the *$w$-density* of $\widehat{H}$. We similarly define the w-density of any hypergraph all of whose hyperedges have sizes between $h-w+1$ and h. It helps to notice, by the definition of w-density, that

$$n\kappa(\widehat{H}) = d(\widehat{H}) - (h-w)m,$$

where $d(\widehat{H})$ denotes the degree sum of $\widehat{H}$ and $m = \sum_{j=0}^{w-1} m_{h-j}$. We say that a hypergraph H has *property $\mathcal{T}$* if $\kappa(\widehat{H}) \le k$, where $\widehat{H}$ is the $(w, k+1)$-core of H. The following theorem, proved using Theorem 2.2, immediately gives the corollary that there is a sharp threshold function for property $\mathcal{T}$.

Theorem 2.3 *Let $H \in \mathcal{M}_{\bar{n},\bar{m},h}$. Let $\bar{\mu}$ be the average degree of H and let $\widehat{H}$ be the $(w, k+1)$-core of H. Then for all sufficiently large k, there exists a strictly increasing function $c(\bar{\mu})$ of $\bar{\mu}$, such that for any fixed $c_2 > c_1 > 1$ and for any $c_1k < \bar{\mu} < c_2k$, a.a.s. $\kappa(\widehat{H}) \sim c(\bar{\mu})$.*

Corollary 2.4 *There exists a sharp threshold function $f(\bar{n})$ for the hypergraph property $\mathcal{T}$ in $\mathcal{M}_{\bar{n},\bar{m},h}$ and $\mathcal{G}_{\bar{n},\bar{m},h}$ provided k is sufficiently large.*

The function $c(\bar{\mu})$ in the theorem, and the threshold function in the corollary, are determined by the solution of the differential equation system referred to in Theorem 2.1.

We have defined a (w,k)-orientation of a uniform hypergraph in Section 1. We can similarly define a (w,k)-orientation of a non-uniform hypergraph G with sizes of hyperedges between $h-w+1$ and h to be a simultaneous $(w-j)$-orientation of each hyperedge of size $h-j$ such that every vertex has indegree at most k. By counting the positive signs in orientations, we see that if property $\mathcal{T}$ fails, there is no (w,k)-orientation of $\widehat{H}$, and hence there is no (w,k)-orientation of H.

For a nonnegative integer vector $\mathbf{m} = (m_{h-w+1}, \ldots, m_h)$, let $\mathcal{M}(n, \mathbf{m}, k+1)$ denote $\mathcal{M}_{n,\mathbf{m}}$ restricted to multihypergraphs with minimum degree at least $k+1$. By Proposition 2.1, $\mathcal{M}(n, \mathbf{m}, k+1)$ has the distribution of the $(w, k+1)$-core of $H \in \mathcal{M}_{\bar{n},\bar{m},h}$ conditioned on the number of vertices being n and the number of hyperedges of each size being given by $\mathbf{m}$. To emphasise the difference, we will use G to denote a not-necessarily-uniform hypergraph in cases where we might use H for a uniform hypergraph.

Given a vertex set S, we say a hyperedge x is *partially contained in S* if $|x \cap S| \ge 2$.

Definition 2.5 *Let $0 < \gamma < 1$. We say that a multihypergraph G has property $\mathcal{A}(\gamma)$ if for all $S \subset V(G)$ with $|S| < \gamma|V(G)|$ the number of hyperedges partially contained in S is strictly less than $k|S|/2w$.*

In the following theorem, $\mathbf{m} = \mathbf{m}(n)$ denotes an integer vector for each n.

Theorem 2.6 *Let γ be any constant between 0 and 1. Then there exists a constant $N > 0$ depending only on γ, such that for all $k > N$ and any $\epsilon > 0$, if $\mathbf{m}(n)$ satisfies $\sum_{j=0}^{w-1}(w-j)m_{h-j}(n) \leq kn - \epsilon n$ for all n, then $G \in \mathcal{M}(n, \mathbf{m}(n), k+1)$ a.a.s. either has a (w,k)-orientation or does not have property $\mathcal{A}(\gamma)$.*

Let $f(\bar{n})$ be the threshold of property $\mathcal{T}$ given in Corollary 2.4. We show in the forthcoming Corollary 4.3 that for certain values of γ, a.a.s. $\widehat{H}$ has property $\mathcal{A}(\gamma)$ if the average degree of H is at most hk/w. We will combine this with Corollary 2.4 and Theorem 2.6 and a relation we will show between $\mathcal{M}_{\bar{n},\bar{m},h}$ and $\mathcal{G}_{\bar{n},\bar{m},h}$ (Lemma 3.1), to obtain the following.

Corollary 2.7 *Let $h > w > 0$ be two given integers and k be a sufficiently large constant. Let $f(\bar{n})$ be the threshold function of property $\mathcal{T}$ whose existence is asserted in Corollary 2.4. Then $f(\bar{n})$ is a sharp threshold for the (w,k)-orientability of $\mathcal{M}_{\bar{n},\bar{m},h}$ and $\mathcal{G}_{\bar{n},\bar{m},h}$.*

For any vertex set $S \subset V(H)$, define the subgraph *w-induced* by S to be the subgraph of G on vertex set S with the set of hyperedges $\{x' = x \cap S : \ x \in H, \ s.t. \ |x'| \geq h - w + 1\}$. Call this hypergraph H_S. It helps to notice that $\widehat{H}$ is the largest w-induced subgraph of H with minimum degree at least $k+1$. From the above results and a relation we will show between $\mathcal{M}_{\bar{n},\bar{m},h}$ and $\mathcal{G}_{\bar{n},\bar{m},h}$, we will obtain the following.

Corollary 2.8 *The following three graph properties of $H \in \mathcal{M}_{\bar{n},\bar{m},h}$ (or $\mathcal{G}_{\bar{n},\bar{m},h}$) have the same sharp threshold.*

(i) H is (w,k)-orientable.

(ii) H has property $\mathcal{T}$.

(iii) There exists no w-induced subgraph $H' \subset H$ with $\kappa(H') \geq k$.

3 Analysing the size and density of the $(w, k+1)$-core

A model of generating random graphs via multigraphs, used by Bollobás and Frieze [4] and Chvatál [8], is described as follows. Let $\mathcal{P}_{\bar{n},\bar{m}}$ be the probability space of functions $g : [\bar{m}] \times [2] \to [\bar{n}]$ with the uniform distribution. Equivalently, $\mathcal{P}_{\bar{n},\bar{m}}$ can be described as the uniform probability space of allocations of $2\bar{m}$ balls into $\bar{n}$ bins. A probability space of random multigraphs can be obtained by taking $\{g(i,1), g(i,2)\}$ as an edge for each i. This model can easily be extended to generate non-uniform random multihypergraphs by letting $\mathbf{m} = (m_2, \ldots, m_h)$ and taking $\mathcal{P}_{\bar{n},\mathbf{m}} = \{g : \cup_{i=2}^{h}[m_i] \times [i] \to [\bar{n}]\}$. Let $\mathcal{M}_{\bar{n},\mathbf{m}}$ be the probability space of random multihypergraphs obtained by taking each $\{g(j,1), \ldots, g(j,i)\}$ as a hyperedge, where $j \in [m_i]$ and $2 \leq i \leq h$. (Loops and multiple edges are possible.) Note that $\mathcal{M}_{\bar{n},\mathbf{m}}$, where $\mathbf{m} = (m_2) = (\bar{m})$, is a random multigraph; it was shown in [8] that if this is conditioned on being simple (i.e. no loops and no multiple edges), it is equal to $\mathcal{G}_{\bar{n},\bar{m},2}$, and that the probability of a multigraph in $\mathcal{M}_{\bar{n},(\bar{m})}$ being simple is $\Omega(1)$ if $\bar{m} = O(\bar{n})$. This result is easily extended to the following result, using the same method of proof.

Lemma 3.1 *Assume $h \geq 2$ is a fixed integer and $\mathbf{m} = (m_2, \dots, m_h)$ is a non-negative integer vector. Assume further that $\sum_{i=2}^{h} m_i = O(\bar{n})$. Then the probability that a hypergraph in $\mathcal{M}_{\bar{n},\mathbf{m}}$ is simple is $\Omega(1)$.*

Cain and Wormald [6] recently introduced a related model to analyse the k-core of a random (multi)graph or (multi)hypergraph, including its size and degree distribution. This model is called the *pairing-allocation* model. The *partition-allocation* model, as defined below, is a generalisation of the pairing-allocation model, and analyses cores of multihypergraphs with given numbers of hyperedges of various sizes. We will use this model to prove Theorem 2.6 and to analyse a randomized algorithm called the RanCore algorithm, defined later in this section, which outputs the $(w, k+1)$-core of an input h-hypergraph.

Given $h \geq 2$, n, $\mathbf{m} = (m_2, \dots, m_h)$, $\mathbf{L} = (l_2, \dots, l_h)$ and a nonnegative integer k such that $D - \ell \geq kn$, where $D = \sum_{i=2}^{h} i m_i$ and $\ell = \sum_{i=2}^{h} l_i$, let V be a set of n bins, and $\mathbf{M}$ a collection of pairwise disjoint sets $\{M_1, \dots, M_h\}$, where M_i is a set of $i m_i$ balls partitioned into parts, each of size i, for all $2 \leq i \leq h$. Let Q be an additional bin to V. It may assist the reader to know that Q 'represents' all the hyperedge incidences at vertices of degree less than k, and l_i is the number of these incidences in edges of size i. The *partition-allocation* model $\mathcal{P}(V, \mathbf{M}, \mathbf{L}, k)$ is the probability space of ways of allocating balls to bins in the following way. Let $\mathcal{C} = \{c_2, \dots, c_h\}$ be a set of colours. Colour balls in M_i with c_i. (The function of the colours is only to denote the size of the part a ball lies in.) Then allocate the D balls uniformly at random (u.a.r.) into the bins in $V \cup \{Q\}$, such that the following constraints are satisfied:

(i) Q contains exactly ℓ balls;

(ii) each bin in V contains at least k balls;

(iii) for any $2 \leq i \leq h$, the number of balls with colour c_i that are contained in Q is l_i.

We call Q the *light* bin and all bins in V *heavy*. To assist with the analysis in some situations, we consider the following algorithm which clearly generates a probability space equivalent to $\mathcal{P}(V, \mathbf{M}, \mathbf{L}, k)$. We call this alternative the *allocation-partition* algorithm since it allocates before partitioning the balls. First, allocate D balls randomly into bins $\{Q\} \cup V$ with the restriction that Q contains exactly ℓ balls and each bin in V contains at least k balls. Then colour the balls u.a.r. with the following constraints:

(i) exactly $i m_i$ balls are coloured with c_i;

(ii) for each $i = 2, \dots, h$, the number of balls with colour c_i contained in Q is exactly l_i.

Finally, take u.a.r. a partition of the balls such that for each $i = 2, \dots, h$, all balls with colour c_i are partitioned into parts of size i.

To prove Theorem 2.2, we will convert the problem to a question about $\mathcal{P}(V, \mathbf{M}, \mathbf{L}, k+1)$, in particular the $(w, k+1)$-core of the hypergraph induced in the obvious way by the bins containing at least $k+1$ balls.

A deletion algorithm producing the k-core of a random multigraph was analysed in [6]. The differential equation method [27] was used to analyse the size and the number of hyperedges of the final k-core. The degree distribution of the k-core was shown to be a truncated multinomial. We now extend this deletion algorithm to find the $(w, k+1)$-core of H in $\mathcal{M}_{\bar{n},\bar{m},h}$ and $\mathcal{G}_{\bar{n},\bar{m},h}$. We describe the algorithm in the setting of representing multihypergraphs using bins for vertices, where each hyperedge x is a set $h(x)$ of $|x|$ balls. Initially let LV be the set of all light vertices/bins, and let $\overline{LV} = V(H) \setminus LV$ be the set of heavy vertices. A *light* ball is any ball contained in LV.

RanCore Algorithm to obtain the $(w, k+1)$-core
Input: an h-hypergraph H. Set $t := 0$.
While neither LV nor $\overline{LV}$ is empty,
$t := t + 1$;
Remove all empty bins;
U.a.r. choose a light ball u. Let x be the hyperedge that contains u and let v be the vertex that contains u;
If $|x| \geq h - w + 2$, update x with $x \setminus \{u\}$,
otherwise, remove this hyperedge x from the current hypergraph. If any vertex $v' \in \overline{LV}$ becomes light, move v' to LV together with all balls in it;
If LV is empty, ouput the remaining hypergraph, otherwise, output the empty graph.

We will prove Proposition 2.1 and Theorem 2.2 by analysing the RanCore algorithm using the partition-allocation model and the allocation-partition algorithm which generates $\mathcal{P}(V, \mathbf{M}, \mathbf{0}, k+1)$. Define

$$f_k(\mu) = \sum_{i \geq k} e^{-\mu} \cdot \frac{\mu^i}{i!} = 1 - \sum_{i=0}^{k-1} e^{-\mu} \cdot \frac{\mu^i}{i!}, \tag{3.1}$$

for any integer $k \geq 0$. By convention, define $f_k(\mu) = 1$ for any $k < 0$. Let $Z_{(\geq k)}$ be a truncated Poisson random variable with parameter λ defined as follows.

$$\mathbf{P}(Z_{(\geq k)} = j) = \frac{e^{-\lambda}}{f_k(\lambda)} \cdot \frac{\lambda^j}{j!}, \quad \text{for any } j \geq k. \tag{3.2}$$

Note that it follows that $\mathbf{P}(Z_{(\geq k)} = j) = 0$ whenever $j < k$. The following proposition will be used in the proof of Theorem 2.2 and in Section 5.

Proposition 3.2 *For $\mu \geq k+2$, there exists a unique real λ satisfying $\lambda f_k(\lambda) = \mu f_{k+1}(\lambda)$. Moreover, $\mu \geq \lambda$, and if $\mu \geq ck$ for a fixed $c > 1$, then $\mu - \lambda \to 0$ and $f_k(\lambda) \to 1$ as $k \to \infty$.*

Proof. Since $y f_k(y)/f_{k+1}(y)$ is monotonic in the domain $y > 0$, as shown in [24, Lemma 1], there exist a unique $\lambda > 0$ that satisfies $\lambda f_k(\lambda) = \mu f_{k+1}$ as long as $\mu \geq \inf_{y>0}\{y f_k(y)/f_{k+1}(y)\}$. Clearly $f_k(1)/f_{k+1}(1) < k+2$, so $\mu \geq k+2$ suffices. Since $f_k(\lambda) \geq f_{k+1}(\lambda)$ for all $k \geq -1$ by the definition of the function $f_k(x)$ in (3.1), it follows directly that $\mu \geq \lambda$. For $k \to \infty$, we use well known simple bounds on tails of the Poisson distribution. Set $c = 1 + 3\alpha$, where $\alpha > 0$. If $\lambda < k + \alpha k$, then

$$\begin{aligned}\lambda f_k(\lambda) &\sim \sum_{j=k}^{\lfloor k+2\alpha k\rfloor} e^{-\lambda}\frac{\lambda^{j+1}}{j!} < (1+o(1))(k+2\alpha k)\sum_{j=k}^{\lfloor k+2\alpha k\rfloor} e^{-\lambda}\frac{\lambda^{j+1}}{(j+1)!} \\ &< ckf_{k+1}(\lambda) < \mu f_{k+1}(\lambda),\end{aligned}$$

a contradiction. So $\lambda \geq k + \alpha k$, whence $f_k(\lambda)$ and $f_{k+1}(\lambda)$ are both $1 + o(1/\lambda)$ and so $\lambda - \mu = o(1)$. ∎

The following is essentially [28, Lemma 4.2].

Lemma 3.3 *Let $c > 0$, δ be constants. Let $(Y_t)_{t\geq 1}$ be independent random variables such that $|Y_t| \leq c$ always and $\mathbf{E}Y_t \leq \delta$ for all $t \geq 1$. Let $X_0 = 0$ and $X_t = \sum_{i\leq t} Y_i$ for all $t \geq 1$. Then for any $\epsilon > 0$, a.a.s. $X_n \leq \delta n + \epsilon|\delta|n$. More precisely, $\mathbf{P}(X_n \geq \delta n + \epsilon|\delta|n) \leq \exp(-\Omega(\epsilon^2 n))$.*

Proof of Proposition 2.1. Consider an element $H \in \mathcal{M}_{\bar{n},\mathbf{m}}$ arising from $P \in \mathcal{P}_{\bar{n},\mathbf{m}}$, where $\mathbf{m} = (0, \ldots, 0, \bar{m})$ with $\bar{m}$ corresponding to the value of the coordinate m_h. If we merge all the bins of P containing k or fewer balls into one bin Q, we obtain in an obvious way an element $P' \in \mathcal{P}(V, \mathbf{M}, \mathbf{L}, k + 1)$ for an appropriate sequence $\mathbf{L} = (0, 0, \ldots, L_0)$ where L_0 is the total degree of light vertices (vertices with degree at most k) in H. Given the parameters $(V, \mathbf{M}, \mathbf{L}, k + 1)$, the number of P that will produce P' is independent of P'. It follows that, conditional on the total degree of the light vertices in H, this generates $\mathcal{P}(V, \mathbf{M}, \mathbf{L}, k+1)$ with the correct distribution. Moreover, the hypergraph induced by the vertices of H of degree at least $k + 1$ is also induced in the obvious way by the heavy bins of P'. Hence, it suffices to study the $(w, k + 1)$-core of this hypergraph, conditional upon any feasible $\mathbf{L}$. Because of the correspondence between $\mathcal{P}(V, \mathbf{M}, \mathbf{L}, k + 1)$ and the random multihypergraphs, we sometimes call bins in V vertices and the *degree sequence* of V denotes the sequence of numbers of balls in bins in V.

To this end, we adapt the RanCore algorithm in the obvious way to be run on $\mathcal{P}(V, \mathbf{M}, \mathbf{L}, k+1)$, as follows. In each step t, the algorithm removes a ball, denoted by u, u.a.r. chosen from all balls in Q. If the colour of u is c_{h-j} for $i < w - 1$, the algorithm recolours the balls in the same part as u with the new colour c_{h-j-1}. If the colour of u is c_{h-w+1}, the algorithm removes all balls contained in the same part as u, and if any heavy bin becomes light (i.e. the number of balls contained in it becomes at most k) because of the removal of balls, the bin is removed and the balls remaining in it are put into Q. This clearly treats the heavy bins of P' in a corresponding way to RanCore treating the heavy vertices of H. Thus, the modified RanCore stops with a final partition-allocation that corresponds to the $(w, k + 1)$-core of H, and this is what we will analyse.

For easier reference, let g_t denote the random partition-allocation derived after t steps of this process. Let V_t denote its set of heavy bins and let $\mathbf{M}_t$ denote the class of sets $\{M_{t,h-w+1}, \ldots, M_{t,h}\}$ such that $M_{t,h-j}$ denotes the set of partitioned balls with colour c_{h-j} in g_t for $0 \leq j \leq w - 1$. Let $m_{t,h-j} = |M_{t,h-j}|/(h - j)$ and $\mathbf{m} = (m_{t,h-w+1}, \ldots, m_{t,h})$. Let $L_{t,h-j}$ denote the number of balls with colour c_{h-j} in Q and let $\mathbf{L} = (L_{t,h-w+1}, \ldots, L_{t,h})$. Let $L_t = \sum_{j=0}^{w-1} L_{t,h-j}$. Initially, $g_0 = P'$, $V_0 = V$ etc.

There is a straightforward way to see, by induction on t, that the partition-allocation g_t, conditional on V_t, $\mathbf{M}_t$ and $\mathbf{L}_t$, is distributed as $\mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1)$. We have already noted that this is true for $t = 0$. For the inductive step, it suffices to note that that for any $t \geq 0$, conditional on the values of $V_t, \mathbf{M}_t, \mathbf{L}_t$ for g_t, the probability that g_{t+1} is any particular member g' of $V_{t+1}, \mathbf{M}_{t+1}, \mathbf{L}_{t+1}$ does not depend on g'. This is because of three facts. Firstly, g_t is uniform conditional on the parameters at step t. Secondly, the change in the parameters determines which type of step the algorithm is taking (e.g. if a heavy bin becomes light). Thirdly, each possibility for g_{t+1} is reachable from the same number of g_t and, given the type of step occurring, each such transition has the same probability of occurring as step $t+1$.

