Counting connected graphs inside-out

Boris Pittel[*]
Department of Mathematics
Ohio State University
Columbus, Ohio 43210-1174
bgp@math.ohio-state.edu

Nicholas C. Wormald[†]
Department of Combinatorics and Optimization
University of Waterloo
Waterloo ON, Canada
nwormald@uwaterloo.ca

Abstract

The theme of this work is an "inside-out" approach to the enumeration of graphs. It is based on a well-known decomposition of a graph into its *2-core*, i.e. the largest subgraph of minimum degree 2 or more, and a forest of trees attached. Using our earlier (asymptotic) formulae for the total number of 2-cores with a given number of vertices and edges, we solve the corresponding enumeration problem for the connected 2-cores. For a subrange of the parameters, we also enumerate those 2-cores by using a deeper inside-out notion of a *kernel* of a connected 2-core.

Using this enumeration result in combination with Caley's formula for forests, we obtain an alternative and simpler proof of the asymptotic formula of Bender, Canfield and McKay for the number of connected graphs with n vertices and m edges, with improved error estimate for a range of m values.

As another application, we study the limit joint distribution of three parameters of the giant component of a random graph with n vertices in the supercritical phase, when the difference between average vertex degree and 1 far exceeds $n^{-1/3}$. The three parameters are defined in terms of the *2-core* of the giant component, i.e. its largest subgraph of minimum degree 2 or more. They are the number of vertices in the 2-core, the excess (#edges − #vertices) of the 2-core, and the number of vertices not in the 2-core. We show that the limit distribution is jointly Gaussian throughout the whole supercritical phase. In particular, for the first time, the 2-core size is shown to be asymptotically normal, in the widest possible range of the average vertex degree.

1 Introduction

Erdős and Rényi first considered the evolution of a random graph, in which n vertices begin life as isolated points and then edges are thrown in randomly one by one. It is well known that this evolving random graph undergoes a phase transition when the number of edges, $m = m(n)$, is around $n/2$: a

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"giant" component suddenly appears. If $(m-n/2)/n^{2/3} \to \infty$ this giant component is unique with probability 1. The interval between $(m-n/2)/n^{2/3} \to \infty$ and $m < cn$ for some $c > 1/2$ we call the supercritical phase. Properties of the giant component in the supercritical phase have been one of the fundamental areas of study in random graph theory. For instance, it was known that the size of the giant component is with high probability close to a known function of m and n, and that, when m reaches $n/2 + \epsilon n$ for any fixed $\epsilon > 0$, the size is asymptotically normally distributed.

The theme of the present paper is to approach this topic with an "inside-out" philosophy. We use the well-known decomposition of a connected graph into its 2-core (largest subgraph with minimum degree at least 2) with a forest of trees attached, and combine the enumeration of the 2-cores and the attached forests.

Our approach pays some dividends. Not only does it give more information about the structure of random connected graphs in the supercritical phase, but it also provides a simpler proof of the asymptotic formula of Bender et al. [4] for the number of connected graphs with n vertices and m edges. (Here, and throughout the paper, graphs are labelled.) By contrast, the approach in [4] was to study a differential equation derived from a recurrence relation for the numbers of these graphs.

The 2-core of a connected graph is necessarily connected. So, to begin with, we need to count connected graphs with minimum degree at least 2, asymptotically according to number of vertices and edges. This is so fundamental to our work that we pay it considerable attention.

For brevity, an (n, m)-*graph* is a graph with n vertices and m edges, and as in [16], we use *2-core* to denote any graph with minimum degree at least 2. We finish this section with a brief description of our methods and results on the three related topics: asymptotic enumeration of connected (n, m) 2-cores, asymptotic enumeration of connected (n, m)-graphs, and properties of the "giant" component in a supercritical random graph.

Denote by $C_2(n, m)$ the total number of 2-cores with n vertices and m edges, and by $C_2^{(1)}(n, m)$ the number of these which are connected. We use the enumerational results on $C_2(n, m)$ from [16] to determine an asymptotic formula for $C_2^{(1)}(n, m)$, under the condition that $m - n \to \infty$ and $m = O(n \log n)$. (The upper bound on m here is a natural one for all the problems we consider, since a random graph with $m = n \log n$ is itself a connected 2-core with probability tending to 1. On the other hand, the questions such as we consider are normally handled by entirely different methods for $m - n$ bounded; see Wright [20].) A key observation is that with high probability the random 2-core is connected if and only if it contains no isolated cycles, and the probability of the latter event can be effectively evaluated via combinatorial inversion. We also give results from an alternative approach which continues the theme of inside-out counting: we count connected 2-cores by taking consideration of their kernel, which is the sub(multi)graph obtained by shrinking the paths made up of vertices of degree 2. This argument uses a number of the intermediate results in [16], and applies for $m - n = o(n)$ (and $m - n \to \infty$). We believe that both approaches will be useful in further work on related study of connected 2-cores.

We use the asymptotic formulae for $C_2^{(1)}(n, m)$ to rederive a classic approximation for the number of connected graphs with parameters n and m, obtained in 1990 by Bender, Canfield and McKay [4]. Our argument is rather simpler, and improves the error bound for a range of m. At the heart of our argument is the fact that, pruning away the degree 1 vertex of a random connected graph, we get a connected subgraph of minimum degree 2 or more, and that—conditioned on the vertex set and the number of edges—this subgraph is distributed uniformly. In a natural way, our derivation leads to the asymptotic distribution of the 2-core size in a uniformly random graph with parameters n and m.

Last, but not least, we obtain results on the giant component of the Erdős-Rényi random graph $\mathcal{G}(n, m)$, and for the associated Bernoulli random graph $\mathcal{G}(n, p)$. (For a definition of these random graph models, see, for instance, [9], where many results on the giant component are given. We give some details in Section 2.) We determine the joint limiting distribution of the size of its 2-core, its *excess* (number of edges minus number of vertices), and its *tree mantle size* (the number of vertices of the giant component not in its 2-core), in the supercritical phase, i.e. when $n^{1/3}(2m/n - 1) \to \infty$, or $n^{1/3}(np - 1) \to \infty$. In essence, we show that the three random variables in question are jointly Gaussian in the limit, with explicit, admittedly complex, formulae for the (co)variances. Quite remarkably, the two covariance matrices have the same leading terms for $\epsilon = 2m/n - 1 \to 0$, and $\epsilon = np - 1 \to 0$. This result considerably extends the previously known results of Stepanov [17] and Pittel [15] ($\liminf(np - 1) > 0$, $\liminf(2m/n - 1) > 0$), and of Janson et al. [8] ($2m/n - 1 \geq n^{-1/4}$). In particular, this is the first time that the distribution of the size of the 2-core has been shown to be asymptotically normal.

In Section 2 we elaborate on the the various items touched upon in this introduction, giving much more detail on related results and background, and stating our main theorems. Section 3 introduces notation and gives background results on the functions of relevance to the enumeration of 2-cores, mainly from [16]. Sections 4 and 5 give a model for random connected 2-cores, and the enumeration results we obtain from it, in the case of excess r being $o(n)$. Then in Section 6 we obtain the asymptotic formula for $C_2^{(1)}(n, m)$ for the full range of m of concern. The remaining two sections give proofs of the asymptotic formula for connected graphs and related results, and the limiting distribution result for the three random variables of the giant component in a random graph.

It seems likely that our approach applies directly to the giant component in random hypergraphs, for which recent results were obtained by Karoński and Łuczak [10]. We plan to extend the methods used in the present paper to the asymptotic enumeration of 2-connected graphs and strongly connected digraphs.

2 Setting and main results

2.1 Asymptotic enumeration of connected 2-cores

Łuczak [13] studied properties of a random graph G with a given degree sequence, when the degrees are bounded below by d and above by $n^{0.02}$. For instance, for $d \geq 3$ it was shown that G is a.a.s. d-connected (extending the result in [19] in which the degrees were bounded above by a constant). For $d = 2$, Łuczak obtained a series of asymptotic results, including the distribution of the number of isolated cycles and the probability of connectedness. The argument was based on the powerful notion of the kernel.

Our approach for counting connected 2-cores is two-pronged. First, extending Łuczak's idea, we define what we call the *kernel configuration model* in Section 4. We use this notion to get the asymptotic formula for $C_2^{(1)}(n, m)$, in the sparse case $r = o(n)$, where, throughout this work, we define

$r = 2(m - n)$. To cover the whole spectrum of m, we develop a complementary approach, based on both combinatorial inversion (inclusion-exclusion formula) and Fourier-analytic inversion, not unlike that used in the proofs of local limit theorems. This technique yields an asymptotic formula for $C_2^{(1)}(n, m)$, valid for $r \to \infty$, $m = O(n \log n)$, which is our target domain. Consequently, the two results overlap for $r = o(n)$, and this will allow us to compare their respective error terms.