Furthermore, it is easy to see that the degree distribution of the $(w, k+1)$-core, conditional on the number of hyperedges of each size, is truncated multinomial. This is because, for any V and $\mathbf{M}$, the allocation-partition algorithm which generates $\mathcal{P}(V, \mathbf{M}, \mathbf{0}, k+1)$ produces a truncated multinomial distribution for the degrees of vertices in V. ∎

The proof of Theorem 2.2 uses the differential equation method (d.e. method). In particular, we use the following special case of [28, Theorem 6.1]. For each $n > 0$ let a sequence of random vectors $(Y_t^{(1)}, \ldots, Y_t^{(l)})_{0 \leq t \leq m}$ be defined on a probability space Ω_n. (We suppress the notation n.) Let U_t denote the history of the process up to step t.

Theorem 3.4 *Suppose that there exists $C > 0$ such that for each i, $Y^{(i)} < Cn$ always. Let $\widehat{\mathcal{D}} \subset \mathbb{R}^{l+1}$ and let the stopping time T be the minimum t such that $(t/n, Y_t^{(1)}/n, \ldots, Y_t^{(l)}) \notin \widehat{\mathcal{D}}$. Assume further that the following three hypotheses are satisfied.*

(a) (Boundedness hypothesis.) There exists a constant $C' > 0$ such that for all $0 \leq t \leq \min\{m, T\}$, $|Y_{t+1} - Y_t| < C'$ always;

(b) (Trend hypothesis.) There exists functions f_i for all $1 \leq i \leq l$ such that for all $0 \leq t \leq \min\{m, T\}$ and all $1 \leq i \leq l$,

$$\mathbf{E}(Y_{t+1}^{(i)} - Y_t^{(i)} \mid U_t) = f_i(t/n, Y_t^{(1)}/n, \ldots, Y_t^{(l)}/n) + o(1);$$

(c) (Lipschitz hypothesis.) For every $1 \leq i \leq l$, the functions f_i are Lipschitz continuous in all their variables on a bounded connected open set $\mathcal{D}$ where $\mathcal{D}$ contains the intersection of $(t, z^{(1)}, \ldots, z^{(l)} : t \geq 0)$ with some neighbourhood of $(0, z^{(1)}, \ldots, z^{(l)} : \mathbf{P}(Y_0^{(i)} = z^{(l)}n, 1 \leq i \leq l) \neq 0$ for some n).

Then the following conclusions hold.

(a) For any $(0, \hat{z}^{(1)}, \ldots, \hat{z}^{(l)}) \in \mathcal{D}$, the differential equation system

$$\frac{d\ z_i}{d\ s} = f_i(s, z_1, \ldots, z_l), \quad i = 1, \ldots, l$$

has a unique solution in $\mathcal{D}$ for $z_l : I\!R \to I\!R$ with the initial conditions

$$z_i(0) = \hat{z}^{(i)}, \quad i = 1, \ldots, l,$$

where the solution is extended arbitrarily close to the boundary of $\mathcal{D}$.

(b) A.a.s.

$$Y_t^{(i)} = nz_i(t/n) + o(n)$$

uniformly for all $0 \leq t \leq \min\{\sigma n, T\}$, where σ is the supremum of all x such that the solution $(z^{(i)}(x))_{1\leq i\leq l}$ to the differential equation system lies inside the domain $\mathcal{D}$.

Our usage of a.a.s. in conjunction with other asymptotic notation such as $o()$ conforms to the conventions in [29]. For more details of the method and proofs, readers can refer to [27, Theorem 1], [28, Theorem 5.1] and [28, Theorem 6.1]. In our case, Ω_n is the probability space of sequences of random partition-allocations generated by running the RanCore algorithm on graphs with n vertices, where t refers to the t-th step of the algorithm and the Y_t are variables defined during the algorithm.)

The idea of the proof of Theorem 2.2 is, roughly speaking, as follows. We use the d.e. method to analyse the asymptotic values of random variables defined on the random process generated by the RanCore algorithm. The difficulty arises from the fact that the natural functions f_i for our application are not Lipschitz continuous at $x = 0$. To avoid this, we artificially modify the f_i in a neighbourhood of the problem point, and show that the solution to the new differential equation system coincides with the original inside a domain D_0 which contains all points relevant to the random process. Theorem 3.4 then applies to show that the asymptotic values of the random variables inside D_0 are approximated by the solution of the system, we analyse the random variables when they leave D_0. We show that provided k is sufficiently large, the algorithm then terminates quickly, which allows us to estimate the size and density of $\widehat{H}$.

Proof of Theorem 2.2. It was shown in the proof of Proposition 2.1 that for every t, conditional on the values of V_t, $\mathbf{M}_t$ and $\mathbf{L}_t$, the partition-allocation g_t is distributed as $\mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1)$. After step t of the RanCore algorithm, define (or recall) the following random variables:

B_t	total number of balls remaining
$B_{t,h-j}$	number of balls coloured c_{h-j}
$A_{t,i}$	number of bins containing exactly i balls
A_t	$A_{t,k+1}$ (number of bins containing exactly $k+1$ balls)
L_t	number of light balls
$L_{t,h-j}$	number of light balls that are coloured c_{h-j}
$H_{t,h-j}$	number of balls contained in heavy bins that are coloured c_{h-j}
HV_t	$\lvert V_t\rvert$, number of heavy bins

and note that $H_{t,h-j} = B_{t,h-j} - L_{t,h-j}$.

Recall that n and m_{h-j} denote the number of vertices and the number of hyperedges of size $h-j$ in $\widehat{H}$, the $(w, k+1)$-core of H, and $\bar{\mu}$ denotes its average degree. We show that if $\bar{\mu} \geq ck$ for some $c > 1$ and k is sufficiently large, then a.a.s. $n \sim \alpha\bar{n}$ and $m_{h-j} \sim \beta_{h-j}\bar{n}$ for some constants $\alpha > 0$, $\beta_{h-j} > 0$ which are determined by the solution of the differential equation system given below, on a domain D_0 defined below (3.14). In particular, we will show that

$$\alpha = z_{HV}(x^*), \quad \beta_{h-j} = z_{H,h-j}(x^*)/(h-j), \tag{3.3}$$

where x^* is the smallest positive root of $z_L(x) = 0$.

The d.e. method relies on a relation between solutions of a differential equation system and the random variables of the process under consideration. We will use subscripts of the real valued functions to indicate their corresponding random variables. For instance, the real function $z_{L,h-j}(x)$ is associated with the random variable $L_{t,h-j}$. The differential equation system is as follows.

$$\begin{aligned}
z'_{L,h-j}(x) &= \frac{z_{L,h-j}}{z_L}\left(-1 - \frac{(h-j-1)z_{L,h-j}}{z_{B,h-j}}\right) \\
&\quad + \frac{z_{L,h-w+1}}{z_L}\left(\frac{(h-w)z_{H,h-w+1}}{z_{B,h-w+1}} \cdot \frac{(k+1)z_A}{z_B - z_L} \cdot k \cdot \frac{z_{H,h-j}}{z_B - z_L}\right) \\
&\quad + \frac{z_{L,h-j+1}}{z_L}\frac{(h-j)z_{L,h-j+1}}{z_{B,h-j+1}}, \quad j = 1, \ldots, w-1, &(3.4) \\
z'_{H,h-j}(x) &= \frac{z_{L,h-j}}{z_L}\left(-\frac{(h-j-1)z_{H,h-j}}{z_{B,h-j}}\right) \\
&\quad - \frac{z_{L,h-w+1}}{z_L}\left(\frac{(h-w)z_{H,h-w+1}}{z_{B,h-w+1}} \cdot \frac{(k+1)z_A}{z_B - z_L} \cdot k \cdot \frac{z_{H,h-j}}{z_B - z_L}\right) \\
&\quad + \frac{z_{L,h-j+1}}{z_L}\frac{(h-j)z_{H,h-j+1}}{z_{B,h-j+1}}, \quad j = 1, \ldots, w-1, &(3.5) \\
z'_L(x) &= -1 + \frac{z_{L,h-w+1}}{z_L}\left(-\frac{(h-w)z_{L,h-w+1}}{z_{B,h-w+1}} + (h-w)k \cdot \frac{z_{H,h-w+1}}{z_{B,h-w+1}} \cdot \frac{(k+1)z_A}{z_B - z_L}\right) &(3.6) \\
z'_B(x) &= -1 - \frac{(h-w)z_{L,h-w+1}}{z_L} &(3.7) \\
z'_{HV}(x) &= -\frac{z_{L,h-w+1}}{z_L}\frac{(h-w)z_{H,h-w+1}}{z_{B,h-w+1}} \cdot \frac{(k+1)z_A}{z_B - z_L} &(3.8) \\
\lambda'(x) &= \frac{((z'_B - z'_L)z_{HV} - (z_B - z_L)z'_{HV})f_{k+1}(\lambda)}{z_{HV}^2(f_k(\lambda) + \lambda e^{-\lambda} \cdot \frac{\lambda^{k-1}}{(k-1)!} - \frac{z_B - z_L}{z_{HV}} \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!})} &(3.9) \\
z_{L,h}(x) &= z_L(x) - \sum_{i=1}^{w-1} z_{L,h-j}(x), \quad z_{H,h}(x) = z_B(x) - z_L(x) - \sum_{i=1}^{w-1} z_{H,h-j}(x), &(3.10) \\
z_{B,h-j}(x) &= z_{L,h-j}(x) + z_{H,h-j}(x), \quad \text{for every } 0 \le j \le w-1, &(3.11) \\
z_A(x) &= \frac{\lambda(x)^{k+1}}{e^{\lambda(x)}(k+1)!f_{k+1}(\lambda(x))} z_{HV}(x), &(3.12)
\end{aligned}$$

where $f_k(\lambda)$ was defined in (3.1). The initial conditions are

$$\begin{aligned}
z_B(0) &= \bar{\mu}, \ z_{L,h-j}(0) = 0, \ z_{H,h-j}(0) = 0, \ \text{for all } 1 \le j \le w-1, &(3.13) \\
z_L(0) &= \bar{\mu}(1 - f_k(\bar{\mu})), \ z_{HV}(0) = 1 - \exp(-\bar{\mu}) \sum_{i=0}^{k} \bar{\mu}^i/i!, \ \lambda(0) = \bar{\mu}. &(3.14)
\end{aligned}$$

Let D_0 be the domain which contains all points such that $x \in \mathbb{R}$, $0 \leq z_{L,h-j} \leq z_{B,h-j}$, $z_{L,h-j} \leq z_L$ for all $0 \leq j \leq w-1$, $z_L > 0$, $z_B - z_L > 0$, $z_{HV} > 0$ and $(z_B - z_L)/z_{HV} > k+2$. We will call the right hand sides of (3.4–3.9) the *derivative functions*, and at present we regard them to be only defined in D_0. It is straightforward to check that for any point $\mathbf{z}^* \in D_0$ such that $z_{L,h-j} = z_{B,h-j} = 0$, the functions specified in the right hand sides of (3.4–3.8) tend to 0 when $\mathbf{z}$ approaches $\mathbf{z}^*$ from the interior of D_0. For example, note that the term

$$\frac{z_{L,h-j}}{z_L} \cdot (h-j-1) \cdot \frac{z_{L,h-j}}{z_{B,h-j}}$$

on the right hand side of (3.4) is bounded above by $(h-j-1)z_{L,h-j}/z_L$ since $|z_{L,h-j}/z_{B,h-j}| \leq 1$ when $\mathbf{z} \in D_0$. Hence, it tends to 0 if $\mathbf{z} \to \mathbf{z}^*$. The same applies to similar terms in (3.4)–(3.8). As part of our definition of the differential equation system (3.4)–(3.14), we now declare the values of these terms at such points $\mathbf{z}^*$ to be 0.

In applying Theorem 3.4, the variable x will be associated with $t/\bar{n}$. As mentioned above, the variable $z_{L,h-j}(x)$ is associated with the variable $L_{t,h-j}/\bar{n}$, which we call the *scaled version* of the random variable $L_{t,h-j}$. We do the same for the other random variables, and call $t/\bar{n}$ the scaled version of t.

There are two kinds of problems with the Lipschitz property required in Theorem 3.4 (c). The first is caused by terms in the equations with denominators z_L or $z_B - z_L$ appearing in the derivative functions, which are potentially 0, causing singularities. These are relatively easy to take care of since they do not become small until near the end of the process. For any fixed constant $\epsilon > 0$, define $D_0(\epsilon)$ to be the connected subset of D_0 obtained by restricting to $\mathbf{z}$ such that $z_L > \epsilon$ and $z_B - z_L > \epsilon$. We will basically restrict consideration to points in $D_0(\epsilon)$. Let T be the (stopping) time that the vector of scaled random variables leaves $D_0(\epsilon)$. Let $t \wedge T$ denote $\min\{t, T\}$. The conclusion of Theorem 3.4 will give information on the scaled random variables up to the step when they reach the boundary of the domain $D_0(\epsilon)$. This gives us information about $(g_{t\wedge T})_{0\leq t\leq \tau}$, where g_t is the partition-allocation obtained after step t. At that point we will need some further observations to show that the process finishes soon afterwards.

The second type of problem comes from denominators containing $z_{B,h-j}$, which can be 0 even right at the start of the process. This poses a difficulty since the theorem requires the derivative functions to be Lipschitz in an open domain containing the starting point. To deal with this, we will, at an appropriate point below, extend the differential equations into a larger connected open domain $D \supset D_0$, and correspondingly extend $D_0(\epsilon)$ to $D(\epsilon)$. We will actually apply Theorem 3.4 with $\mathcal{D} = D(\epsilon)$ and $\widehat{\mathcal{D}} = D_0(\epsilon)$.

We first verify hypotheses (a) and (b) of Theorem 3.4, which are unrelated to the choice of $D(\epsilon)$. It is easy to see that the change of each random variable in every step of the algorithm is bounded. This is because in every step, the number of balls deleted (or recoloured, or moved from heavy bins to the light bin Q) is bounded. Thus, Theorem 3.4(a) clearly holds.

To verify hypothesis (b), we will need to show that the expected one-step change of each random variable, such as $L_{t,h-j}$, can be approximated to within $o(1)$ error by some function of the scaled variables. Replacing the scaled variables in these functions by their associated real variables will give the derivative functions in (3.4)–(3.8).

Let g_t be the partition allocation obtained after step t. At step $t+1$, a partition-allocation g_{t+1} is to be obtained by applying the RanCore algorithm to g_t. Let v be the ball randomly chosen by the algorithm from Q. Let $C(v)$ be the colour of v, so $C(v) = h - j$ for some j. If $j < w - 1$, the algorithm removes another $h - j - 1$ balls that are uniformly distributed among all balls with colour c_{h-j} since $g_t \in \mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1)$ as proved in Proposition 2.1. If $j = w - 1$, then the algorithm removes v together with another $h - w$ balls which are chosen u.a.r. from all balls of colour c_{h-w+1}. If the removal of the $h - w$ balls results in some heavy bins turning into light bins, these bins are removed and the balls remaining in these bins are put into Q.

Now we estimate the expected value of $L_{t+1,h-j} - L_{t,h-j}$ for any $1 \leq j \leq w - 1$ and for any $0 \leq t < \tau$ conditional on V_t, $\mathbf{M}_t$, $\mathbf{L}_t$ and the event $g_t \in \mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1)$. Given j, the probability that $C(v) = c_{h-j}$ is $L_{t,h-j}/L_t$. If $C(v) = c_{h-j}$, one ball of colour c_{h-j} contained in Q is removed, and another $h - j - 1$ balls of colour c_{h-j} are recoloured with c_{h-j-1} (or removed if $j = w - 1$). So the expected number of those balls that are contained in Q is

$$\frac{(h-j-1)L_{t,h-j}}{B_{t,h-j}}(1 + o(1)),$$

provided $B_{t,h-j} \geq \log n$ (say). Hence

$$\frac{L_{t,h-j}}{L_t}\left(-1 - \frac{(h-j-1)L_{t,h-j}}{B_{t,h-j}}\right) + o(1)$$

is the negative contribution to $\mathbf{E}(L_{t+1,h-j} - L_{t,h-j} \mid V_t, \mathbf{M}_t, \mathbf{L}_t, g_t \in \mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1))$. Note that we reach the same conclusion if $B_{t,h-j} < \log n$ because in that case

$$L_{t,h-j}/L_t \leq B_{t,h-j}/L_t < \log n/\epsilon n = o(1).$$

The positive contribution to $\mathbf{E}(L_{t+1,h-j} - L_{t,h-j} \mid V_t, \mathbf{M}_t, \mathbf{L}_t, g_t \in \mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1))$ comes from the following two cases.

Case 1: $C(v) = c_{h-w+1}$. Here, the algorithm removes v and another $h - w$ balls of colour c_{h-w+1}. $\mathbf{P}(C(v) = h - w + 1) = L_{t,h-w+1}/L_t$. We first note that, for any $2 \leq i \leq h - w$, the contribution from the case that i of the $h - w$ removed balls lie in a bin containing at most $k + i$ balls is at most $\big((k+i)/(B_t - L_t)\big)^{i-1} = o(1)$, since the definition of $\widehat{\mathcal{D}} = D_0(\epsilon)$ ensures that the denominator is at least ϵn for $t \leq T$.

It only remains to consider the contribution from the case that a ball in a bin containing exactly $k + 1$ balls is removed. For each ball removed, the probability that it is in a bin containing exactly $k + 1$ balls is

$$\frac{H_{t,h-w+1}}{B_{t,h-w+1}} \cdot \frac{(k+1)A_{t,k+1}}{B_t - L_t} + o(1).$$

The removal of such a ball causes the bin to become light. Since $g_t \in \mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1)$, the balls of each colour are uniformly distributed among all balls in the heavy bins, and thus the expected number of balls of colour c_{h-j}, for $0 \leq j \leq w - 1$, among the remaining k balls in the bin is

$$k \cdot \frac{H_{t,h-j}}{B_t - L_t} + o(1).$$

In total, $h-w$ balls of colour c_{h-w+1} are removed, other than v. Hence the expected contribution to $\mathbf{E}(L_{t+1,h-j} - L_{t,h-j} \mid V_t, \mathbf{M}_t, \mathbf{L}_t, g_t \in \mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1))$ is

$$(h-w) \cdot \frac{L_{t,h-w+1}}{L_t} \cdot \frac{H_{t,h-w+1}}{B_{t,h-w+1}} \cdot \frac{(k+1)A_{t,k+1}}{B_t - L_t} \cdot k \cdot \frac{H_{t,h-j}}{B_t - L_t} + o(1).$$

Case 2: $C(v) = c_{h-j+1}$. The algorithm removes v, chooses another $h-j$ balls u.a.r. from those of colour c_{h-j+1}, and recolours them with c_{h-j}. Since $\mathbf{P}(C(v) = c_{h-j+1}) = L_{t,h-j+1}/L_t$, conditional on $C(v) = c_{h-j+1}$, the expected number of balls of colour c_{h-j+1} that are in the light bins and are recoloured is

$$(h-j) \cdot \frac{L_{t,h-j+1}}{B_{t,h-j+1}} + o(1),$$

provided $B_{t,h-j+1} \geq \log n$. Hence the positive contribution to $\mathbf{E}(L_{t+1,h-j} - L_{t,h-j} \mid V_t, \mathbf{M}_t, \mathbf{L}_t, g_t \in \mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1))$ is

$$\frac{L_{t,h-j+1}}{L_t} \cdot (h-j) \cdot \frac{L_{t,h-j+1}}{B_{t,h-j+1}} + o(1)$$

in this case. The same conclusion holds when $B_{t,h-j+1} < \log n$ for the same reason as discussed before. Therefore

$$\begin{aligned}
&\mathbf{E}(L_{t+1,h-j} - L_{t,h-j} \mid V_t, \mathbf{M}_t, \mathbf{L}_t, g_t \in \mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1)) \\
&= \frac{L_{t,h-j}}{L_t}\left(-1 - \frac{(h-j-1)L_{t,h-j}}{B_{t,h-j}}\right) + \frac{L_{t,h-j+1}}{L_t} \cdot \frac{(h-j)L_{t,h-j+1}}{B_{t,h-j+1}} \\
&\quad + \frac{L_{t,h-w+1}}{L_t}\left(\frac{(h-w)H_{t,h-w+1}}{B_{t,h-w+1}} \cdot \frac{(k+1)A_{t,k+1}}{B_t - L_t} \cdot k \cdot \frac{H_{t,h-j}}{B_t - L_t}\right) + o(1),
\end{aligned} \tag{3.15}$$

for $j = 1, \ldots, w-1$. Replacing the random variables in the right hand side of (3.15) by their associated real variables (noting that the scaling cancels out) gives the right hand side of (3.4). Using a similar approach to computing the expected changes of $H_{t,h-j}$, B_t, D_t, HV_t, conditional on V_t, $\mathbf{M}_t$, $\mathbf{L}_t$ and the event $g_t \in \mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1)$, we easily obtain the derivative functions in (3.5)–(3.8). The equations

$$\begin{aligned}
L_{t,h} &= L_t - \sum_{i=1}^{w-1} L_{t,h-j}, \quad H_{t,h} = B_t - L_t - \sum_{i=1}^{w-1} H_{t,h-j}, \\
B_{t,h-j} &= L_{t,h-j} + H_{t,h-j}, \quad \text{for every } h-w+1 \leq j \leq h
\end{aligned}$$

are obvious and lead to (3.10) and (3.11).