A quantity quite basic to our work is the following function of n and m

$$Q(n, m) = \sum_{\substack{d_1,\ldots,d_n \geq 2 \\ d_1+\cdots+d_n=2m}} \prod_{j=1}^{n} \frac{1}{d_j!}, \tag{2.1}$$

which was estimated in [16] (as we will see in (3.8) and (3.16)). Using the kernel approach we obtain asymptotics for the number of 2-cores with small excess in terms of Q, as follows.

Theorem 1 *If $r \to \infty$ and $r = o(n)$ then*

$$C_2^{(1)}(n, m) = \sqrt{\frac{3r}{2m}}(2m - 1)!!Q(n, m)(1 - O(r^{-1} + rn^{-1}))$$

The leading term of the asymptotics of $C_2^{(1)}(n, m)$ for larger r could be obtained by coupling the results of Łuczak [13], on connectedness of 2-cores with a given degree sequence (which depends on the number of 2's in the sequence) with our knowledge of the distribution of degrees in a random 2-core coming from [16]. What Łuczak showed was that, for a given degree sequence, 2-cores are typically disconnected because of the existence of an isolated cycle. Inspired by Łuczak's insight, we achieve the following result for the whole spectrum of m by using the full power of our enumeration formula from [16] and two inversions, combinatorial and Fourier-analytic respectively. These inversion steps are needed to estimate, with good error bounds, the total number of 2-cores without cycles, or equivalently the probability of no isolated cycle in a random 2-core.

Theorem 2 *Let $r \to \infty$ and $m = O(n \log n)$. Then for any fixed $\epsilon > 0$,*

$$C_2^{(1)}(n, m) = (1 + O(n^{-1/2+\epsilon} + r^{-1}))h(\sigma)C_2(n, m), \tag{2.2}$$

where

$$h(x) = (1 - x)^{1/2} \exp(x/2 + x^2/4), \tag{2.3}$$

$$\sigma = \frac{\lambda}{e^\lambda - 1} \tag{2.4}$$

and λ is the unique positive root of

$$\frac{\lambda(e^\lambda - 1)}{e^\lambda - 1 - \lambda} = \frac{m}{n}. \tag{2.5}$$

From [16, Theorem 3] (given as Theorem 8 in the present paper) we obtain the following, noticing that the error term ξ is absorbed by the error term in (2.2).

Corollary 1 *Let $r \to \infty$ and $m = O(n \log n)$. Then for any fixed $\epsilon > 0$,*

$$C_2^{(1)}(n,m) = (1 + O(n^{-1/2+\epsilon} + r^{-1}))h(\sigma)\frac{(2m-1)!!Q(n,m)}{e^{\bar{\eta}/2+\bar{\eta}^2/4}}, \tag{2.6}$$

where

$$\bar{\eta} = \sigma e^{\lambda} \tag{2.7}$$

and $h(x)$, σ and λ are as in Theorem 2.

Some properties of λ and $\bar{\eta}$ are given in Section 3.

Note (a) Since $r = o(n)$ is covered by both Theorem 1 and Corollary 1, it is natural to compare their respective accuracies. The task is easy since the common factor $Q(n,m)$ is irrelevant. So we compare $r^{-1} + r/n$ and $n^{-1/2+\epsilon} + r^{-1}$, and conclude: for every $\epsilon > 0$, Theorem 1 and Corollary 1 provide equivalent error terms if $r \le n^{1/2-\epsilon}$, Theorem 1 is the winner if $n^{1/2-\epsilon} \ll r \ll n^{1/2+\epsilon}$, and Corollary 1 takes over for $r \ge n^{1/2+\epsilon}$.

(b) It is known (see Lemma 1(a)) that $\lambda = 3r/n + O(r^2/n^2)$ if $r = o(n)$. So, combining the two results, the error term in Corollary 1 may be replaced by

$$O(\min(n^{-1/2+\epsilon}, rn^{-1}) + r^{-1})$$

for any fixed $\epsilon > 0$.