Let μ_t denote $(B_t - L_t)/HV_t$, the average degree of heavy vertices after step t. Correspondingly we define a function $\mu(x)$ associated with the random variable μ_t to be

$$\mu(x) = (z_B(x) - z_L(x))/z_{HV}(x). \tag{3.16}$$

Then by Proposition 3.2, we may define $\lambda(x)$ by

$$\lambda(x) f_k(\lambda(x)) = \mu(x) f_{k+1}(\lambda(x)) \tag{3.17}$$

provided that $\mu(x) > k+2$, which is guaranteed inside $D_0(\epsilon)$. Let $\lambda_t = \lambda(t/\bar{n})$, so that λ_t is the unique positive root of

$$\frac{\lambda_t f_k(\lambda_t)}{f_{k+1}(\lambda_t)} - \mu_t = 0. \tag{3.18}$$

Since $g_t \in \mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1)$ for every t, by considering the allocation-partition algorithm that generates $\mathcal{P}(V_t, \mathbf{M}_t, \mathbf{L}_t, k+1)$, the degree sequence of the heavy vertices has the truncated multinomial distribution. Hence, by [6, Lemma 1],

$$A_{t,k+1} \sim \frac{e^{-\lambda_t}\lambda_t^{k+1}}{(k+1)! f_{k+1}(\lambda_t)} HV_t, \tag{3.19}$$

where λ_t satisfies (3.18). This gives (3.14).

Now (3.9), which gives the derivative of $\lambda(x)$, follows by taking the derivative of both sides of (3.17),

$$\lambda'(x) f_k(\lambda(x)) + \lambda(x) \frac{df_k(\lambda)}{d\lambda}\bigg|_{\lambda=\lambda(x)} \lambda'(x) = \mu'(x) f_{k+1}(\lambda(x)) + \mu(x) \frac{df_{k+1}(\lambda)}{d\lambda}\bigg|_{\lambda=\lambda(x)} \lambda'(x),$$

where, by the definitions of $f_k(\lambda)$ in (3.1) and $\mu(x)$ in (3.16),

$$\frac{df_k(\lambda)}{d\lambda} = e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} \quad \text{and} \quad \mu'(x) = \frac{(z_B' - z_L')z_{HV} - (z_B - z_L)z_{HV}'}{z_{HV}^2}.$$

Now we justify hypothesis (c). We will first extend the derivative functions (which, up until this point, we restricted to D_0) into a larger domain D, which defines an extended d.e. system, and show that these extended functions are continuous and Lipschitz inside an open domain $D(\epsilon)$ extended from $D_0(\epsilon)$. Later we will show that the solution of the extended d.e. system for $0 \le x \le T/\bar{n}$, with the same initial conditions as the original system, is contained inside the domain $D_0(\epsilon)$ and is thus the solution to the original d.e. system.

We begin with the domain $D_0(\epsilon)$, which was defined by restricting the points in D_0 to $z_L > \epsilon$ and $z_B - z_L > \epsilon$. Recalling our treatment of the possible singularity $z_{L,h-j} = z_{B,h-j} = 0$ just after (3.14), each derivative function is continuous in $D_0(\epsilon)$. The only potential problems for the Lipschitz property are the constant multiples of the function

$$f(z_{L,h-j}, z_L, z_{B,h-j}) = \frac{z_{L,h-j}}{z_L} \cdot \frac{z_{L,h-j}}{z_{B,h-j}}. \tag{3.20}$$

However, recalling that $z_L > \epsilon$, $0 \le z_{L,h-j} \le z_{B,h-j}$ and $z_{L,h-j} \le z_L$ in $D_0(\epsilon)$, we have that the partial derivatives of $f(z_{L,h-j}, z_L, z_{B,h-j})$ with respect to $z_{L,h-j}$, z_L and $z_{B,h-j}$ are all $O(1/\epsilon)$, from which it follows that the derivative functions are Lipschitz in $D_0(\epsilon)$.

Let $\mathbf{z}_0$ denote the initial condition vector given by (3.13) and (3.14): $x = 0$, $z_L = \bar{\mu}(1 - f_k(\bar{\mu}))$, $z_B = \bar{\mu}$, $z_{HV} = 1 - \exp(-\bar{\mu})\sum_{i=0}^{k} \bar{\mu}^i/i!$, $z_{L,h-j} = z_{B,h-j} = 0$ for all $1 \le j \le w-1$. Note that $\mathbf{z}_0$ lies on the boundary of both D_0 and $D_0(\epsilon)$. Define $D := \{(x, z_{L,h-w+1}, \ldots, z_{L,h-1}, z_{B,h-w+1}, \ldots, z_{B,h-1}, z_L, z_B, z_{HV}) : z_L > 0, z_B - z_L > 0, z_B - z_L > (k+2)z_{HV}\}$, and let $D(\epsilon)$ be the domain obtained by restricting points in D to those with

$z_L > \epsilon$ and $z_B - z_L > \epsilon$. Thus $D(\epsilon)$ is the corresponding extension of $D_0(\epsilon)$. Clearly $\mathbf{z}_0$ is an interior point in D and $D(\epsilon)$. To extend the derivative functions to D, it is enough to extend the function f in (3.20). Define

$$f^*(z_{L,h-j}, z_{B,h-j}, z_L) = \begin{cases} f(z_{L,h-j}, z_{B,h-j}, z_L) & \text{if } 0 \le z_{L,h-j} \le z_{B,h-j}, z_{B,h-j} > 0 \\ 0 & \text{if } z_{L,h-j} = z_{B,h-j} = 0, \\ z_{B,h-j}/z_L & \text{if } z_{L,h-j} > z_{B,h-j} \ge 0 \\ f(|z_{L,h-j}|, |z_{B,h-j}|, z_L) & \text{otherwise.} \end{cases} \tag{3.21}$$

We have already shown that f is Lipschitz continuous on $D_0(\epsilon)$, which is the first case of (3.21). Since $z_L > \epsilon$ and $z_B - z_L > \epsilon$ in $D(\epsilon)$, f^* is Lipschitz continuous on $D(\epsilon)$. Hence, if we modify the differential equation system (3.4)–(3.14) by replacing each expression equivalent to f by f^*, we obtain derivative functions that are Lipschitz continuous in the open domain $D(\epsilon)$. Thus, hypothesis (c) holds for this system, which we call the *extended differential equation system*.

We may now apply Theorem 3.4, to deduce that a.a.s. uniformly for every $0 \le t \le T$, $L_t = \bar{n} z_L(t/\bar{n}) + o(\bar{n})$, and the same applies to all the other random variables under consideration. We claim that the stopping time T coincides with the time at which $L_t/\bar{n}$ or $(B_t - L_t)/\bar{n}$ decreases to ϵ. This follows by the following two observations, whose verifications are only sketched here since they require straightforward analysis. (See [15, pp. 86,87] for details.)

(i) The solution of the extended differential equation system is interior to $D_0(\epsilon)$ for all sufficiently small $x > 0$. For instance, all functions taking the value 0 at $x = 0$ have positive derivatives for sufficiently small $x > 0$. Thus, these functions become positive for any sufficiently small x and thus the solution is inside $D_0(\epsilon)$.)

(ii) Once the solution is interior to $D_0(\epsilon)$, the only boundaries of $D_0(\epsilon)$ it can reach are $z_L = \epsilon$, $z_B - z_L = \epsilon$, $z_{HV} = 0$ and $(z_B - z_L)/z_{HV} = k + 2$. The other boundaries of this domain are $z_{L,h-j} = 0$, $z_{L,h-j} = z_{B,h-j}$ (i.e. $z_{H,h-j} = 0$), and $z_{L,h-j} = z_L$ for any $j \ge 0$. For example, it cannot reach $z_{L,h-j} = 0$ because the only negative contribution to the derivative of $z_{L,h-j}$ is proportional to $z_{L,h-j}$ itself. In view of this, $z_{L,h-j} < z_L$ for any $0 \le j \le w - 1$ and the last-listed boundary cannot be reached.

Let $x(\epsilon)$ be the smallest value of x such that $z_L(x) = \epsilon$, $z_B(x) - z_L(x) = \epsilon$, $z_{HV}(x) = 0$ or $z_B(x) - z_L(x) = (k+2) z_{HV}(x)$, i.e., $\mu = k + 2$, considering the definition (3.16). Then the solution to the extended differential equation system for all $0 \le x \le x(\epsilon)$ is also the solution to the original differential equation system. Let x^* be the smallest real number such that $z_L(x^*) = 0$, $z_B(x^*) - z_L(x^*) = 0$, $z_{HV} = 0$ or $\mu = (k+2)$. Then the solution of the original differential equation system can be extended arbitrarily close to x^*.

By the theorem's hypothesis, $\bar{\mu} \ge ck$ for some $c > 1$. We next show that for sufficiently large k (depending on the value of c), the function $z_L(x)$ reaches 0 before $z_B(x) - z_L(x)$ or $z_{HV}(x)$ reach 0 or μ reaches $k + 2$, and we also provide an upper bound of the value of x^*. Let $z_H(x) = z_B(x) - z_L(x)$. Clearly $z_L(x) = \sum_{i=0}^{w-1} z_{L,h-j}(x)$ and $z_H(x) = \sum_{i=0}^{w-1} z_{H,h-j}(x)$. So (3.6), (3.7) and (3.8) immediately lead to

$$z'_L(x) \le -1 + \frac{hk(k+1)z_A}{z_H}, \quad z'_H \ge -h - \frac{hk(k+1)z_A}{z_H}, \quad z'_{HV}(x) \ge -\frac{h(k+1)z_A}{z_H}. \tag{3.22}$$

Let $\delta = (1 - f_k(\bar{\mu}))\bar{\mu}$. Then the initial conditions give $z_L(0) = \delta$ and $z_H(0) = \bar{\mu} - \delta$. Since $\bar{\mu} \geq ck$ for some $c > 1$, $\delta = \exp(-\Omega_c(k))$. By Proposition 3.2, we may assume that as long as $\mu(x) \geq c'k$ for some $c' > 1$ and k sufficiently large, $\lambda(x)$, is well defined by (3.17), and $|\mu(x) - \lambda(x)| \leq 1$, which implies that $z_A(x)/z_H(x) = \exp(-\Omega_{c'}(k))$ by (3.19). We next observe that $\mu(0) \geq \lambda(0) = \bar{\mu}$ by (3.14) and Proposition 3.2. Let $[0, x_0]$ be an interval such that $\mu(x) \geq \bar{\mu} - 4h$ for all $0 \leq x \leq x_0$. Certainly $\mu(x) \geq c'k$ for some $c' > 1$ for all $0 \leq x \leq x_0$. We may choose k sufficiently large (depending only on the value of c') that $\delta \leq 1$ and for all $0 \leq x \leq x_0$ we have $|\lambda(x) - \mu(x)| \leq 1$, $hk(k+1)z_A(x)/z_H(x) \leq 1/2$ and $h(k+1)z_A(x)/z_H(x) \leq 1/8$. Then for all $0 \leq x < x_0$

$$z_L'(x) \leq -1/2, \quad 0 \geq z_H'(x) \geq -h - 1/2, \quad z_{HV}'(x) \geq -1/8. \tag{3.23}$$

Note that $z_{HV}(0) \leq 1$, and $z_{HV}'(x) < 0$ from (3.8). Thus $\mu(x) = z_H(x)/z_{HV}(x) \geq z_H(x)$ for any $0 \leq x < x^*$. Hence, provided $z_H(x) \geq \bar{\mu} - 4h$, we have $\mu(x) \geq \bar{\mu} - 4h$ and so the inequalities (3.23) hold. Then $z_H(x) \geq z_H(0) + x\left(-h - \frac{1}{2}\right) = \bar{\mu} - \delta + x(-h - \frac{1}{2}) > \bar{\mu} - 4h$ provided $x \leq 3\delta$ say, since δ is arbitrarily small for large k. It follows that $\mu(x) \geq z_H(x) \geq \bar{\mu} - 4h$ for $x \leq \min\{x^*, 3\delta\}$. Thus we may choose $x_0 \geq \min\{x^*, 3\delta\}$, and so (3.23) implies, for any $0 \leq x < \min\{x^*, 3\delta\}$, that

$$z_{HV}(x) \geq z_{HV}(0) - \frac{3\delta}{8} > 0, \quad z_L(x) \leq \delta - \frac{x}{2}. \tag{3.24}$$

So $x^* < 3\delta$ and $z_{HV}(x^*) > 0$, since otherwise $3\delta \leq x^*$ and $z_L(3\delta) \leq \delta - 3\delta/2 < 0$, contradicting the definition of x^*. Combining this with $\mu(x) \geq z_H(x) \geq \bar{\mu} - 4h$, which is greater than $k+2$ for sufficiently large k, we conclude that $z_L(x)$ reaches 0 before $z_H(x)$ or $z_{HV}(x)$ reaches 0 and before $\mu(x)$ reaches $k+2$ (in fact, before $\mu(x)$ reaches $\bar{\mu} - 4h$) and $x^* \leq 3\delta$. We also have that $z_L'(x) \leq -1/2$ for all $x < x^*$.

For notational convenience, define the following limits from below (which we know to exist from the above bounds on the functions and their derivatives):

$$z_{H,h-j}(x^*) := \lim_{x \to (x^*)^-} z_{H,h-j}(x) \text{ and } z_{HV}(x^*) := \lim_{x \to (x^*)^-} z_{HV}(x). \tag{3.25}$$

Note that this definition yields continuous functions $z_{H,h-j}(x)$ and $z_{HV}(x)$ on the closed interval $[0, x^*]$.

Given any sufficiently small $\epsilon > 0$, let $x(\epsilon)$ be the root of $z_L(x) = \epsilon$ and let $t(\epsilon) = \lfloor x(\epsilon)\bar{n} \rfloor$. Let Y_t denote any of the random variables $H_{t,h-j}$ or HV_t, and $y(x)$ its associated real function. We have shown that a.a.s.

$$Y_{\lfloor x\bar{n} \rfloor} = \bar{n}y(x) + o(\bar{n}) \tag{3.26}$$

for $0 \leq x \leq x(\epsilon)$. Also, we have $|Y_{\lfloor x\bar{n} \rfloor} - Y_{\lfloor x(\epsilon)\bar{n} \rfloor}| = O((x - x(\epsilon))\bar{n})$ for all $x(\epsilon) \leq x \leq x^*$ since the change of each variable in every step is bounded by $O(1)$. Let $\delta_1(\epsilon)$ denote the number of light balls remaining at step $t(\epsilon)$. Then $\delta_1(\epsilon) = \epsilon\bar{n} + o(\bar{n})$. Applying Lemma 3.3 with $X_0 = L_{t(\epsilon)}$, $X_n = L_{t(\epsilon)+4\delta_1(\epsilon)}$, $n = 4\delta_1(\epsilon)$, $\delta = -1/2$ and $c = h$, we have a.a.s. $L_{t(\epsilon)+4\delta_1(\epsilon)} \leq \delta_1(\epsilon) - (4\delta_1(\epsilon)/2)/2 = 0$. Hence, the time τ that the RanCore algorithm terminates a.a.s. satisfies $\tau \leq t(\epsilon) + 4\delta_1(\epsilon)$. If it terminates before $\bar{n}x^*$, we may artificially let it run to that point, with the variables remaining static, thereby defining them on the interval $t \leq \lfloor x^*\bar{n} \rceil$.

Then, letting $\epsilon \to 0$ shows that the conclusion (3.26) above applies for $0 \le x \le x^*$, noting that the function $y(x)$ is continuous on $[0, x^*]$ as noted below (3.25). We may also conclude, since $\delta_1(\epsilon) \to 0$ as $\epsilon \to 0$, that

$$\tau = x^*\bar{n} + o(n) \text{ a.a.s.} \tag{3.27}$$

In particular, we conclude that a.a.s. $H_{\lfloor x\bar{n}\rfloor, h-j} = \bar{n}z_{H,h-j}(x) + o(\bar{n})$ and $HV_{\lfloor x\bar{n}\rfloor} = \bar{n}z_{HV}(x^*) + o(\bar{n})$. Since $z_{HV}(x^*) > 0$ as shown above, a.a.s. H has a non-empty $(w, k+1)$-core $\widehat{H}$. Recall that n and m_{h-j} denote the number of vertices and hyperedges of size $h-j$ in $\widehat{H}$. Then a.a.s. the number of vertices in $\widehat{H}$ is $\bar{n}z_{HV}(x^*) + o(\bar{n})$, and the number of hyperedges of size $h-j$ in $\widehat{H}$ is $\bar{n}z_{H,h-j}(x^*)/(h-j) + o(\bar{n})$. Since $z_{HV}(x^*) > 0$ and $z_{H,h-j}(x^*) > 0$, we have a.a.s. $n \sim \alpha\bar{n}$ and $m_{h-j} \sim \beta_{h-j}\bar{n}$, where $\alpha = z_{HV}(x^*)$ and $\beta_{h-j} = z_{H,h-j}(x^*)/(h-j)$.

This proves the assertions about $H \in \mathcal{M}_{\bar{n},\bar{m},h}$. Lemma 3.1 transfers them to $\mathcal{G}_{\bar{n},\bar{m},h}$. ∎

There are several useful results that we will now derive recalling various pieces of the proof of Theorem 2.2. As noted at the start of that proof, the partition-allocation g_τ output by the RanCore algorithm, if it is nonempty, is distributed as $\mathcal{P}(V_\tau, \mathbf{M}_\tau, \mathbf{0}, k+1)$ conditional on V_τ and $\mathbf{M}_\tau$. Let n denote $|V_\tau|$ and m_{h-j} denote $|M_{\tau,h-j}|/(h-j)$ for all $0 \le j \le w-1$. Without loss of generality, by relabeling elements in V_τ and $\mathbf{M}_\tau$ in a canonical way, we can simplify the notation $\mathcal{P}(V_\tau, \mathbf{M}_\tau, \mathbf{0}, k+1)$ to $\mathcal{P}([n], \mathbf{M}, \mathbf{0}, k+1)$, where $\mathbf{M} = (M_{h-w+1}, \dots, M_h)$ and $M_i = [m_i] \times [i]$. The space $\mathcal{P}([n], \mathbf{M}, \mathbf{0}, k+1)$ is used in the proof of Theorem 2.6 in Section 5.

Lemma 3.5 *Assume $c_1k < h\bar{m}/\bar{n} < c_2k$ for some constants $c_2 > c_1 > 1$. Let H be a random multihypergraph in $\mathcal{M}_{\bar{n},\bar{m},h}$. Then, provided k is sufficiently large, a.a.s. H has a nonempty $(w, k+1)$-core with average degree $O(k)$.*

Proof. Let $\bar{\mu} = h\bar{m}/\bar{n}$. Since $\bar{\mu} > c_1k$ for some $c_1 > 1$, the existence of a non-empty $(w, k+1)$-core has been shown in Theorem 3.4. Let x^* be as defined in the statement of Theorem 2.2 and let $\delta = L_0/\bar{n}$. We have shown that $\delta = O(e^{-\Omega(k)})$ below (3.22) and $x^* \le 3\delta$ below (3.24). Let $z_B(x)$ and $z_{HV}(x)$ be defined the same as those functions in (3.4)–(3.14) for $0 \le x \le x^*$. Then clearly $z_B(x^*) \le z_B(0)$ since $z'_B(x) \le -1$ for all $0 \le x \le x^*$. We also have $z'_{HV}(x) \ge -1/8$ for all $0 \le x \le x^*$ when k is large enough, as shown in the argument below (3.22). So $z_{HV}(x^*) \ge z_{HV}(0) - x^*/8$ for sufficiently large k. Since $z_{HV}(0) = f_{k+1}(\bar{\mu}) = 1 - O(e^{-\Omega(k)})$ and $x^* = O(e^{-\Omega(k)})$, we have $z_{HV}(x^*) = 1 - O(e^{-\Omega(k)})$. Recall that $\mu(x) = (z_B(x) - z_L(x))/z_{HV}(x)$. Recall also that $z'_B(x) - z'_L(x) \le 0$ by the argument below (3.22). Thus, we have $\mu(0) = O(k)$ since $h\bar{m}/\bar{n} < c_2k$ and

$$\mu(x^*) \le \frac{z_B(0) - z_L(0)}{z_{HV}(x^*)} = \frac{z_B(0) - z_L(0)}{z_{HV}(0)}(1 + O(e^{-\Omega(k)})) = O(k).$$

By Theorem 2.2, the average degree of the $(w, k+1)$-core of H is asymptotically $\mu(x^*)$, which is bounded by $O(k)$. ∎

The following lemma gives a lower bound on the size of the $(w, k+1)$-core of a random h-multihypergraph.