2.2 Derivation of the Bender-Canfield-McKay formula

Given n and m, let $C^{(1)}(n,m)$ denote the total number of connected graphs with n vertices and m edges. It is a well known result of Erdős and Rényi that $C^{(1)}(n,m)$ is asymptotically equal to the total number of graphs on n vertices and m edges provided $2m/n - \log n \to \infty$ as $n \to \infty$. Of course, the number is 0 if $m < n-1$ and is the number of n-trees if $m = n-1$. Wright [21] obtained formulae in the case that the excess of m over n is rather small. Then Bender, Canfield and McKay [4] filled in the gap, providing a formula spanning all m. Since $me^{-2m/n} \to \infty$ if $2m/n - \log n - \log\log n \to -\infty$ and $m \to \infty$, the following result provides the Bender-Canfield-McKay formula covering all m at the bottom of the Erdős-Rényi range as well as all below it (provided $m - n \to \infty$). A second result will say a little more about slightly larger m.

Theorem 3 *Let $m, n \to \infty$ in such a way that $m - n \to \infty$ and $me^{-2m/n} \to \infty$. Then*

$$C^{(1)}(n,m) = \frac{(1+O(\beta))\alpha n^m}{\sqrt{2\pi n}} \left(\frac{2\sinh \overline{\lambda}/2}{(\overline{\lambda})^c}\right)^n, \tag{2.8}$$

where

$$\begin{aligned} \alpha &= e^{-(c+1)\overline{\lambda}/2}\sqrt{\frac{2(e^{\overline{\lambda}}-1-\overline{\lambda})^2}{\overline{\lambda}(e^{2\overline{\lambda}}-1-2\overline{\lambda}e^{\overline{\lambda}})}}, \\ \beta &= ((m-n)e^{-2m/n})^{-a} \quad \textit{for any } a < 1/2, \\ c &= c(n,m) = m/n \end{aligned} \tag{2.9}$$

and $\overline{\lambda} = \overline{\lambda}(n, m)$ *is the unique positive root of the equation*

$$\frac{\lambda}{2} \coth \frac{\lambda}{2} = c. \tag{2.10}$$

Note 1 We may interpret $\overline{\lambda}$ as the parameter of the Poisson random variable Z conditioned on the event $\{Z \geq 2\}$. Its distribution is the limit of the degree distribution of a generic vertex in the uniformly random 2-core with number of edges and vertices close to those for the 2-core of the most *frequent* connected graphs with m edges and n vertices.

Note 2 Let us rewrite (2.8). The fraction of connected graphs with n vertices and m edges is $C^{(1)}(n, m)/\binom{N}{m}$, where $N = \binom{n}{2}$. Here, within a factor $1 + O(n^{-1} \log^3 n)$,

$$\begin{aligned} \binom{N}{m} &\approx \frac{\binom{n}{2}^m}{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m} \cdot e^{-m^2/n^2} \\ &\approx \frac{1}{\sqrt{2\pi cn}} \left(\frac{ne}{2c}\right)^{cn} \cdot e^{-c-c^2}. \end{aligned}$$

So, using (2.8),

$$\begin{aligned} \frac{C^{(1)}(n, m)}{\binom{N}{m}} &= (1 + O(\beta))g(c)h(c)^n; \qquad (2.11) \\ h(c) : &= (e^{\overline{\lambda}/2} - e^{-\overline{\lambda}/2}) \left(\frac{2c}{e\overline{\lambda}}\right)^c, \\ g(c) : &= \sqrt{2c} \exp\left(c + c^2 - \frac{(c+1)\lambda}{2}\right) \frac{e^{\overline{\lambda}} - 1 - \overline{\lambda}}{(\overline{\lambda})^{1/2}(e^{2\overline{\lambda}} - 1 - 2\overline{\lambda}e^{\overline{\lambda}})^{1/2}}. \end{aligned}$$

From (2.10), $\overline{\lambda} < 2c$, so we can introduce a function $z = z(c) \in (0, 1)$ by setting $\overline{\lambda} = 2cz$. Then equation (2.10) becomes

$$2cz = \log \frac{1+z}{1-z}.$$

In terms of z instead of $\overline{\lambda}$,

$$h(c) = \left(\sqrt{\frac{1+z}{1-z}} - \sqrt{\frac{1-z}{1+z}}\right) \left(\frac{1}{ez}\right)^c = \frac{2e^{-c}z^{1-c}}{\sqrt{1-z^2}}, \tag{2.12}$$

and finally

$$\begin{aligned} \log g(c) &= \frac{1}{2} \log c + c + c^2 - cz(c+1) + \frac{1}{2} \log \frac{\frac{8z^2}{(1-z)^2}(1 - c(1-z))^2}{\frac{8cz^2}{(1-z)^2}(1 - c(1-z^2))} \\ &= c(1+c)(1-z) + \log(1 - c(1-z)) - \frac{1}{2} \log(1 - c(1-z^2)). \qquad (2.13) \end{aligned}$$