Lemma 3.6 *Assume $c_1k < h\bar{m}/\bar{n} < c_2k$ for some constants $c_2 > c_1 > 1$. Let H be a random multihypergraph in $\mathcal{M}_{\bar{n},\bar{m},h}$. Then a.a.s. the number of vertices in the $(w, k+1)$-core of H is $(1 - O(e^{-\Omega(k)}))\bar{n}$.*

Proof. Let n denote the number of vertices in the $(w, k+1)$-core of H. We showed just after (3.24) that $x^* < 3\delta$ where we had $\delta = \exp(-\Omega_c(k))$. Since in each step at most h heavy bins can disappear, the result follows from (3.27). ∎

We need the following lemma before proving Theorem 2.3.

Lemma 3.7 *Assume $c_1 k < h\bar{m}/\bar{n} < c_2 k$ for some constants $c_2 > c_1 > 1$. Let $\epsilon > 0$ be fixed. Let H_1 be a random multihypergraph in $\mathcal{M}_{\bar{n},\bar{m},h}$ and $H_2 \in \mathcal{M}_{\bar{n},\bar{m}+\epsilon\bar{n},h}$. Let n_1 and n_2 be the number of vertices in the $(w, k+1)$-core of H_1 and H_2 respectively. Then a.a.s. we have $|n_1 - n_2| = O(e^{-\Omega(k)}\epsilon\bar{n})$.*

Proof. Since $c_1 k < h\bar{m}/\bar{n} < c_2 k$, by Lemma 3.5, the $(w, k+1)$-core $\widehat{H}_1$ of H_1 exists and the average degree of $\widehat{H}_1$ is $O(k)$. Let H_2 be a random uniform multihypergraph obtained from $H_1 \cup \mathcal{E}$, where $\mathcal{E}$ is a set of $\epsilon\bar{n}$ hyperedges, each of which is a multiset of h vertices, each of which u.a.r. chosen from $[\bar{n}]$. Then $H_2 \in \mathcal{M}_{\bar{n},\bar{m}+\epsilon\bar{n},h}$. We say that the hyperedges in $\mathcal{E}$ are *marked*, and the other hyperedges in H_2 are *unmarked*. Define a random process $(H_t^{(1)}, H_t^{(2)})_{t\geq 0}$ as follows.

(i) The process starts with $(H_0^{(1)}, H_0^{(2)}) = (H_1, H_2)$.

(ii) The RanCore algorithm is applied to $H_t^{(2)}$ for every $t \geq 0$. The process $(H_t^{(1)}, H_t^{(2)})_{t\geq 0}$ stops when the RanCore algorithm running on $(H_t^{(2)})_{t\geq 0}$ terminates.

(iii) For every $t \geq 0$, if a marked hyperedge x in $H_{t-1}^{(2)}$ is updated to x', then x' remains marked in $H_t^{(2)}$ and $H_t^{(1)}$ is defined as $H_{t-1}^{(1)}$; if a marked hyperedge x is removed, also let $H_t^{(1)} = H_{t-1}^{(1)}$.

(iv) For every $t \geq 0$, if an unmarked hyperedge x in $H_{t-1}^{(2)}$ is updated or removed, do the same operation to x in $H_{t-1}^{(1)}$ and define $H_t^{(1)}$ to be the resulting hypergraph.

We call the random process $(H_t^{(i)})_{t\geq 0}$ for $i = 1, 2$ generated by $(H_t^{(1)}, H_t^{(2)})_{t\geq 0}$ the H_i-*process*. Note that the H_1-process is not equivalent to running the RanCore algorithm on H_1, since the light balls are not chosen u.a.r. in each step.

Instead of analysing $(H_t^{(1)}, H_t^{(2)})_{t\geq 0}$ directly, we consider $(g_t^{(1)}, g_t^{(2)})_{t\geq 0}$, the corresponding process obtained by considering the pairing-allocation model. Recall that H_1 can be represented as dropping $h\bar{m}$ unmarked balls u.a.r. into $\bar{n}$ bins with balls evenly partitioned into $\bar{m}$ groups randomly and H_2 can be represented as dropping $h\epsilon\bar{n}$ partitioned marked balls into H_1. The partition-allocation $g_0^{(i)}$ for $i = 1, 2$ is obtained by putting all balls contained in light bins of H_i into one light bin. Define $L_t^{(i)}$, $HV_t^{(i)}$, $\mathbf{m}_t^{(i)}$ and $\mathbf{L}_t^{(i)}$, etc., for $i = 1, 2$ and for $t \geq 0$, the same way as in the proof of Theorem 2.2, for the H_i-process. Conditional on $L_0^{(i)}$, $V_0^{(i)}$, $\mathbf{M}_0^{(i)}$ and $\mathbf{L}_0^{(i)}$, $g_0^{(i)}$ is distributed as $\mathcal{P}(V_0^{(i)}, \mathbf{M}_0^{(i)}, \mathbf{L}_0^{(i)}, k+1)$ for $i = 1, 2$ and all balls in $g_0^{(1)}$ are unmarked.

Let $\bar{\mu}$ denote the average degree of H_1 and let τ be the time the H_2-process terminates. It is easy to show that $g_\tau^{(1)}$ is distributed as $\mathcal{P}(V_\tau^{(1)}, \mathbf{M}_\tau^{(1)}, \mathbf{L}_\tau^{(1)}, k+1)$ conditional on the values of $V_\tau^{(1)}$, $\mathbf{M}_\tau^{(1)}$ and $\mathbf{L}_\tau^{(1)}$, since whenever a light ball is chosen, even not uniformly at random, it results in recolouring or removal of heavy balls that are uniformly chosen at random. We will later let the RanCore algorithm be run on $g_\tau^{(1)}$ in the following steps and apply the d.e. method to analyse the asymptotic behavior of this process.

First we show that $\tau = O(e^{-k}\bar{n})$. The solution of the differential equation system (3.4)–(3.14) tells the asymptotic value of $L_t^{(2)}$ in every step t. Let $x^*_{(2)}$ be the smallest root of $z_L^{(2)}(x) = 0$. Since $z_L^{(2)}(0) = O(e^{-\Omega(k)})$ and by the argument below (3.22), $z_L^{\prime(2)}(x) < -1/2$ for all $0 \leq x < x^*_{(2)}$ provided k sufficiently large, we have $x^*_{(2)} = O(e^{-\Omega(k)})$ and so $\tau = O(e^{-\Omega(k)}\bar{n})$.

Next we show that $n_2 - n_1 = O(e^{-\Omega(k)}\epsilon\bar{n})$, assuming the following three statements.

(S1) The number of balls that are unmarked and light in $g_0^{(1)}$ but not in $g_0^{(2)}$ is bounded by $O(e^{-\Omega(k)}\epsilon\bar{n})$.

(S2) The number of bins that begin heavy in the H_1-process and become light in that process but remain heavy in the H_2-process up to step τ is $O(e^{-\Omega(k)}\epsilon\bar{n})$.

(S3) $L_\tau^{(1)} = O(e^{-\Omega(k)}\epsilon\bar{n})$.

Run the Rancore algorithm on $g_\tau^{(1)}$. The differential equation system (3.4)–(3.14) tells the asymptotic values of the various random variables in $g_t^{(1)}$ for all $t \geq \tau$. Let $x^*_{(1)}$ be the smallest positive root of $z_L^{(1)}(x) = 0$. Since $z_L^{(1)}(\tau/\bar{n}) = O(e^{-k}\epsilon)$ by (S3) and by the argument below (3.22), $z_L^{\prime(1)}(x) \leq -1/2$ for all $\tau/\bar{n} \leq x < x^*_{(1)}$ provided k sufficiently large, we have $x^*_{(1)} - \tau/\bar{n} = O(e^{-k}\epsilon)$. We also have $-1/8 \leq z'_{HV}(x) \leq 0$ for sufficiently large k for all $\tau/\bar{n} \leq x < x^*_{(1)}$ as explained in Lemma 3.5. So $HV_\tau^{(1)} - n_1 = O(e^{-k}\epsilon\bar{n})$. Since $n_2 - HV_\tau^{(1)}$ counts the number of bins that are, or become light in the H_1-process but stay heavy in the H_2-process, it follows from (S1) and (S2) that $n_2 - HV_\tau^{(1)} = O(e^{-k}\epsilon\bar{n})$. So $|n_1 - n_2| = O(e^{-k}\epsilon\bar{n})$.

It only remains to prove (S1)–(S3). We first show that (S3) follows directly from (S1) and (S2). $L_\tau^{(1)}$ counts two types of light balls. The first type comes from balls that are unmarked and light in $g_0^{(1)}$ but not in $g_0^{(2)}$. By (S1), the number of these balls is a.a.s. $O(e^{-\Omega(k)}\epsilon\bar{n})$. The second type comes from balls that begin heavy and become light in the H_1-process but stay heavy in the H_2-process. By (S2), the number of these balls is a.a.s. $k \cdot O(e^{-\Omega(k)}\epsilon\bar{n}) = O(e^{-\Omega(k)}\epsilon\bar{n})$. Thereby (S3) follows.

Next we show (S1). At step 0, clearly the set of unmarked light balls in $g_0^{(2)}$ is a subset of those in $g_0^{(1)}$. The number of light balls in $g_0^{(1)}$ is a.a.s. $(1 - f_k(\bar{\mu}))\bar{\mu}\bar{n} = O(e^{-\Omega(k)}\bar{n})$ as shown in the proof of Theorem 2.2 and hence the number of light vertices of H_1 is a.a.s. $O(e^{-\Omega(k)}\bar{n})$. Since each multihyperedges in $\mathcal{E}$ is a random multihyperedges, the expected number of those which contains a light vertex in H_1 is $O(e^{-\Omega(k)}\epsilon\bar{n})$, hence the number of light vertex in H_1 that become heavy after the hyperedges in $\mathcal{E}$ being dropped is a.a.s. $O(e^{-\Omega(k)}\epsilon\bar{n})$ and each of these vertex/bin contains at most k unmarked balls. Thus (S1) follows.

Now we show (S2). Recall that H_1 is represented as dropping $h\bar{m}$ unmarked balls u.a.r. into $\bar{n}$ bins and H_2 is obtained by dropping $h\epsilon\bar{n}$ extra marked balls u.a.r. into the $\bar{n}$ bins in H_1. Recall that $\widehat{H}_1$ denotes the $(w, k+1)$-core of H_1. The number of bins that begin heavy in the H_1-process and become light in that process but remain heavy in the H_2-process up to step τ is at most the number of bins/vertices not in $\widehat{H}_1$ which receive at least one marked balls after dropping $h\epsilon\bar{n}$ marked balls u.a.r. into the $\bar{n}$ bins. By Lemma 3.6, the number of vertices/bins in $\widehat{H}_1$ is a.a.s. $(1-O(e^{-\Omega(k)}))\bar{n}$. Then for each marked ball, the probability that it is dropped into a bin not in $\widehat{H}_1$ is $O(e^{-\Omega(k)})$. By Lemma 3.3, the number of marked balls dropped into bins not in $\widehat{H}_1$ is a.a.s. $O(e^{-\Omega(k)}\epsilon\bar{n})$. Hence the number of bins that are not in $\widehat{H}_1$ and receive at least one marked balls is a.a.s. $O(e^{-\Omega(k)}\epsilon\bar{n})$. ∎

Proof of Theorem 2.3. Let H_1 be a random uniform multihypergraph with average degree $\bar{\mu}$ and let H_2 be a random uniform multihypergraph obtained from $H_1 \cup \mathcal{E}$, where $\mathcal{E}$ is a set of $\epsilon\bar{n}$ hyperedges, each of which is a multiset of h vertices, each of which is uniformly chosen from $[\bar{n}]$.

For $i = 1, 2$, let $\widehat{H}_i$ be the $(w, k+1)$-core of H_i and let $m_{h-j}^{(i)}$ be the number of hyperedges with size $h-j$ in $\widehat{H}_i$. We first show that

$$\sum_{j=0}^{w-1}(w-j)m_{h-j}^{(2)} - \sum_{j=0}^{w-1}(w-j)m_{h-j}^{(1)} \geq w\epsilon\bar{n}/2.$$

Clearly $\widehat{H}_1$ is a subgraph of $\widehat{H}_2$. Let n_i denote the number of vertices in $\widehat{H}_i$ and let $[n_i]$ denote the set of vertices in $\widehat{H}_i$. By Lemma 3.6, a.a.s. $n_1 = (1-O(e^{-\Omega(k)}))\bar{n}$. Then for any hyperedge $x \in \mathcal{E}$, the probability that all vertices in x are contained in $[n_1]$ is $1-O(e^{-\Omega(k)})$. So the expected number of hyperedges in $\mathcal{E}$ lying completely in $[n_1]$ is $(1-O(e^{-\Omega(k)}))\epsilon\bar{n}$. By the Chernoff bound, originally given in [7, Theorem 1], we have a.a.s. the number of hyperedges in $\mathcal{E}$ lying completely in $[n_1]$ is at least $\epsilon\bar{n}/2$ for sufficiently large k. So it follows immediately that a.a.s.,

$$\sum_{j=0}^{w-1}(w-j)m_{h-j}^{(2)} - \sum_{j=0}^{w-1}(w-j)m_{h-j}^{(1)} \geq w\epsilon\bar{n}/2.$$

For simplicity, let $S(i)$ denote $\sum_{j=0}^{w-1}(w-j)m_{h-j}^{(i)}$ for $i = 1, 2$. Recall that $\kappa(\widehat{H}_i)$ denotes $S(i)/n_i$. Then a.a.s.,

$$\kappa(\widehat{H}_2) - \kappa(\widehat{H}_1) = \frac{S(2)}{n_2} - \frac{S(1)}{n_1} \geq \frac{(S(1) + w\epsilon\bar{n}/2) - S(1)\cdot n_2/n_1}{n_2}.$$

By Lemma 3.7, a.a.s. $n_2 - n_1 = O(e^{-\Omega(k)})\epsilon\bar{n}$, i.e. $n_2/n_1 - 1 \leq f(k)\epsilon$ for some function $f(k) = O(e^{-\Omega(k)})$. Then a.a.s.,

$$\kappa(\widehat{H}_2) - \kappa(\widehat{H}_1) \geq \frac{w\epsilon\bar{n}/2 - O(f(k)\epsilon S(1))}{n_2} \geq w\epsilon/4 > 0, \tag{3.28}$$

for sufficiently large k and for every $\epsilon > 0$, since $S(1) = O(k)\bar{n}$ and $n_2 = (1-O(e^{-\Omega(k)}))\bar{n}$.

By Theorem 2.2, for given $h > w > 0$ and sufficiently large k, a.a.s. $\kappa(\widehat{H}) = c(\bar{\mu}) + o(1)$, where $c(\bar{\mu})$ is a constant depending only on $\bar{\mu}$. The inequality (3.28) implies that $c(\bar{\mu})$ is an increasing function of $\bar{\mu}$. ∎

Proof of Corollary 2.4. By Theorem 2.3, there exists a unique critical value of $\bar{\mu}$ such that a.a.s. $\kappa(\widehat{H}) = k + o(1)$ and so there exists a threshold function $\bar{m} = f(\bar{n})$ of $\mathcal{M}_{\bar{n},\bar{m},h}$ for the graph property $\mathcal{T}$. Then this holds as well in $\mathcal{G}_{\bar{n},\bar{m},h}$ by Lemma 3.1. ∎

The differential equations in Theorem 2.2 are only used in a theoretical way to show properties of the $(w, k+1)$-core, and we do not have an analytic solution. However, they can be numerically solved when the values of h, w, k and μ are given. Table 3 gives the results of some computations, where h, w and k are given, $\widetilde{\mu}$ denotes the expected average degree of the hypergraph H at the threshold for $\mathcal{T}$ given in Corollary 2.4, and $\widehat{\mu}$ denotes the corresponding average degree of its core $\widehat{H}$. Even though our results on the concentration of the size and density of the $(w, k+1)$-core and the threshold of property $\mathcal{T}$ only cover for the case of sufficiently large k, our numerical computation results as shown in the table do coincide with our simulation results. Hence we believe that Theorem 2.2, 2.3 and Corollary 2.4 actually hold for all $k \geq 1$. By Corollary 2.7, discussed in the next section, $\widetilde{\mu}$ is also our main target, the threshold for orientability. Note that $\widehat{\mu}$ must be at least hk/w by the definition of property $\mathcal{T}$, and that it follows from the trivial upper bound of the orientability threshold given in the introduction part that $\widetilde{\mu}$ is at most hk/w.

h	w	k	$\widetilde{\mu}$	$\widehat{\mu}$
3	2	4	5.485	6.65086
3	2	10	14.766	15.5872
3	2	40	59.991	60.0773
10	2	4	19.99999	20.0003

Table 1: Some numerical computation results

4 The (w, k)-orientability of the $(w, k+1)$-core

In this section we prove Corollary 2.7 assuming Theorem 2.6, and study the basic network flow formulation of the problem that is used in the next section to prove Theorem 2.6.

The following lemma is in preparation for proving that $\widehat{H}$, the $(w, k+1)$-core of $H \in \mathcal{M}_{\bar{n},\bar{m},h}$, if not empty, a.a.s. has property $\mathcal{A}(\gamma)$ for some $0 < \gamma < 1$.