With $h(c), g(c)$ defined by (2.12) and (2.13), equation (2.11) is the main Bender-Canfield-McKay formula [4, Theorem 1], except the remainder terms. For $m = O(n)$, $\beta = O((m-n)^{-a})$ for any

$a < 1/2$, whereas the one in [4] is $(m-n)^{-1} + (m-n)^{1/16} n^{-9/50}$. The new result has a smaller error as long as $n^{8/25+\epsilon} < m - n = O(n)$.

The proof of this theorem does not require Theorem 1, but uses Theorem 2.

In Section 7 we also give a result on the asymptotics of $C^{(1)}(n,m)$ for a higher range of m (Theorem 9).

Our approach to the enumeration of connected graphs provides an answer to the following question. Let $G^c(n,m)$ denote a graph chosen uniformly at random among all $C^{(1)}(n,m)$ connected graphs on n vertices and with m edges. What is the limiting distribution of X_{nm}, the number of vertices in the 2-core of $G^c(n,m)$?

Theorem 4 *Under the conditions of Theorem 3, X_{nm} is in the limit Gaussian, with mean*

$$\overline{y} = \frac{e^{\overline{\lambda}} - 1 - \overline{\lambda}}{e^{\overline{\lambda}} - 1}$$

and variance $n\sigma^2$, where

$$\sigma^2 = \frac{\overline{\lambda}[(e^{\overline{\lambda}} - 1)^2 - (\overline{\lambda})^2 e^{\overline{\lambda}}]}{(e^{2\overline{\lambda}} - 1 - 2\overline{\lambda} e^{\overline{\lambda}})(e^{\overline{\lambda}} - 1 - \overline{\lambda})}.$$

More precisely,

$$\mathbf{P}(X_{nm} = \nu) = (1 + O(\beta)) \frac{\exp\left(-\frac{(\nu - n\overline{y})^2}{2n\sigma^2}\right)}{\sqrt{2\pi n\sigma^2}}$$

uniformly for

$$\left| \frac{\nu - n\overline{y}}{\sqrt{n\sigma^2}} \right| \le d^{1/2-\epsilon}$$

where

$$d = \begin{cases} m - n, & \text{if } \lim c = 1; \\ m e^{-2m/n}, & \text{if } \lim c > 1. \end{cases} \quad ∎$$

2.3 Properties of the giant component in the supercritical random graph

Our original motivation for this study was the distribution of the size of the 2-core in the giant component of the supercritical random graph. We will consider the joint distribution of three variables in a supercritical random graph $\mathcal{G}(n, p = c/n)$ or $\mathcal{G}(n, m = cn/2)$ with $c = c(n) > 1$. (*Warning:* for results on the giant component and on connected cores, our definition of c in terms of m is $2m/n$ rather than m/n for the results on connected graphs. This is for historical reasons.)

Referring to the largest component in the random graph, its *2-core size* is the number of vertices in its 2-core, and its tree mantle size and excess were defined in Section 1. It is well known that for $n^{1/3}(c-1) \to \infty$, $\mathcal{G}(n,m)$ a.a.s. has a unique largest ("giant") component which is much larger than all other components (see Theorem 5 below). For this range of c, we introduce X_n, Y_{n1}, Y_{n2} and Y_{n3}, where X_n is the size (i.e. number of vertices) of the giant component, Y_{n1} is the size (number of vertices) of the 2-core of the giant component, Y_{n2} is its tree mantle size, and Y_{n3} is its excess. Thus $X_n = Y_{n1} + Y_{n2}$, the number of edges in the 2-core of the giant component is $Y_{n1} + Y_{n3}$ (since the excess of a component equals the excess of its 2-core), and the total number of edges in the giant component is $Y_{n1} + Y_{n2} + Y_{n3}$.