Lemma 4.1 *Let $H \in \mathcal{M}_{\bar{n},\bar{m},h}$ and let $\widehat{H}$ be the $(w, k+1)$-core of H. Let $c_1 > 1$ be a constant that can depend on k. Then there exists a constant $0 < \gamma = \varphi(k, c_1)$ depending only on k and c_1, such that a.a.s. there exists no $S \subset V(H)$ with $|S| < \gamma\bar{n}$ and at least $c_1|S|$ hyperedges partially contained in S. More specifically, when $c_1 \geq 2$ and $c_1 < h^2e^2\bar{\mu}$, we may choose $\gamma = \varphi(k, c_1) = (c_1/h^2e^2\bar{\mu})^2$.*

Proof. Let s be any integer such that $0 < s < n$ and let $r = s/n$. Let Y denote the number of S with $|S| = s$ and at least c_1s hyperedges partially contained in S. The probability for a given hyperedge to be partially contained in S is at most $\binom{h}{2}(s/\bar{n})^2 < h^2r^2$. Then the probability that there are at least c_1s such hyperedges is at most

$$\binom{\bar{m}}{c_1 s}(hr)^{2c_1 s}.$$

Since there are $\binom{\bar{n}}{s}$ ways to choose S,

$$\begin{aligned}
\mathbf{E}(Y) &= \sum_{s\le\gamma\bar{n}} \binom{\bar{n}}{s}\binom{\bar{m}}{c_1 s}(hr)^{2c_1 s} \\
&\le \sum_{\ln\bar{n}\le s\le\gamma\bar{n}} \left(\frac{e\bar{n}}{s}\right)^s \left(\frac{e\bar{m}}{c_1 s}\right)^{c_1 s} (hr)^{2c_1 s} + \sum_{1\le s\le\ln\bar{n}} \bar{n}^s \bar{m}^{c_1 s}\left(\frac{hs}{\bar{n}}\right)^{2c_1 s} \\
&= \sum_{\ln\bar{n}\le s\le\gamma\bar{n}} \left(h^{2c_1} e^{1+c_1} r^{c_1-1}\left(\frac{\bar{\mu}}{c_1}\right)^{c_1}\right)^s + \sum_{1\le s\le\ln\bar{n}} \left(\frac{(\bar{\mu}h^2 s^2)^{c_1}}{\bar{n}^{c_1-1}}\right)^s \\
&\le \sum_{\ln\bar{n}\le s\le\gamma\bar{n}} \left(\bar{C} r^{c_1-1}\bar{\mu}^{c_1}\right)^s + \ln\bar{n}\cdot\frac{(\bar{\mu}h^2\ln^2\bar{n})^{c_1}}{\bar{n}^{c_1-1}} \\
&= \sum_{\ln\bar{n}\le s\le\gamma\bar{n}} \left(\bar{C} r^{c_1-1}\bar{\mu}^{c_1}\right)^s + o(1),
\end{aligned}$$

for some constant $0<\bar{C}=\bar{C}(c_1)\le (h^2/c_1)^{c_1}\, e^{c_1+1}$. Choose

$$\gamma < \left(\frac{c_1}{h^2 e\bar{\mu}}\right)^{\frac{c_1}{c_1-1}} e^{-\frac{1}{c_1-1}}.$$

Then $\bar{C}\gamma^{c_1-1}\bar{\mu}^{c_1}<1$. So there exist $0<\beta<1$, such that $\bar{C}\gamma^{c_1-1}\bar{\mu}^{c_1}<\beta$, for all $r\le\gamma$. When $c_1\ge 2$ and $c_1/h^2e^2\bar{\mu}<1$,

$$\left(\frac{c_1}{h^2 e\bar{\mu}}\right)^{\frac{c_1}{c_1-1}} e^{-\frac{1}{c_1-1}} > \left(\frac{c_1}{h^2e^2\bar{\mu}}\right)^{c_1/(c_1-1)} > \left(\frac{c_1}{h^2e^2\bar{\mu}}\right)^2.$$

Hence we may simply choose $\gamma=(c_1/h^2e^2\bar{\mu})^2$. Then

$$\sum_{\ln\bar{n}\le s\le\gamma\bar{n}} \left(\bar{C} r^{c_1-1}\bar{\mu}^{c_1}\right)^s < \sum_{\ln\bar{n}\le s\le\gamma\bar{n}} \beta^s = O(\beta^{\ln\bar{n}}) = o(1).$$

Hence we have $\mathbf{E}(Y)=o(1)$. ∎

The following corollary shows that the same property is shared by $\widehat{H}$.

Corollary 4.2 *Let $H\in\mathcal{M}_{\bar{n},\bar{m},h}$ and let $\widehat{H}$ be the $(w,k+1)$-core of H. Let c_1 be a constant that can depend on k, with the constraint that $2\le c_1<h^2e^2\bar{\mu}$. Let $0<\gamma=\varphi(k,c_1)=(c_1/h^2e^2\bar{\mu})^2$. Then a.a.s. for all $S\subset V(\widehat{H})$ with $|S|<\gamma n$, the number of hyperedges partially contained in S is less than $c_1|S|$.*

Proof. Let n be the number of vertices in $\widehat{H}$ and D the sum of degrees of vertices in $\widehat{H}$. For any hyperedge $x\in\widehat{H}$, let x^+ denote its corresponding hyperedge in H. Obviously $n\le\bar{n}$. Combining with Lemma 4.1 and the fact that for any $S\subset V(\widehat{H})$, a hyperedge x is partially contained in S only if x^+ is partially contained S in H, Corollary 4.2 follows. ∎

We next show that $\widehat{H}$, if not empty, a.a.s. has property $\mathcal{A}(\gamma)$, defined in Definition 2.5, for some certain value of γ.

Corollary 4.3 *Assume that $H \in \mathcal{M}_{\bar{n},\bar{m},h}$ with $\bar{m} \leq hk/w$, and that $\widehat{H}$ is the $(w, k+1)$-core of H. Let $\gamma = e^{-4}h^{-6}/4$. Then provided $k \geq 4w$, a.a.s. either $\widehat{H}$ is empty or $\widehat{H}$ has property $\mathcal{A}(\gamma)$.*

Proof. Apply Lemma 4.2 with $c_1 = k/2w$. Clearly $c_1 < ch^2e^2k$, and $c_1 \geq 2$ provided $k \geq 4w$. Then $\gamma \leq \phi(k, c_1)$. By Definition 2.5, $\widehat{H}$ a.a.s. has property $\mathcal{A}(\gamma)$. ∎

Proof of Corollary 2.7 Let $\widehat{H}$ be the $(k+1)$-core of the random multihypergraph $H \in \mathcal{M}_{\bar{n},\bar{m},h}$. Let $\epsilon > 0$ be any constant. By Theorem 2.3, there exists a constant $\delta > 0$, such that a.a.s. if $\bar{m} \leq f(\bar{m}) - \epsilon\bar{n}$, then $\sum_{j=0}^{w-1}(w-j)m_{h-j} \leq kn - \delta n$. By Theorem 2.6 and Corollary 4.3, there exists a constant N depending only on h and w such that provided $k > N$, $\widehat{H}$ a.a.s. has a (w,k)-orientation. On the other hand, if $\bar{m} \geq f(\bar{m}) + \epsilon\bar{n}$, then a.a.s. $\sum_{j=0}^{w-1}(w-j)m_{h-j} \geq kn + \delta n$, and hence clearly $\widehat{H}$ is not (w,k)-orientable. Therefore $f(\bar{n})$ is a sharp threshold function for the (w,k)-orientation of $\mathcal{M}_{\bar{n},\bar{m},h}$. By Lemma 3.1, $f(\bar{n})$ is also a sharp threshold function for the (w,k)-orientation of $\mathcal{G}_{\bar{n},\bar{m},h}$. ∎

Let G be a non-uniform multihypergraph with the sizes of hyperedges between $h-w+1$ to h. In the rest of the chapter, we will use the following notations. Let $E_{h-j} := \{x \in E(G) : |x| = h-j\}$. For any given $S \subset [n]$, let $m_{h-j,i}(S) := |\{x \in E_{h-j} : |x \cap S| = i\}|$ for any $0 \leq i \leq h-j$. When the context is clear of which set S is referred to, we may drop S from the notation. Let $\overline{S}$ denote the set $[n] \setminus S$ and let $d(S)$ denote the sum of degrees of vertices in S.

Recall from above the statement of Corollary 2.8 in Section 2 that for any $S \subset V(G)$, G_S denotes the subgraph w-induced by S. The following Lemma generalises Hakimi's theorem [17, Theorem 4] for graphs. It is proved using network flow and the max-flow min-cut theorem, along the lines of the standard techniques discussed in [9, 26]. This setting was used before in connection with the load balancing problem in [25, Section 3.3].

Lemma 4.4 *A multihypergraph G with sizes of hyperedges between $h-w+1$ and h has a (w,k)-orientation if and only if $\kappa(G_S) \leq k$ for all $S \subset V(G)$.*

Proof. Formulate a network flow problem on a network G^* as follows. Let L be a set of vertices, each of which represents a hyperedge of G, and R be a set of n vertices, each of which represents a vertex in G. For any $u \in L$, and $v \in R$, uv is an edge in G^* if and only if $v \in u$ in G. Add vertices a and b to G^*, such that a is linked to every vertex in L, and b is linked to every vertex in R. Let $c : E(G^*) \to \mathbf{N}^+$ be defined as $c(au) = w-j$ for every $u \in L$ such that the degree of u is $h-j$, $c(vb) = k$ for every $v \in R$, and $c(uv) = 1$ for every $uv \in E(G^*)$. Then G has a (w,k)-orientation if and only if G^* has a flow of size $\sum_{j=0}^{w-1}(w-j)m_{h-j}$ from a to b. By the max-flow min-cut Theorem, G^* has a flow with all edges incident with a saturated if and only if

$$c(\delta(C)) \geq \sum_{j=0}^{w-1}(w-j)m_{h-j}, \quad \text{for all (a,b)-cuts } C. \tag{4.1}$$

As an example in Figure 1, $A \subset L$ is a set of hyperedges in G, and $S \subset R$ is a set of vertices in G. Let $C = \{a\} \cup \overline{A} \cup S$ define a cut of G^*. Then the condition in (4.1) is equivalent

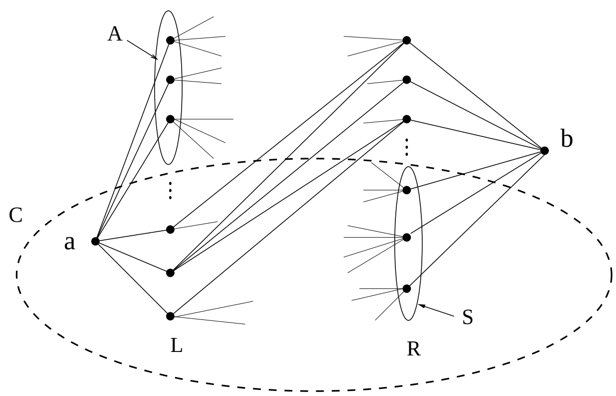

Figure 1: *A cut C in the graph G^**

to

$$\forall C,\quad c(\delta(C)) = k|S| + \sum_{j=0}^{w-1}\left(\sum_{x\in A\cap E_{h-j}}(w-j) + \sum_{x\in E_{h-j}\setminus A}|x\cap \overline{S}|\right) \geq \sum_{j=0}^{w-1}(w-j)m_{h-j}, \quad (4.2)$$

Let $A^* := \{x \in E_{h-j} : |x\cap S| \leq h-w\}$. Clearly A^* minimizes $c(\delta(C))$ for a given S. Therefore we only need to check (4.2) when $A = A^*$. The condition in (4.2) is then equivalent to

$$\sum_{j=0}^{w-1}\left(\sum_{x\in E_{h-j}\setminus A}(w-j) - \sum_{x\in E_{h-j}\setminus A}|x\cap \overline{S}|\right) \leq k|S|.$$

For any hypergraph G, let $\beta(G)$ denote the number of hyperedges in G. Recall from the statement above Theorem 2.3 that $|S|\kappa(G_S) = d(G) - (h-w)\beta(G)$. Since

$$\begin{aligned}\sum_{j=0}^{w-1}\left(\sum_{x\in E_{h-j}\setminus A}(w-j) - \sum_{x\in E_{h-j}\setminus A}|x\cap \overline{S}|\right) &= \sum_{j=0}^{w-1}\sum_{x\in E_{h-j}\setminus A}(w-j) - (h-j-|x\cap S|)\\ &= \sum_{x\notin A}|x\cap S| - \sum_{x\notin A}(h-w) = d(G_S) - (h-w)\beta(G_S) = |S|\kappa(G_S), \quad (4.3)\end{aligned}$$

Lemma 4.4 follows. ∎

The next corollary follows immediately.

Corollary 4.5 *A hypergraph H in $\mathcal{G}_{\bar{n},\bar{m},h}$ has a (w,k)-orientation if and only if for every $S \subset V(H)$, $\kappa(H_S) \leq k$.*

Proof of Corollary 2.8. This follows directly from Corollary 2.7 and Corollary 4.5. ∎

For any vertex set S, define

$$\partial^*(S) = d(S) - \sum_{j=0}^{w-1} \sum_{i=w-j+1}^{h-j} (i-(w-j))m_{h-j,i}, \tag{4.4}$$

which measures a type of expansion in the hypergraph. For each hyperedge x of size $h-j$ which intersects S with i vertices, its contribution to $\partial^*(S)$ is $w-j \geq 0$ if $i \geq w-j+1$ and $i \geq 0$ otherwise. Therefore $\partial^*(S) \geq 0$ for any S. The following lemma characterises the existence of the (w,k)-orientation of G in terms of $\partial^*(S)$.

Lemma 4.6 *Let G be a multihypergraph whose hyperedges all have sizes between $h-w+1$ and h inclusively. Then the following two properties of G are equivalent:*

(i) $\kappa(G_S) \leq k$, *for all* $S \subset V(G)$;

(ii) $\partial^*(S) \geq k|S| + \left(\sum_{j=0}^{w-1}(w-j)m_{h-j}\right) - kn$, *for all* $S \subset V(G)$.

Proof. Let $\beta(G_S)$ denote the number of hyperedges in G_S. We show that for any $S \subset V(G)$, $\kappa(G_S)|S| = d(G_S)-(h-w)\beta(G_S) \leq k|S|$ if and only if $\partial^*(\overline{S}) \geq k|\overline{S}|+\left(\sum_{j=0}^{w-1}(w-j)m_{h-j}\right)-kn$. Then Lemma 4.6 follows immediately. Note from the definition of A^*, we have for any $x \in E_{h-j} \setminus A^*$, $|x \cap S| \geq h-w+1$ and hence $|x \cap \overline{S}| \leq (h-j)-(h-w+1) = w-j-1$. By (4.3), for any $S \subset V(G)$,

$$\begin{aligned}
& d(G_S) - (h-w)\beta(G_S) \leq k|S| \\
& \quad \Longleftrightarrow \sum_{j=0}^{w-1}\left(\sum_{x\in E_{h-j}\setminus A^*}(w-j) - \sum_{x\in E_{h-j}\setminus A^*}|x\cap \overline{S}|\right) \leq kn - k|\overline{S}| \\
& \quad \Longleftrightarrow \sum_{j=0}^{w-1}\sum_{i=0}^{w-j-1}(w-j-i)m_{h-j,i}(\overline{S}) \leq kn - k|\overline{S}| \\
& \quad \Longleftrightarrow \sum_{j=0}^{w-1}(w-j)m_{h-j} - \sum_{j=0}^{w-1}\left(\sum_{i=w-j}^{h-j}(w-j)m_{h-j,i}(\overline{S}) + \sum_{i=0}^{w-j-1} i m_{h-j,i}(\overline{S})\right) \leq kn - k|\overline{S}| \\
& \quad \Longleftrightarrow \partial^*(\overline{S}) \geq k|\overline{S}| + \left(\sum_{j=0}^{w-1}(w-j)m_{h-j}\right) - kn. \quad \blacksquare
\end{aligned}$$

It follows from Lemma 4.4 and Lemma 4.6 that G is (w,k)-orientable if and only if Lemma 4.6 (ii) holds.

Without loss of generality, we assume $\sum_{j=0}^{w-1}(w-j)m_{h-j} - kn \leq 0$. Otherwise, condition (4.1) is violated by taking $C = \{a\} \cup L \cup R$. The following lemma shows that, instead of checking conditions in Lemma 4.6 (ii), we can check that certain other events do not occur.

For any $S \subset V(G)$, let

$$q_{h-j}(S) = \sum_{i=1}^{h-j} i m_{h-j,i}, \quad \eta(S) = \sum_{j=0}^{w-1} \sum_{i=1}^{h-j-1} m_{h-j,i}. \tag{4.5}$$

In other words, $q_{h-j}(S)$ denotes the contribution to $d(S)$ from hyperedges of size $h-j$ and $\eta(S)$ denotes the number of hyperedges which intersect both S and $\overline{S}$. When the context is clear, we may use q_{h-j} and η instead to simplify the notation.

Recall that given a vertex set S, a hyperedge x is partially contained in S if $|x \cap S| \geq 2$. Let $\rho(S)$ denote the number of hyperedges partially contained in S and let $\nu(S)$ denote the number of hyperedges intersecting S.

Lemma 4.7 *Suppose that for some $S \subset V(G)$,*

$$\partial^*(S) < k|S| + \left(\sum_{j=0}^{w-1}(w-j)m_{h-j}\right) - kn. \tag{4.6}$$

Then all of the following hold:

(i) $\rho(\overline{S}) > k|\overline{S}|/w$;

(ii) $\nu(S) < k|S|$;

(iii) $(h-w)\rho(S) > d(S) - k|S|$;

(iv) if, in addition, $\sum_{j=0}^{w-1} \frac{w-j}{h-j} q_{h-j}(S) \geq (1-\delta)k|S|$ for some $\delta > 0$, then $\eta(S) < h^2\delta k|S|$.

Proof. Let s and $\bar{s}$ denote $|S|$ and $|\overline{S}|$ respectively. If (4.6) is satisfied, then

$$d(S) - \sum_{j=0}^{w-1} \sum_{i=w-j+1}^{h-j} (i-(w-j))m_{h-j,i} < ks - \left(kn - \sum_{j=0}^{w-1}(w-j)m_{h-j}\right) = \sum_{j=0}^{w-1}(w-j)m_{h-j} - k\bar{s}.$$

Hence

$$\begin{aligned}
k\bar{s} &< \sum_{j=0}^{w-1}(w-j)m_{h-j} - \sum_{j=0}^{w-1}\sum_{i=1}^{h-j} i m_{h-j,i} + \sum_{j=0}^{w-1}\sum_{i=w-j+1}^{h-j}(i-(w-j))m_{h-j,i} \\
&= \sum_{j=0}^{w-1}(w-j)m_{h-j,0} + \sum_{j=0}^{w-1}\sum_{i=1}^{w-1-j}(w-j-i)m_{h-j,i} \\
&\leq w\sum_{j=0}^{w-1}\left(m_{h-j,0} + \sum_{i=1}^{w-1-j} m_{h-j,i}\right).
\end{aligned}$$

Since

$$m_{h-j,0} = |\{x \in E_{h-j} : |x \cap \overline{S}| = h - j\}|,$$

and

$$\sum_{i=1}^{w-1-j} m_{h-j,i} \leq w|\{x \in E_{h-j} : 2 \leq |x \cap \overline{S}| \leq h - j - 1\}|,$$

(this is because $1 \leq i \leq w-1-j$ and so $h-j-i \leq h-j-1$ and $h-j-i \geq h-(w-1) \geq 2$), we have

$$k\bar{s} < w|\{x \in E(G) : |x \cap \overline{S}| \geq 2\}|.$$

This proves part (i). Again, if (4.6) is satisfied, then

$$\sum_{j=0}^{w-1} \sum_{i=w-j+1}^{h-j} (i - (w-j))m_{h-j,i} > d(S) - ks + \left(kn - \sum_{j=0}^{w-1} (w-j)m_{h-j}\right).$$

Since

$$\sum_{j=0}^{w-1} \sum_{i=w-j+1}^{h-j} (i - (w-j))m_{h-j,i} \leq \sum_{i=2}^{h} (i-1)|\{x : |x \cap S| = i\}| = d(S) - \nu(S),$$

we have

$$d(S) - \nu(S) > d(S) - ks + \left(kn - \sum_{j=0}^{w-1} (w-j)m_{h-j}\right).$$

Since $kn - \sum_{j=0}^{w-1}(w-j)m_{h-j} > 0$, this directly leads to part (ii). Since

$$\sum_{j=0}^{w-1} \sum_{i=w-j+1}^{h-j} (i - (w-j))m_{h-j,i} \leq (h-w)|\{x : |x \cap S| \geq 2\}|,$$

we have

$$|\{x : |x \cap S| \geq 2\}| > d(S) - ks + \left(kn - \sum_{j=0}^{w-1} (w-j)m_{h-j}\right).$$

Since $kn - \sum_{j=0}^{w-1}(w-j)m_{h-j} > 0$, this proves part (iii). Now we prove part (iv). Let $t_{h-j} = 1 - (h-j)m_{h-j,h-j}/q_{h-j}$. Note that $d(S) = \sum_{j=0}^{w-1} q_{h-j}$ and $q_{h-j} = \sum_{i=1}^{h-j} i m_{h-j,i}$. For each hyperedge x of size $h-j$ which intersects S with i vertices, its contribution to q_{h-j} (and thus to $\partial^*(S)$) is

- $i \cdot (w-j)/(h-j)$, if $i = h-j$;
- $i \cdot (w-j)/i \geq i \cdot (w-j)/(h-j-1)$, if $w-j+1 \leq i \leq h-j-1$;
- $i \geq i \cdot (w-j)/(h-j-1)$, if $1 \leq i \leq w-j$;

Then

$$\begin{aligned}
\partial^*(S) &\geq \sum_{j=0}^{w-1}\left(\frac{w-j}{h-j}q_{h-j}(1-t_{h-j})+\frac{w-j}{h-j-1}q_{h-j}t_{h-j}\right)\\
&= \sum_{j=0}^{w-1}\frac{w-j}{h-j}q_{h-j}+\sum_{j=0}^{w-1}\frac{w-j}{(h-j)(h-j-1)}q_{h-j}t_{h-j}\\
&\geq \sum_{j=0}^{w-1}\frac{w-j}{h-j}q_{h-j}+\frac{1}{h^2}\sum_{j=0}^{w-1}(w-j)q_{h-j}t_{h-j}.
\end{aligned}$$

If $\sum_{j=0}^{w-1}\frac{w-j}{h-j}q_{h-j}\geq(1-\delta)ks$ for some $\delta>0$, then (4.6) implies that

$$\frac{1}{h^2}\sum_{j=0}^{w-1}(w-j)q_{h-j}t_{h-j}<\delta ks.$$

Therefore $\eta(S)\leq\sum_{j=0}^{w-1}q_{h-j}t_{h-j}<h^2\delta ks$. This proves part (iv). ∎

5 Proof of Theorem 2.6

Recall from Section 2 that $\mathcal{M}(n,\mathbf{m},k+1)$ is $\mathcal{M}_{n,\mathbf{m}}$, which is a random multihypergraph with given edge sizes, restricted to multihypergraphs with minimum degree at least $k+1$. In this section we prove the only remaining theorem, Theorem 2.6. This theorem relates the orientability of $\mathcal{M}(n,\mathbf{m},k+1)$ to its w-density. Recall that this probability space was important because, by Proposition 2.1, it gives the distribution of the $(w,k+1)$-core $\widehat{H}$ of $H\in\mathcal{M}_{\bar{n},\bar{m},h}$ conditioned on the values of n, the number of vertices and m_{h-j}, the number of hyperedges of size $h-j$ for each j, in the core.

It is clear, that given values of n and $\mathbf{m}$, the probability space of random multihypergraphs generated by $\mathcal{P}([n],\mathbf{M},\mathbf{0},k+1)$, with $|M_{h-j}|=(h-j)m_{h-j}$ $(h=0,\ldots,w-1)$, is equivalent to $\mathcal{M}(n,\mathbf{m},k+1)$. So we may, and do, make use of the partition-allocation model for proving results about $\mathcal{M}(n,\mathbf{m},k+1)$.

For the rest of the chapter, let $\epsilon>0$ and $k\geq 2$ be fixed. Without loss of generality, we may assume that $\epsilon<\frac{1}{2}$ since ϵ may be taken arbitrarily small. By the hypothesis of Theorem 2.6, we consider only $\mathbf{m}$ such that $\sum_{j=0}^{w-1}(w-j)m_{h-j}\leq kn-\epsilon n$. We may also assume that $\sum_{j=0}^{w-1}(w-j)m_{h-j}\geq kn-2\epsilon n$ since otherwise, by Theorem 2.2, we can simply add a set of random hyperedges so that the assumption holds. This is valid because (w,k)-orientability is a decreasing property (i.e. it holds in all subgraphs of G whenever G has the property). Let

$$D=\sum_{j=0}^{w-1}(h-j)m_{h-j},\quad m=\sum_{j=0}^{w-1}m_{h-j},\quad \mu=\frac{D}{n}. \tag{5.1}$$

Since

$$D\cdot\frac{1}{h-w+1}\leq\sum_{j=0}^{w-1}(w-j)m_{h-j}\leq D\cdot\frac{w}{h},\quad m\leq\sum_{j=0}^{w-1}(w-j)m_{h-j}\leq wm,$$

and

$$kn - 2\epsilon n \leq \sum_{j=0}^{w-1}(w-j)m_{h-j} \leq kn - \epsilon n, \tag{5.2}$$

we have

$$h(k-1)/w \leq \mu = D/n \leq (h-w+1)k, \quad \frac{(k-1)n}{w} \leq m \leq \Big(k - \frac{1}{2}\Big)n. \tag{5.3}$$

In the rest of the paper, whenever we refer to the probability space $\mathcal{H}(n, \mathbf{m}, k+1)$ or $\mathcal{M}(n, \mathbf{m}, k+1)$, we assume $\mathbf{m}$ satisfies (5.2).

Since h and w are given, we consider them as absolute constants. Therefore, whenever we refer to $g = O(f)$, it means that there exists a constant C such that $g \leq Cf$, where C can depend on h and w. We also use notation $g = O_\gamma(f)$, which means that there exists a constant C depending on γ only such that $g \leq Cf$. The same convention applies to $o(f)$, $\Omega(f)$, $\Theta(f)$ and $o_\gamma(f)$, $\Omega_\gamma(f)$ and $\Theta_\gamma(f)$.

By Proposition 2.1, conditioned on the values of n, the number of vertices and m_{h-j}, the number of hyperedges of size $h-j$, of the $(w, k+1)$-core $\widehat{H}$ of $H \in \mathcal{M}_{\bar{n},\bar{m},h}$, $\widehat{H}$ is distributed as $\mathcal{M}(n, \mathbf{m}, k+1)$. Recall that $\epsilon > 0$ and $k \geq 2$ are fixed. Let $\mathbf{m}$ be an integer vector with the constraint (5.2). Given $\mathbf{m}$, let D, μ be as defined in (5.1). Let G be a random multihypergraph from the probability space $\mathcal{M}(n, \mathbf{m}, k+1)$.

We next sketch the proof of Theorem 2.6. Let $q_{h-j}(S)$ and $\eta(S)$ be defined as in (4.5). The partition-allocation model gives a good foundation for proving that a.a.s. certain properties hold concerning the distribution of vertex degrees and intersections of hyperedge sets with vertex sets. Using this and various other probabilistic tools, we show that

(a) the probability that $G \in \mathcal{M}(n, \mathbf{m}, k+1)$ has property $\mathcal{A}(\gamma)$ and contains some set S with $|S| < \gamma n$ for which both Lemma 4.7(ii) and (iii) holds is $o(1)$;

(b) there exists $\delta > 0$, such that when k is large enough, a.a.s. $\sum_{j=0}^{w-1} \frac{w-j}{h-j} q_{h-j} \geq (1-\delta)k|S|$, and the probability of $G \in \mathcal{M}(n, \mathbf{m}, k+1)$ containing some set S with $\gamma n \leq |S| \leq (1-\gamma)n$ and $\eta(S) < h^2\delta k|S|$ is $o(1)$.

We also show the deterministic result that

(c) no multihypergraph G with property $\mathcal{A}(\gamma)$ contains any sets S with $|S| > (1-\gamma)n$ for which Lemma 4.7(i) holds.

It follows that the probability that G has property $\mathcal{A}(\gamma)$ and contains some set S for which all parts (i)–(iv) of Lemma 4.7 hold is $o(1)$. Then by Lemmas 4.6 and 4.7,

$$\mathbf{P}(G \in \mathcal{A}(\gamma) \wedge G \text{ is not } (w,k)\text{-orientable}) = o(1).$$

Finally, Lemma 3.1 shows that the result applies to random (simple) hypergraphs as well.

We start with a few concentration properties. As discussed in Section 3, the degree sequence of $G \in \mathcal{M}(n, \mathbf{m}, k+1)$ obeys the multinomial distribution. The following lemma bounds the probability of rare degree (sub)sequences where the degree distribution is independent truncated Poisson. We will use this result to bound the probability of rare degree sequences in $\mathcal{M}(n, \mathbf{m}, k+1)$.

Lemma 5.1 *Let $s \geq w(n)$ for some $w(n) \to \infty$ as $n \to \infty$ and let $Y_1, \ldots, Y_s$ be independent copies of Z defined in (3.2) with λ satisfying $\lambda f_k(\lambda) = \mu f_{k+1}(\lambda)$. Let $0 < \delta < 1$ be any constant. Then there exist $N > 0$ and $0 < \alpha < 1$ both depending only on δ, such that provided $k > N$,*

$$\mathbf{P}\left(\Big|\sum_{i=1}^{s} Y_i - \mu s\Big| \geq \delta\mu s\right) \leq \alpha^{\mu s}.$$

Proof. Let $G(x)$ be the probability generating function of Y_i. Then

$$G(x) = \sum_{j\geq k+1} \mathbf{P}(Z = j)x^j = \frac{e^{-\lambda}}{f_{k+1}(\lambda)}\left(e^{\lambda x} - \sum_{j=0}^{k}\frac{(\lambda x)^j}{j!}\right) \leq \frac{e^{\lambda x-\lambda}}{f_{k+1}(\lambda)},$$

for all $x \geq 0$. For any nonnegative integer ℓ,

$$\mathbf{P}\left(\sum_{i=0}^{s} Y_i = l\right) \leq \frac{G(x)^s}{x^\ell}, \quad \forall x \geq 0.$$

Putting $x = \ell/s\lambda$ gives

$$\mathbf{P}\left(\sum_{i=0}^{s} Y_i = \ell\right) \leq \frac{e^{\ell-\lambda s}}{(\ell/(\lambda s))^\ell f_{k+1}(\lambda)^s} = \left(\frac{es\lambda}{\ell}\right)^\ell \left(\frac{e^{-\lambda}}{f_{k+1}(\lambda)}\right)^s. \tag{5.4}$$

It is easy to check that the right hand side of (5.4) is an increasing function of l when $l \leq \lambda s$ and decreasing function of l when $l \geq \lambda s$. By Proposition 3.2, there exists a constant N_0 depending only on δ such that provided $k > N_0$, $(1-\delta)\mu < \lambda$. Thus, for any $\ell \leq (1-\delta)\mu s$,

$$\mathbf{P}\left(\sum_{i=0}^{s} Y_i = \ell\right) \leq \left(\frac{es\lambda}{(1-\delta)\mu s}\right)^{(1-\delta)\mu s}\left(\frac{e^{-\lambda}}{f_{k+1}(\lambda)}\right)^s,$$

and so

$$\mathbf{P}\left(\sum_{i=1}^{s} Y_i \leq (1-\delta)\mu s\right) \leq \mu s\left(\frac{e\lambda}{(1-\delta)\mu}\right)^{(1-\delta)\mu s}\left(\frac{e^{-\lambda}}{f_{k+1}(\lambda)}\right)^s.$$

The expectation of Y_1 is $\lambda f_k(\lambda)/f_{k+1}(\lambda) = \mu$. By Proposition 3.2, we have $\mu \geq \lambda$ and $\mu - \lambda \to 0$ as $k \to \infty$. Therefore

$$\begin{aligned}\mathbf{P}\left(\sum_{i=1}^{s} Y_i \leq (1-\delta)\mu s\right) &\leq \mu s\left(\frac{\exp(\mu-\lambda-\delta\mu)}{(1-\delta)^{(1-\delta)\mu}f_{k+1}(\lambda)}\right)^s \\ &= \mu s\left(\frac{\exp(\mu-\lambda)}{f_{k+1}(\lambda)}\cdot\left(\frac{\exp(-\delta)}{(1-\delta)^{(1-\delta)}}\right)^\mu\right)^s.\end{aligned}$$

Since $0 < \delta < 1$,

$$0 < \frac{\exp(-\delta)}{(1-\delta)^{(1-\delta)}} < 1.$$

Since

$$\exp(\mu - \lambda) \to 1, \quad f_{k+1}(\lambda) \to 1, \quad \text{as } k \to \infty$$

by Proposition 3.2, there exists $N_1 > 0$ and $0 < \alpha_1 < 1$, both depending only on δ, such that provided $k > N_1$,

$$\mathbf{P}\left(\sum_{i=1}^{s} Y_i \leq (1-\delta)\mu s\right) \leq \alpha_1^{\mu s}.$$

Now we bound the upper tail of $\sum_{i=1}^{s} Y_i$. Let $j = 1, 2, \ldots$. For any ℓ satisfying $(1+j)\mu s \leq \ell < (2+j)\mu s$, as with the lower tail bound,

$$\mathbf{P}\left(\sum_{i=0}^{s} Y_i = \ell\right) \leq \left(\frac{es\lambda}{(1+j)\mu s}\right)^{(1+j)\mu s} \left(\frac{e^{-\lambda}}{f_{k+1}(\lambda)}\right)^{s} = \left(\frac{\exp(\mu-\lambda)}{f_{k+1}(\lambda)} \cdot \left(\frac{e^{j}}{(1+j)^{(1+j)}}\right)^{\mu}\right)^{s},$$

and so

$$\mathbf{P}\left((1+j)\mu s \leq \sum_{i=1}^{s} Y_i < (2+j)\mu s\right) \leq \mu s \left(\frac{\exp(\mu-\lambda)}{f_{k+1}(\lambda)} \cdot \left(\frac{e^{j}}{(1+j)^{(1+j)}}\right)^{\mu}\right)^{s}. \qquad (5.5)$$

Similarly we have

$$\mathbf{P}\left((1+\delta)\mu s \leq \sum_{i=1}^{s} Y_i < 2\mu s\right) \leq \mu s \left(\frac{\exp(\mu-\lambda)}{f_{k+1}(\lambda)} \cdot \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}\right)^{s}. \qquad (5.6)$$

Since $0 < e^{\delta}/(1+\delta)^{(1+\delta)} < 1$ for any $\delta > 0$, we may bound the right side of (5.5) and (5.6) by $\alpha_2^{\mu s}$ where $0 < \alpha_2 < 1$ is some constant depending only on δ. Also, since $e/(1+j) < 1$ for all $j \geq 2$, the right side of (5.5) is at most $\exp(-\Omega(\mu)js)$ for $j \geq 2$ provided k is large enough. Hence there exists $N_2 > 0$ and $0 < \alpha_3 < 1$ depending only on δ, such that provided $k > N_2$,

$$\mathbf{P}\left(\sum_{i=1}^{s} Y_i \geq (1+\delta)\mu s\right) \leq \alpha_3^{\mu s}.$$

The lemma follows by choosing $\alpha = \max\{\alpha_1, \alpha_3\}$ and $N = \max\{N_1, N_2\}$. ∎

Lemma 5.2 *Let $k \geq -1$ be an integer. Drop D balls independently at random into n bins. Let $\mu = D/n$ and let λ be defined as $\lambda f_k(\lambda) = \mu f_{k+1}(\lambda)$. Assume $D - (k+1)n \to \infty$ as $n \to \infty$. Then the probability that each bin contains at least $k+1$ balls is $\Omega(f_{k+1}(\lambda)^n)$.*

Proof. Let $\mathbf{d}$ denote $(d_1, \ldots, d_n)$. Let $\mathscr{D} = \{\mathbf{d} : d_i \geq k+1 \ \forall i \in [n], \ \sum_{i=1}^{n} d_i = D\}$. Let $\mathbf{P}(B)$ denote the probability that each bin contains at least $k+1$ balls. Then

$$\mathbf{P}(B) = \sum_{\mathbf{d} \in \mathscr{D}} \binom{D}{d_1, \ldots, d_n} \Big/ n^D = \frac{D!}{n^D} \sum_{\mathbf{d} \in \mathscr{D}} \prod_{i=1}^{n} \frac{1}{d_i!}.$$

Let $Y_1, \ldots, Y_n$ be n independent truncated Poisson variables which are copies of $Z_{(\geq k+1)}$ as defined in (3.2) with parameter λ satisfying $\lambda f_k(\lambda) = \mu f_{k+1}(\lambda)$. Then

$$\mathbf{P}\left(\sum_{i=1}^{n} Y_i = D\right) = \sum_{\mathbf{d}\in\mathscr{D}} \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{d_i}}{f_{k+1}(\lambda) d_i!} = \frac{e^{-\lambda n}\lambda^D}{f_{k+1}(\lambda)^n} \sum_{\mathbf{d}\in\mathscr{D}} \prod_{i=1}^{n} \frac{1}{d_i!}.$$

Since $D - (k+1)n \to \infty$ as $n \to \infty$, $\mathbf{P}\left(\sum_{i=1}^n Y_i = D\right) = \Omega(D^{-1/2})$ (see [24, Theorem 4(a)] for a short proof),

$$\sum_{\mathbf{d}\in\mathscr{D}} \prod_{i=1}^{n} \frac{1}{d_i!} = \Omega\left(\frac{e^{\lambda n} f_{k+1}(\lambda)^n}{\lambda^D D^{1/2}}\right).$$

So, using Stirling's formula,

$$\begin{aligned} \mathbf{P}(B) &= \Omega\left(\frac{D!}{n^D} \cdot \frac{e^{\lambda n} f_{k+1}(\lambda)^n}{\lambda^D D^{1/2}}\right) = \Omega\left(\sqrt{D}\left(\frac{D}{en}\right)^D \cdot \frac{e^{\lambda n} f_{k+1}(\lambda)^n}{\lambda^D D^{1/2}}\right) \\ &= \Omega\left(\left(\frac{\mu}{\lambda} e^{\lambda/\mu - 1}\right)^{\mu n} f_{k+1}(\lambda)^n\right). \end{aligned} \tag{5.7}$$

Since $(\mu/\lambda) \cdot e^{\lambda/\mu-1} \geq 1$, $\mathbf{P}(B) = \Omega(f_{k+1}(\lambda)^n)$. ∎

Corollary 5.3 *Let $k \geq -1$ be an integer. Let $\mathscr{D} = \{\mathbf{d} : d_i \geq k+1, \forall i \in [n], \sum_{i=1}^n d_i = D\}$ and let A_n be any subset of $\mathscr{D}$. Let $\mu = D/n$. Let $\mathbf{P}(A_n)$ denote the probability that the degree sequence $\mathbf{d}$ of $G \in \mathcal{M}(n, \mathbf{m}, k+1)$ is in A_n and let $\mathbf{P}_{TP}(A_n)$ be the probability that $(Y_1, \ldots, Y_n) \in A_n$ where Y_i are independent copies of the random variable $Z_{(\geq k+1)}$ as defined in (3.2) with the parameter λ satisfying $\lambda f_k(\lambda) = \mu f_{k+1}(\lambda)$. Assume $D - (k+1)n \to \infty$ as $n \to \infty$. Then*

$$\mathbf{P}(A_n) = O\left(\sqrt{D}\right) \mathbf{P}_{TP}(A_n).$$

Proof. Let A_n be any subset of $\mathscr{D}$ and let $\mathbf{P}(B)$ denote the probability that each bin contains at least $k+1$ balls by dropping D balls independently and randomly into n bins. Consider the partition-allocation model that generates $\mathcal{M}(n, \mathbf{m}, k+1)$, which allocates the partitioned D balls randomly into n bins with the restriction that each bin contains at least $k+1$ balls. Then

$$\mathbf{P}(A_n) = \sum_{\mathbf{d}\in A_n} \frac{1}{\mathbf{P}(B)} \cdot \binom{D}{d_1, \ldots, d_n} \Big/ n^D = \frac{D!}{n^D \mathbf{P}(B)} \sum_{\mathbf{d}\in A_n} \prod_{i=1}^{n} \frac{1}{d_i!},$$

and

$$\mathbf{P}_{TP}(A_n) = \sum_{\mathbf{d}\in A_n} \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{d_i}}{f_{k+1}(\lambda) d_i!} = \frac{e^{-\lambda n}\lambda^D}{f_{k+1}(\lambda)^n} \sum_{\mathbf{d}\in A_n} \prod_{i=1}^{n} \frac{1}{d_i!}.$$

Therefore

$$\mathbf{P}(A_n) = \frac{D! e^{\lambda n} f_{k+1}^n}{n^D \mathbf{P}(B) \lambda^D} \mathbf{P}_{TP}(A_n) = O\left(\sqrt{D}\right) \mathbf{P}_{TP}(A_n),$$

since $\mathbf{P}(B) = \Omega\left(\left(\frac{\mu}{\lambda}e^{\lambda/\mu-1}\right)^{\mu n} f_{k+1}(\lambda)^n\right)$ by Lemma 5.2 (5.7). ∎

A significant difficulty in this work is to ensure that various constants do not depend on the choice of ϵ. In particular, we emphasize that the constants such as α and N in the following results do not depend on ϵ.

The next is a corollary of Lemma 5.1 and Corollary 5.3.

Corollary 5.4 *Let μ be defined as in (5.1). Let $0 < \delta < 1$ be any constant. Then there exist two constants $N > 0$ and $0 < \alpha < 1$, both depending only on δ, such that, provided $k > N$, for any vertex set $S \subset V(G)$ with $|S| \geq \log^2 n$,*

$$\mathbf{P}(|d(S) - \mu|S|| \geq \delta\mu|S|) \leq \alpha^{\mu|S|}.$$

Proof. Let $Y_1, \ldots, Y_n$ be independent copies of the truncated Poisson random variable Z as defined in (3.2). Let $S \subset V(G)$ and let $s = |S|$. Then by Lemma 5.1, there exist $N > 0$ and $0 < \hat{\alpha} < 1$, both depending only on δ, such that provided $k > N$,

$$\mathbf{P}\left(\left|\sum_{i\in S} Y_i - \mu s\right| \geq \delta\mu s\right) \leq \hat{\alpha}^{\mu s},$$

By Corollary 5.3,

$$\mathbf{P}(|d(S) - \mu s| \geq \delta\mu s) \quad \leq \quad O(D^{1/2})\hat{\alpha}^{\mu s} = \left(\exp\left(\frac{\ln\Theta(\sqrt{\mu n})}{\mu s}\right)\hat{\alpha}\right)^{\mu s}.$$

Since $s \geq \log^2 n$ and so

$$\frac{\ln\Theta(\sqrt{\mu n})}{\mu s} \to 0, \quad \text{as } n \to \infty.$$

Let $\alpha = 1/2 + \hat{\alpha}/2$. Then $0 < \hat{\alpha} < \alpha < 1$ and α depends only on δ. Then provided $k > N$, $\mathbf{P}(|d(S) - \mu s| \geq \delta\mu s) \leq \alpha^{\mu s}$. ∎

The following corollary shows that $d(S)$ is very concentrated when S is not too small.

Corollary 5.5 *Let $\delta > 0$ and $0 < \gamma < 1$ be arbitrary constants. Then there exists a constant N depending only on δ and γ, such that provided $k > N$,*

$$\mathbf{P}(\exists S \subset V(G), s \geq \gamma n, |d(S) - \mu s| \geq \delta\mu s) = o(1).$$

Proof. For any $S \subset V(G)$, let $s = |S|$. By Corollary 5.4, there exists $N_1 > 0$ and $0 < \alpha < 1$, both depending only on δ, such that provided $k > N_1$, for any $S \subset V(G)$,

$$\mathbf{P}(|d(S) - \mu s| \geq \delta\mu s) \leq \alpha^{\mu s}.$$

Let N_2 be the smallest integer such that $e\alpha^{N_2}/\gamma < 1/2$. Let $N = \max\{N_1, N_2\}$. Then N depends only on δ and γ. For all $\mu > N$,

$$\mathbf{P}(\exists S \subset [n], s \geq \gamma n, |d(S) - \mu s| \geq \delta\mu s) \leq \sum_{\gamma n \leq s \leq n} \binom{n}{s} \alpha^{\mu s} \leq \sum_{\gamma n \leq s \leq n} \left(\frac{en}{s} \cdot \alpha^{\mu}\right)^s$$
$$\leq \sum_{\gamma n \leq s \leq n} \left(\frac{e}{\gamma} \cdot \alpha^{\mu}\right)^s = O\left(2^{-\gamma n}\right) = o(1). \quad \blacksquare$$

The following lemma will be used later to prove that a.a.s. $\sum_{j=0}^{w-1}(w-j)q_{h-j}/(h-j) \geq (1-\delta)ks$ provided k is large enough.

Lemma 5.6 *Let $\mathcal{C} = \{c_0, \ldots, c_{w-1}\}$ be a set of colours. Suppose that D balls are each coloured with some colour in $\mathcal{C}$, and let p_j denote the proportion of balls that are coloured c_j ($0 \leq j \leq w-1$). Randomly choose a subset of q of the balls. Let q_j be the number of balls chosen that are coloured with c_j. Then for any $0 \leq j \leq w-1$ and $0 < \delta < 1$,*

$$\mathbf{P}(|q_j - p_j q| \geq \delta p_j q) \leq \exp(-\Omega(\delta^2 p_j q)).$$

Proof. For any $0 \leq j \leq w-1$ any $\ell > 0$,

$$\mathbf{P}(q_j = \ell) = \binom{p_j D}{\ell}\binom{D - p_j D}{q - \ell} \Big/ \binom{D}{q}.$$

Let p_ℓ denote $\mathbf{P}(q_j = \ell)$. Put $\ell_0 = p_j q$, $\ell_1 = (1-\delta/2)p_j q$ and $\ell_2 = (1-\delta)p_j q$. Then for any $\ell \leq \ell_1$,

$$\frac{p_{\ell-1}}{p_\ell} = \frac{\ell(D(1-p_j) - q + \ell)}{(p_j D - \ell + 1)(q - \ell + 1)} \leq \frac{\ell_1(D(1-p_j) - q + \ell_0)}{(p_j D - \ell_0)(q - \ell_0)} = 1 - \frac{\delta}{2}.$$

Then

$$p_{\ell_2} \leq (1-\delta/2)^{\delta p_j q/2} p_{\ell_1} \leq \exp\left(\frac{\delta p_j q}{2} \ln\left(1 - \frac{\delta}{2}\right)\right) \leq \exp(-\delta^2 p_j q/4).$$

So

$$\mathbf{P}(q_j \leq (1-\delta)p_j q) = \sum_{\ell \leq \ell_2} p_\ell \leq \frac{1}{\delta} p_{\ell_2} \leq \exp(-\Omega(\delta^2 p_j q)).$$

Similarly we can bound the upper tail and then Lemma 5.6 follows. $\blacksquare$

Lemma 5.7 *Let $0 < \delta < 1$ and $0 < \gamma < 1$ be two arbitrary constants. Given $S \subset V(G)$, let $q_{h-j} = q_{h-j}(S)$ be as defined in (4.5). Then there exists $N > 0$ depending only on δ and γ such that for all $k > N$,*

$$\mathbf{P}\left(\exists S \subset V(G), |S| \geq \gamma n, \sum_{j=0}^{w-1} \frac{w-j}{h-j} q_{h-j} < (1-\delta)k|S|\right) = o(1).$$

Proof. For any $0 \leq j \leq w-1$, let p_j denoted $(h-j)m_{h-j}/D$. Let $J := \{j : p_j > \delta/8w\}$. We first show that given $S \subset V(G)$ with $|S| \geq \gamma n$, if

$$\sum_{j=0}^{w-1} \frac{w-j}{h-j} q_{h-j} < (1-\delta)k|S|, \tag{5.8}$$

then there exists $j \in J$ such that $q_{h-j}(S) \le (1-\delta/8)p_j d(S)$. Assume there is no such j by contradiction. Then

$$
\begin{aligned}
\sum_{j=0}^{w-1} \frac{w-j}{h-j} q_{h-j}(S) &\ge \sum_{j\in J} \frac{w-j}{h-j} q_{h-j}(S) > (1-\delta/8)d(S) \sum_{j\in J} \frac{w-j}{h-j} p_j \\
&= (1-\delta/8)d(S) \left(\sum_{j=0}^{w-1} \frac{w-j}{h-j} p_j - \sum_{j\notin J} \frac{w-j}{h-j} p_j \right) \ge (1-\delta/8)d(S) \left(\sum_{j=0}^{w-1} \frac{w-j}{h-j} p_j - \frac{w}{h}\frac{\delta}{8w} \right) \\
&\ge (1-\delta/8)d(S) \sum_{j=0}^{w-1} \frac{w-j}{h-j} p_j (1-\delta/8) \ge (1-\delta/4)d(S) \sum_{j=0}^{w-1} \frac{w-j}{h-j} p_j. \qquad (5.9)
\end{aligned}
$$

Let $s = |S|$ and let $r = s/n$. Then by Corollary 5.5, there exists $N_2 > 0$ depending on δ and γ only, such that a.a.s. $d(S) \ge (1-\delta/4)Dr$ whenever $k > N_2$. Therefore, combining with (5.9), we get a.a.s. provided $k > \max\{N_1, N_2\}$,

$$
\sum_{j=0}^{w-1} \frac{w-j}{h-j} q_{h-j}(S) > (1-\delta/2)Dr \sum_{j=0}^{w-1} \frac{w-j}{h-j} p_j \ge (1-\delta/2)r(kn-2\epsilon n) = (1-\delta/2)(k-2\epsilon)s.
$$

For any $k > 2/\delta \ge 4\epsilon/\delta$, we have $(1-\delta/2)(k-2\epsilon)s > (1-\delta)ks$. Take $N = \max\{N_1, N_2, 2/\delta\}$. Then for any $k > N$, we have a.a.s.

$$
\sum_{j=0}^{w-1} \frac{w-j}{h-j} q_{h-j}(S) > (1-\delta)ks,
$$

which contradicts (5.8). It follows that there exists $j \in J$ such that $q_{h-j}(S) \le (1-\delta/8)p_j d(S)$.

Consider the partition-allocation model that generates $\mathcal{P}([n], \mathbf{M}, \mathbf{0}, k+1)$. Let $\mathcal{C} = \{c_0, \ldots, c_{w-1}\}$ be a set of colours. For balls partitioned into parts that are of size $h-j$ for some $0 \le j \le w-1$, colour them with c_j. Then the w colours are distributed u.a.r. among the D balls. By Lemma 5.6, for any $S \subset V(G)$,

$$
\mathbf{P}\big(q_{h-j}(S) \le (1-\delta/8)p_j d(S)\big) \le \exp\big(-\Omega(\delta^2 p_j d(S))\big).
$$

Then there exists a constant N_1 depending only on δ and γ such that,

$$
\begin{aligned}
&\mathbf{P}\big(\exists S, j \in J, s \ge \gamma n, q_{h-j}(S) \le (1-\delta/8)p_j d(S)\big) \\
&\quad \le w 2^n \exp\big(-\Omega(\delta^3 d(S))\big) \le w\left(2\exp(-\Omega(\delta^3 \gamma k))\right)^n = o(1).
\end{aligned}
$$

Note that the inequality holds because $|J| \le w$, the number of sets S with $|S| \ge \gamma n$ is at most 2^n, $\delta/8w \le p_j < 1$ for all $j \in J$ and $d(S) \ge (k+1)|S| > k\gamma n$. It follows that a.a.s. there exists no set S with $|S| \ge \gamma n$ for which there exists $j \in J$ such that $q_{h-j}(S) \le (1-\delta/8)p_j d(S)$. Lemma 5.7 then follows. ∎

Recall that $\rho(S)$ is the number of hyperedges partially contained in S and $\nu(S)$ is the number of hyperedges intersecting S by the definition above Lemma 4.7.

Lemma 5.8 *Let $\delta > 0$ be any constant and let $\mu = \mu(G) = D/n$ as defined in (5.1). Then there exists a constant $N > 0$ depending only on δ such that, provided $k > N$, a.a.s. there exists no $S \subset V(G)$ for which $\log^2 n \le |S| \le n$, $d(S) < (1-\delta)\mu|S|$, and $\nu(S) < k|S|$.*

This lemma will be proved after the proof of Theorem 2.6.

Proof of Theorem 2.6. By Lemma 4.6 and 4.7, it is enough to show that the expected number of sets S contained in a hypergraph $G \in \mathcal{M}(n, \mathbf{m}, k+1)$ with property $\mathcal{A}(\gamma)$ for which all of Lemma 4.7 (i)–(iv) are satisfied is $o(1)$. We call a set $S \subset V(G)$ is interesting if it lies in a hypergraph G with property $\mathcal{A}(\gamma)$. Let X be the number of interesting sets $S \subset V(G)$ such that (4.6) holds. Similarly, let $X_{<a}$ (or $X_{>b}$ or $X_{[a,b]}$) for any $0 < a < b < n$ denote the number of interesting $S \subset [n]$ such that (4.6) holds and $|S| < a$ (or $|S| > b$ or $a \le |S| \le b$) respectively. For any set S under discussion, let s denote $|S|$ and $\bar{s}$ denote $|\overline{S}|$.

Case 1: $s < \epsilon n/k$. By theorem's hypothesis

$$\left(\sum_{j=0}^{w-1}(w-j)m_{h-j}\right) - kn < -\epsilon n,$$

any S satisfying (4.6) must satisfy

$$\partial^*(S) < ks - \epsilon n. \tag{5.10}$$

When $s < \epsilon n/k$, $ks - \epsilon n < 0$. However $\partial^*(S) \ge 0$ as observed below (4.4). Hence (5.10) cannot hold. Thus $X_{<\epsilon n/k} = 0$.

Case 2: $s > (1-\gamma)n$. part (i) of Lemma 4.7 says that (4.6) holds only if the number of hyperedges partially contained in $\overline{S}$ is at least $k\bar{s}/w$. But X counts only interesting sets, i.e. sets that lie in a hypergraph with property $\mathcal{A}(\gamma)$. By the definition of property $\mathcal{A}(\gamma)$, there are no such interesting sets and so $X_{\ge(1-\gamma)n} = 0$.

Case 3: $\epsilon n/k \le s < \gamma n$. Let $\delta_1 = (h-w)/2h$. By Lemma 5.8, there exists $N_1 > 0$ such that provided $k > N_1$, the expected number of S with $d(S) < (1-\delta_1)\mu s$ for which Lemma 4.7 (ii) is satisfied and $\epsilon n/k \le s \le n$ is $o(1)$. We now show that there exists no interesting sets $S \subset V(G)$ with $|S| < \gamma n$ for which Lemma 4.7 (iii) holds and $d(S) \ge (1-\delta_1)\mu s$. If $d(S) \ge (1-\delta_1)\mu s$, $d(S) \ge \frac{h+w}{2w}ks$ provided $k \ge h+w$ since $\mu \ge h(k-1)/w$ by (5.3). Then it follows that

$$\frac{d(S) - ks}{h-w} \ge \frac{ks}{2w}.$$

Lemma 4.7 (iii) implies that (4.6) holds only if the number of hyperedges partially contained in S is at least $ks/2w$. By the definition of property $\mathcal{A}(\gamma)$, there is no such interesting sets S when $s < \gamma n$. So provided $k > \max\{N_1, h+w\}$, a.a.s. there exists no interesting sets S, with $s < \gamma n$ for which both Lemma 4.7 (ii) and (iii) hold. Then $\mathbf{E}(X_{<\gamma n}) = o(1)$.

Note that $k|S|/2w$ in the definition of property $\mathcal{A}(\gamma)$ can be modified to be $Ck|S|$ for any positive constant C, and it can be checked straightforwardly that there exists a constant γ depending on C only, such that Corollary 4.3 holds. Therefore, any $0 < \delta_1 < 1 - w/h$ would work here by choosing some appropriate C to modify the definition of property $\mathcal{A}(\gamma)$.

Case 4: $\gamma n \le s \le (1-\gamma)n$. Let $0 < \delta_2 < 1$ be chosen later. By Lemma 5.7, there exists $N_2 > 0$ depending only on δ_2 such that provided $k \ge N_2$, a.a.s.

$$\sum_{j=0}^{w-1} \frac{w-j}{h-j} q_{h-j} \ge (1-\delta_2)ks \quad \text{for all } S \text{ with } \gamma n \le |S| \le (1-\gamma)n.$$

For any $S \subset V(G)$, let $\eta = \eta(S)$ be as defined in (4.5). Then by Lemma 4.7, to show $\mathbf{E}(X_{[\gamma n,(1-\gamma)n]}) = o(1)$, it is enough to show that the expected number of sets S with $\gamma n \le s \le (1-\gamma)n$ for which $\eta(S)$ is at most $h^2\delta_2 ks$, is $o(1)$. Consider the probability space $\mathcal{M}(n, \mathbf{m}, 0)$, which is generated by placing each hyperedge uniformly and randomly on the n vertices. Let B be the event that all bins contain at least $k+1$ balls. Then $\mathcal{M}(n, \mathbf{m}, k+1)$ equals $\mathcal{M}(n, \mathbf{m}, 0)$ conditioned on the event B. By Lemma 5.2 $\mathbf{P}(B) = \Omega(f_{k+1}(\lambda)^n)$ where $\lambda f_k(\lambda) = \mu f_{k+1}(\lambda)$. Given any set S, let $r = s/n$. For any hyperedge of size $h-j$, the probability for it to intersect both S and $\overline{S}$ is $p_{j,r} = 1 - r^{h-j} - (1-r)^{h-j}$. Then $p_{j,r} \ge 1 - \gamma^{h-j} - (1-\gamma)^{h-j} \ge 1 - \gamma^{h-w+1} - (1-\gamma)^{h-w+1}$ for any set S and any $0 \le j \le w-1$. Recall from (5.1) that m is the total number of hyperedges in G. Then $\mathbf{E}\eta(S) = \sum_{j=0}^{w-1} p_{j,r} m_{h-j} \ge m(1 - \gamma^{h-w+1} - (1-\gamma)^{h-w+1})$ for any given S. Since $m \ge (k-1)n/w$ by (5.3),

$$\mathbf{E}\eta(S) \ge (1-\gamma^{h-w+1}-(1-\gamma)^{h-w+1})(k-1)n/w = \Theta_\gamma(k)n, \text{ for any } S \text{ with } \gamma n \le |S| \le (1-\gamma)n.$$

Choose

$$\delta_2 = \frac{1-\gamma^{h-w+1}-(1-\gamma)^{h-w+1}}{4wh^2(1-\gamma)}.$$

Then δ_2 depends only on γ and so N_2 also depends only on γ. By the Chernoff bound [7],

$$\mathbf{P}(\eta(S) < h^2\delta_2 ks) \le \mathbf{P}(\eta(S) < h^2\delta_2 k(1-\gamma)n) \le \mathbf{P}\left(\eta(S) < \frac{1}{2}\mathbf{E}\eta(S)\right) \le \exp(-\mathbf{E}\eta(S)/16).$$

Note that the second inequality holds because of the choice of δ_2. So there exists some constant $C > 0$ s.t.

$$\mathbf{P}(\eta(S) < h^2\delta_2 ks \mid B) \le C\exp\left(-\mathbf{E}\eta(S)/16\right) f_{k+1}(\lambda)^{-n} = C\left(\exp\left(-\frac{\mathbf{E}\eta(S)}{16n} - \ln f_{k+1}(\lambda)\right)\right)^n.$$

The number of sets S with $\gamma n \le |S| \le (1-\gamma)n$ is at most 2^n. So the expected number of sets S with $\gamma n \le s \le (1-\gamma)n$ and $\eta(S) < h^2\delta_2 ks$ in $\mathcal{M}(n, \mathbf{m}, k+1)$ is at most

$$C\left(2\exp\left(-\frac{\mathbf{E}\eta(S)}{16n} - \ln f_{k+1}(\lambda)\right)\right)^n.$$

Clearly $f_{k+1}(\lambda) \to 1$ as $k \to \infty$ and $\mathbf{E}\eta(S) = \Theta_\gamma(k)n$ as observed before. Then there exists a constant $N_3 > 0$ depending only on γ such that provided $k > N_3$,

$$2\exp\left(-\frac{\mathbf{E}\eta(S)}{16n} - \ln f_{k+1}(\lambda)\right) < 1.$$

Then provided $k \geq \max\{N_2, N_3\}$,

$$\mathbf{E}(X_{[\gamma n,(1-\gamma)n]}) = o(1).$$

Combining all cases, let $N = \max\{N_1, N_2, N_3, h+w\}$. Then N depends only on γ. We have shown that provided $k > N$, $\mathbf{E}X = o(1)$. Then Theorem 2.6 follows. ∎

Proof of Lemma 5.8. The idea of the proof is as follows. When S is big, by Corollary 5.5 there are no such sets with $d(S) < (1-\delta)\mu|S|$. We will see later that $\nu(S) < k|S|$ requires a lot of hyperedges partially contained in S, which is unlikely to happen when S is small enough.

Let $G \in \mathcal{M}(n, \mathbf{m}, k+1)$. Let $D = \sum_{j=0}^{w-1}(h-j)m_{h-j}$ and $\mu = D/n$ as defined in (5.1). For any S, let $\rho(S,i)$ denotes the number of hyperedges with exactly i vertices contained in S. Then $\nu(S) < ks$ if and only if $\sum_{i=2}^{h}(i-1)\rho(S,i) > d(S) - ks$. By Corollary 5.5, there exists $N_1 > 0$ depending only on δ such that provided $k > N_1$, a.a.s. there is no S such that $s > n/h$ and $d(S) < (1-\delta)\mu s$. So we only need to consider sets S with $|S| \leq n/h$. We call a vertex set $S \in G$ *bad* if $\log^2 n \leq |S| \leq n/h$, $d(S) < (1-\delta)\mu|S|$ and $\sum_{i=2}^{h}(i-1)\rho(S,i) > d(S) - k|S|$. Let s denote $|S|$.

For any given S, let $p(S)$ denote the probability of S being bad. By Corollary 5.4, there exists $N_2 > 0$ and $0 < \alpha < 1$, both depending only on δ, such that provided $k > N_2$, the probability that $d(S) < (1-\delta)\mu s$ is at most $\alpha^{\mu s}$. Let $p(q,t)$ be the probability that that $\sum_{i=2}^{h}(i-1)\rho(S,i)$ is at least t conditional on $d(S) = q$. Then

$$p(S) = \sum_{(k+1)s \leq q \leq (1-\delta)\mu s} p(q, q-ks)\mathbf{P}(d(S) = q). \tag{5.11}$$

For the small value of q (or s), we need the following claim, to be proved later.

Claim 5.9 *If $q < D/h$, then*

$$p(q,t) \leq \left(\exp\left(\frac{h \ln t}{t}\right) \frac{eh(h-1)^2 q^2}{4tD}\right)^t.$$

In particular, if $t \to \infty$ as $n \to \infty$, then

$$p(q,t) \leq \left(\frac{eh^3 q^2}{4tD}\right)^t.$$

Case 1: $s < 2n/eh^3(k+1)$. Since $(k+1)s \leq q \leq (1-\delta)\mu s < D/h$, we have

$$\frac{q}{q-ks} \leq \frac{(k+1)s}{(k+1)s - ks} = k+1, \qquad \frac{q}{D} \leq \frac{(1-\delta)\mu s}{\mu n} < \frac{s}{n}, \quad q - ks \geq s \geq \log^2 n.$$

So $q - ks \to \infty$ as $n \to \infty$. By (5.11) and the particular case of Claim 5.9, we have

$$\begin{aligned} p(S) &\leq \sum_{(k+1)s \leq q \leq (1-\delta)\mu s} \left(\frac{eh^3(k+1)s}{4n}\right)^{q-ks} \mathbf{P}(d(S) = q) \\ &\leq \left(\frac{eh^3(k+1)s}{4n}\right)^s \mathbf{P}\Big((k+1)s \leq d(S) \leq (1-\delta)\mu s\Big) \leq \left(\frac{eh^3(k+1)s}{4n}\right)^s \alpha^{\mu s}. \end{aligned}$$

Note that the second inequality above holds because $q - ks \geq s$ and $0 < eh^3(k+1)s/4n < 1$ since $s < 2n/eh^3(k+1)$.

Then the expected number of bad sets S with $|S| = s$, for any fixed $\log^2 n \leq s < 2n/eh^3(k+1)$, is at most

$$\binom{n}{s}\left(\frac{eh^3(k+1)s}{4n}\right)^s \alpha^{\mu s} \leq \left(\frac{en}{s} \cdot \alpha^{\mu} \cdot \frac{eh^3(k+1)s}{4n}\right)^s = \left(e^2h^3(k+1)\alpha^{\mu}/4\right)^s.$$

Since $\mu \geq h(k-1)/w$ by (5.3), this is at most $\exp(-s)$ provided $k \geq N_3$ for some $N_3 > 0$ depending only on α.

Case 2: $s \geq 2n/eh^3(k+1)$. Take $p(q, q-ks) \leq 1$ since $p(q, q-ks)$ is a probability. So the expected number of bad sets S with $|S| = s$, for any fixed $2n/eh^3(k+1) \leq s \leq (1-\delta)\mu s$, is at most

$$\binom{n}{s} \cdot \alpha^{\mu s} = \left(\frac{en}{s}\alpha^{\mu}\right)^s \leq \left(e^2h^3(k+1)\alpha^{\mu}/2\right)^s \leq \exp(-s),$$

whenever $k > N_4$ for some $N_4 > 0$ depending only on α. Since α depends only on δ, N_3 and N_4 also depend only on δ. Let $N = \max\{N_1, N_2, N_3, N_4\}$. Then N depends only on δ and provided $k > N$, the expected number of bad S is at most

$$\sum_{\log^2 n \leq s \leq n/h} \exp(-s) = o(1).$$

Lemma 5.8 follows. ∎

It only remains to prove Claim 5.9.

Proof of Claim 5.9. To illustrate the method of computing $p(q,t)$, we show in detail the case $h = 2$ first. Conditional on that $d(S) = q$, we want to estimate the probability that there are at least t edges in S. Consider the alternative algorithm that generates the probability space of the partition-allocation model $\mathcal{P}([n], [m_2], 0, k+1)$. Fix any allocation which allocates q balls into bins representing vertices in S with each bin containing at least $k+1$ balls. There are at most

$$\binom{q}{2t}\frac{(2t)!}{2^t t!}$$

partial partitions that contain t parts within S. The probability of every such partial partition to occur is

$$\prod_{i=0}^{t-1} \frac{1}{D-1-2i}.$$

So

$$p(q,t) \leq \binom{q}{2t}\frac{(2t)!}{2^t t!} \cdot \prod_{i=0}^{t-1}\frac{1}{D-1-2i},$$

which is at most

$$\frac{[q]_t}{2^t t!}\prod_{i=0}^{t-1}\frac{q-t-i}{D-1-2i} \leq \left(\frac{eq}{2t}\right)^t \left(\frac{q-t}{D-1}\right)^t \leq \left(\frac{eq}{2t} \cdot \frac{q}{D}\right)^t$$

Note that the second inequality holds since $q < D/2$ and so $q - t < (D-1)/2$.

Now we estimate $p(q,t)$ in the general case $h \geq 2$. Let $\mathbf{M} = ([2m_2], \ldots, [hm_h])$. Consider the alternative algorithm that generates the probability space of the partition-allocation model $\mathcal{P}([n], \mathbf{M}, \mathbf{0}, k+1)$, defined in Section 3. Fix any allocation that allocates exactly q balls into S with each bin containing at least $k+1$ balls. The algorithm uniformly randomly partitions balls into parts such that there are exactly m_{h-j} parts with size $h-j$ for $j = 0, \ldots, w-1$. Let $\mathcal{U} = \{(u_2, \ldots, u_h) \in I\!N^{(h-1)} : \sum_{i=2}^{h}(i-1)u_i = t\}$. Let $\mathbf{u} = (u_2, \ldots, u_h)$ be an arbitrary vector from $\mathcal{U}$. We over estimate the probability that $\rho(S,i)$ is at least u_i for all $i = 2, \ldots, h$, conditional on $d(S) = q$. Let $p(q,\mathbf{u})$ denote this probability. Then clearly $p(q,t) \leq \sum_{\mathbf{u} \in \mathcal{U}} p(q,\mathbf{u})$. The number of partial partitions that contain u_i partial parts of size i within S is

$$\binom{q}{u_1, 2u_2, 3u_3, \ldots, hu_h} \frac{(2u_2)!}{2!^{u_2} u_2!} \cdots \frac{(hu_h)!}{h!^{u_h} u_h!}, \tag{5.12}$$

where $u_1 = q - \sum_{i=2}^{h} iu_i$. For any such partial partition we compute the probability that it occurs. The algorithm starts from picking a ball v unpartitioned in S and then it chooses at most $h-1$ balls that are u.a.r. chosen from all the unpartitioned balls to be partitioned into the part containing v.

The probability of the occurrence of a given u_2 partial parts of size 2 within S is at most

$$\prod_{i=0}^{u_2}(h-1)\frac{1}{D-1-hi} = (h-1)^{u_2}\frac{1}{D-1} \cdot \frac{1}{D-h-1} \cdots \frac{1}{D-1-h(u_2-1)}.$$

The probability of the occurrence of a given u_3 partial parts of size 3 within S is at most

$$\prod_{i=0}^{u_3-1}\binom{h-1}{2}\frac{1}{D-hu_2-hi-1} \cdot \frac{1}{D-hu_2-hi-2} \leq (h-1)^{2u_3}\prod_{i=0}^{u_3-1}\frac{1}{(D-hu_2-hi-1)^2}.$$

Note that the above inequality holds because $h\sum_{i=2}^{h} u_i \leq hq/2 < D/2$. Keeping the analysis in this procedure, we obtain that the probability of a particular partial partition with u_i partial parts of size i within S is at most

$$\begin{aligned}(h-1)^{u_2+2u_3+\cdots+(h-1)u_h} &\times \prod_{i=0}^{u_2-1}\frac{1}{D-hi-1}\prod_{i=0}^{u_3-1}\frac{1}{(D-hu_2-hi-1)^2}\\ &\times \cdots \times \prod_{i=0}^{u_{h-1}-1}\frac{1}{(D-h\sum_{j=2}^{h-2}u_j-hi-1)^{h-1}}\prod_{i=0}^{u_{h-1}-1}\frac{1}{(D-h\sum_{j=2}^{h-1}u_j-hi-1)^{h-1}}.\end{aligned}$$

The product of this and (5.12) gives an upper bound of $p(q,\mathbf{u})$, which is at most

$$\frac{[q]_{\sum_{i=2}^{h} iu_i}(h-1)^t}{u_2!u_3!\cdots u_h!2!^{u_2}\cdots h!^{u_h}!}\prod_{i=0}^{u_2-1}\frac{1}{D-hi-1}\cdots\prod_{i=0}^{u_h-1}\frac{1}{(D-h\sum_{j=2}^{h-1}u_j-hi-1)^{h-1}}.$$

Since $2!^{u_2}\cdots h!^{u_h}! \geq 2^t$, this is at most

$$\frac{[q]_t(h-1)^t}{u_2!u_3!\cdots u_h!2^t}\prod_{i=0}^{u_2-1}\frac{q-t-i}{D-hi-1}\times\cdots\times\prod_{i=0}^{u_h-1}\frac{q-t-\sum_{j=2}^{h-1}u_j-i}{(D-h\sum_{j=2}^{h-1}u_j-hi-1)^{h-1}}.$$

Since $q < D/h$ and so $q - t \le (D-1)/h$, this is at most

$$\frac{(eq(h-1)/2)^t}{u_2^{u_2}\cdots u_h^{u_h}}\left(\frac{q-t}{D-1}\right)^{u_2}\left(\frac{q-t-u_2}{(D-hu_2-1)^2}\right)^{u_3}\cdots\left(\frac{q-t-\sum_{j=2}^{h-1}u_j}{(D-h\sum_{j=2}^{h-1}u_j-1)^{h-1}}\right)^{u_h}$$
$$= \frac{(eq(h-1)/2)^t}{u_2^{u_2}\cdots u_h^{u_h}(q-t-u_2)^{u_3}\cdots(q-t-\sum_{j=2}^{h-1}u_j)^{(h-2)u_h}}$$
$$\times\left(\frac{q-t}{D-1}\right)^{u_2}\left(\frac{q-t-u_2}{D-hu_2-1}\right)^{2u_3}\cdots\left(\frac{q-t-\sum_{j=2}^{h-1}u_j}{D-h\sum_{j=2}^{h-1}u_j-1}\right)^{(h-1)u_h}$$
$$\le \frac{(eq(h-1)/2)^t}{u_2^{u_2}(u_3(q-t-u_2))^{u_3}\cdots(u_h(q-t-\sum_{j=2}^{h-1}u_j)^{h-2})^{u_h}}\left(\frac{q-t}{D-1}\right)^t$$
$$\le \frac{(eq(h-1)/2)^t}{u_2^{u_2}(u_3(q-t-u_2))^{u_3}\cdots(u_h(q-t-\sum_{j=2}^{h-1}u_j)^{h-2})^{u_h}}\left(\frac{q}{D}\right)^t.$$

Since $q \ge \sum_{i=2}^h iu_i$ and $t = \sum_{i=2}^h (i-1)u_i$, $q - t - \sum_{j=2}^i u_j \ge \sum_{j=i+1}^h u_j \ge u_{i+1}$ for all $2 \le i \le h-1$, and so

$$u_2^{u_2}(u_3(q-t-u_2))^{u_3}\cdots\left(u_h\left(q-t-\sum_{j=2}^{h-1}u_j\right)^{h-2}\right)^{u_h} \ge u_2^{u_2}u_3^{2u_3}\cdots u_h^{(h-1)u_h}.$$

We prove the following claim later.

Claim 5.10 *Let $t = \sum_{j=2}^h (j-1)u_j$. Then*

$$u_2^{u_2}u_3^{2u_3}\cdots u_h^{(h-1)u_h} \ge \left(\frac{2t}{h(h-1)}\right)^t.$$

By Claim 5.10, for any $h \ge 2$,

$$p(q,\mathbf{u}) \le \left(\frac{eqh(h-1)^2}{4t}\cdot\frac{q}{D}\right)^t, \quad \forall \mathbf{u} \in \mathcal{U}.$$

Since $|\mathcal{U}| < t^h$, we have

$$p(q,t) \le t^h\left(\frac{eh(h-1)^2q^2}{4tD}\right)^t = \left(\exp\left(\frac{h\ln t}{t}\right)\frac{eh(h-1)^2q^2}{4tD}\right)^t.$$

In particular, if $t \to \infty$ as $n \to \infty$, then $h\ln t/t \to 0$ and so $\exp(h\ln t/t) \le (h/(h-1))^2$ provided n is large enough. So

$$p(q,t) \le \left(\frac{eh^3q^2}{4tD}\right)^t. \quad \blacksquare$$

Proof of Claim 5.10. We solve the following optimization problem

$$\begin{array}{lll} (P_1) & \min & m_2^{m_2} m_3^{2m_3} \cdots m_h^{(h-1)m_h} \\ & s.t. & m_2 + 2m_3 + \cdots + (h-1)m_h = t \\ & & m_2, m_3, \ldots, m_h \geq 0 \end{array}$$

Letting $x_i = (i-1)m_i$ for $2 \leq i \leq h$, and taking the logarithm of the objective function, (P_1) is equivalent to the following optimization problem.

$$\begin{array}{lll} (P_2) & \min & x_2 \ln x_2 + x_3 \ln(x_3/2) + \cdots + x_h \ln(x_h/(h-1)) \\ & s.t. & x_2 + x_3 + \cdots + x_h = t \\ & & x_2, x_3, \ldots, x_h \geq 0 \end{array}$$

For convention, let $x \ln x = 0$ if $x = 0$. Applying the Lagrange multiplier yields $\mathbf{x}^* = (x_2^*, x_3^*, \ldots, x_h^*)$ with $x_i = 2t(i-1)/h(h-1)$, which is a feasible solution of (P_2). In order to show that this is an optimal solution, we need to show that the optimal solution does not appear on the boundary.

Let $\mathbf{x}$ be any solution on the boundary of (P_2). Then there exists $2 \leq i \leq h$ such that $x_i = 0$. There also exists j with $x_j > 0$. Consider $\mathbf{x}'$ with $x_i' = (i-1)x_j/h$, $x_j' = x_j - (i-1)x_j/h$ and $x_l' = x_l$ for any $l \neq i, j$. Then $\mathbf{x}'$ is feasible and it is straightforward to check that

$$x_i' \ln(x_i'/(i-1)) + x_j' \ln(x_j'/(j-1)) < x_i \ln(x_i/(i-1)) + x_j \ln(x_j/(j-1)).$$

Hence $\mathbf{x}'$ cannot be an optimal solution. This proves that $\mathbf{x}^*$ is the minimizer and so the optimal value of (P_1) is $\exp(t \ln(2t/(h(h-1)))) = (2t/(h(h-1)))^t$. ∎

References

[1] N. Alon and J. Spencer. The probabilistic method. *Wiley-Interscience Series in Discrete Mathematics and Optimization.* John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2008.

[2] Y. Azar, A. Z. Broder, A. R. Karlin, and E. Upfal. Balanced allocations. *SIAM J. Comput.*, 29(1), pages 180–200, 1999.

[3] P. Berenbrink, A. Czumaj, A. Steger, and B. Vőcking. Balanced allocations: The heavily loaded case. *32th Annual ACM Symposium on Theory of Computing*, pages 745–754, 2000.

[4] Béla Bollobás and Alan M. Frieze. On matchings and hamiltonian cycles in random graphs. *Random graphs '83 (Poznan', 1983)*, pages 23–46. North-Holland Math. Stud., 118, North-Holland, Amsterdam, 1985.

[5] J. Cain, P. Sanders, and N. Wormald. The random graph threshold for k-orientiability and a fast algorithm for optimal multiple-choice allocation. *Symposium on Discrete Algorithms (SODA).* pages 469–476. January 2007.

[6] J. Cain and N. Wormald. Encores on cores. *Electronic Journal of Combinatorics*, 13, RP 81, 2006.

[7] Herman Chernoff. A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *Ann. Math. Statistics*, 23, pages 493-507, 1952.

[8] V. Chvátal. Almost all graphs with $1.44n$ edges are 3-colorable. *Random Structures Algorithms*, 2(1), pages 11-28, 1991.

[9] W.J. Cook, William.H. Cunningham, William R. Pulleyblank, and Alexander Schrijver. *Combinatorial optimization*. Wiley-Interscience Series in Discrete Mathematics and Optimization. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1998.

[10] M. Dietzfelbinger, A. Goerdt, M. Mitzenmacher, A. Montanari, R. Pagh and M. Rink. Tight Thresholds for Cuckoo Hashing via XORSAT. preprint available at `http://arxiv.org/abs/0912.0287`.

[11] M. Dietzfelbinger and F. Meyer auf der Heide. Simple, efficient shared memory simulations. *5th ACM Symposium on Parallel Algorithms and Architectures, Velen, Germany, SIGACT and SIGARCH*, pages 110–119, June 30-July 2, 1993.

[12] A. Frieze and P. Melsted. Maximum Matchings in Random Bipartite Graphs and the Space Utilization of Cuckoo Hashtables. preprint available at `http://arxiv.org/abs/0910.5535`.

[13] N. Fountoulakis and K. Panagiotou. Sharp Load Thresholds for Cuckoo Hashing. preprint available at `http://arxiv.org/abs/0910.5147`.

[14] D. Fernholz and V. Ramachandran. The k-orientability thresholds for $G_{n,p}$. *Proc. ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 459–468, January 2007.

[15] P. Gao. Generation and properties of random graphs and analysis of randomized algorithms. Ph.D. dissertation. Department of Combinotorics and Optimization, University of Waterloo. `http://hdl.handle.net/10012/4987`

[16] P. Gao and N.C. Wormald. Load Balancing and Orientability Thresholds for Random Hypergraphs. STOC'10, June 5–8, 2010, Cambridge, Massachusetts, USA. pages 97–103.

[17] S. L. Hakimi. On the degrees of the vertices of a directed graph. *J. Franklin Inst.*, 279, pages 290–308, 1965.

[18] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58, pages 13–30, 1963.

[19] R. M. Karp, M. Luby, and F. Meyer auf der Heide. Efficient pram simulation on a distributed memory machine. *24th ACM Symp. on Theory of Computing*, pages 318–326, May 1992.

[20] B.D. McKay and N.C. Wormald. Asymptotic enumeration by degree sequence of graphs with degrees $o(n^{1/2})$. *Combinatorica*, 11(4), pages 369–382, 1991.

[21] F. Meyer auf der Heide, C. Scheideler, and V. Stemann. Exploiting storage redundancy to speed up randomized shared memory simulations. *Theoret. Comput. Sci.*, 162(2), pages 245–281, August 1996.

[22] M. Mitzenmacher, A. Richa, and R. Sitaraman. The power of two random choices: a survey of techniques and results. *Handbook of randomized computing*, volume I, II, pages 255–312. Comb. Optim., 9, Kluwer Acad. Publ., Dordrecht, 2001.

[23] B. Pittel, J. Spencer and N. Wormald, Sudden emergence of a giant k-core in a random graph, *J. Combinatorial Theory, Series B*, 67, 1996, pages 111–151.

[24] B. Pittel and N. Wormald. Asymptotic enumeration of sparse graphs with a minimum degree constraint. *J. Combin. Theory Ser. A*, 101(2), pages 249–263, 2003.

[25] P. Sanders, S. Egner, and J. Korst. Fast concurrent access to parallel disks. *Algorithmica*, 35(1), pages 21–55, 2003.

[26] Alexander Schrijver. *Combinatorial optimization. Polyhedra and efficiency*, volume B. Algorithms and Combinatorics, 24, Springer-Verlag, Berlin, 2003.

[27] N.C. Wormald. Differential equations for random processes and random graphs. *The Annals of Applied Probability*, 5(4), pages 1217–1235, November 1995.

[28] N.C. Wormald. The differential equation method for random graph processes and greedy algorithms. M. Karonski and H.J. Proemel, editors, *Lectures on Approximation and Randomized Algorithms*, pages 73–155. PWN, Warsaw, 1999.

[29] N.C. Wormald, Random graphs and asymptotics. Section 8.2 in *Handbook of Graph Theory*, J.L. Gross and J. Yellen (eds), pp. 817–836. CRC, Boca Raton, 2004.