Erdős and Rényi [7] discovered that, for $\liminf c > 1$, $X_n/(nb) \to 1$ in probability where

$$b = b(c) := 1 - t/c, \tag{2.14}$$

and $t = t(c)$ is the unique root of the equation

$$te^{-t} = ce^{-c}, \quad t \in (0,1). \tag{2.15}$$

Later Stepanov [17] proved, for the random graph $\mathcal{G}(n, p = c/n)$, that X_n is asymptotically normal, with mean nb and variance $n\sigma_p^2$, where

$$\sigma_p^2 = \frac{t(1 - t/c)}{c(1-t)^2}. \tag{2.16}$$

(A generic vertex belongs to the giant component with limiting probability t/c. So, had it not been for the factor $(1-t)^{-2}$, we might have been able to interpret the Stepanov's result as stating that X_n is asymptotic to the number of "successes" in n *independent* trials, with b being the probability of success in each trial.) Stepanov actually stated and attempted to prove more, namely that X_n obeys the normal local limit law, but the proof of this, though very technical, was not quite complete. Still later, Pittel [15] proved asymptotic normality of X_n for $\mathcal{G}(n,m)$, and used it to obtain a new proof of asymptotic normality of X_n for $\mathcal{G}(n,p)$. The variance for the $\mathcal{G}(n,m)$ case was shown to be asymptotic to $n\sigma_m^2$, where

$$\sigma_m^2 = \sigma_p^2(1 - 2t(1 - c^{-1}t)), \tag{2.17}$$

thus *smaller* than σ_p^2. The bulk of the argument was a proof that a process counting tree components by their sizes weakly converged to an infinite Gaussian sequence. This gave access to the giant component distribution, since apart from that component and the forest of many small tree components, the graph a.a.s. only contains few unicyclic components whose total size is bounded in probability. However, none of these distributional results applied for $c \to 1$.

We will make use of the following important concentration result proved by Bollobás for the case $n^{1/3}(c-1)/\sqrt{\log n} \to \infty$, and extended to the stated range by Łuczak; see also Janson et al. [9, Theorem 5.12].

Theorem 5 **[6, 11]** *Let $n^{1/3}(c-1) \to \infty$. With probability $1 - O(n^{-1/9}(c-1)^{-1/3})$, in $\mathcal{G}(n,m)$,*

$$X_n = bn + O(n^{2/3})$$

where $b = 1 - t/c$, and the second-largest component has size smaller than $n^{2/3}$.

Note that for c as in this theorem, $n^{2/3} = o(bn)$ by (2.14) and (2.15). On the other hand, if $n^{1/3}(c-1) = O(1)$, the phase transition of the random graph is still under way, and with non-zero probability there are many components close to the size of the largest.

It was also shown in [15] for both $\mathcal{G}(n,p)$ and $\mathcal{G}(n,m)$ that

$$\frac{Y_{n1}}{n} \to (1-t)b \quad \text{in probability} \tag{2.18}$$

for $\liminf c > 1$. This was done by analyzing a simple "pruning" algorithm, which consists of consecutive deletions of the pendant vertices of the trees rooted at the core vertices. A result of Łuczak [12, Theorem 10] also implies this in $\mathcal{G}(n,m)$, for $n^{1/3}(c-1) \to \infty$ but $c-1 = o(1)$. However, distributional results for the size of the 2-core appeared to be out of sight.

Information is also known already on Y_{n3}, the excess of the giant component or of its 2-core. It was shown by Łuczak [11] that in $\mathcal{G}(n,m)$, this is sharply concentrated for $n^{1/3}(c-1) \to \infty$ but $c-1 = o(1)$. Janson et al. [8, Theorem 13] gave a 2-dimensional local limit theorem for the excess and another variable, called the "deficiency" of the 2-core, which implies a local limit theorem for the excess alone. This showed that the excess is asymptotically normal provided $c-1 \geq \Theta(n^{-1/4})$. However, in the absence of distributional information on Y_{n1}, this does not imply a distributional result for the number $Y_{n1} + Y_{n3}$ of edges in the 2-core, or for the number $Y_{n1} + Y_{n2} + Y_{n3}$ of edges in the giant component. As to information on X_n for $c \to 1$, in [9, Section 5.3] there is mention of a way to prove asymptotic normality using the main enumeration theorem of Bender et al. [4], but this approach is described as long and not very exciting.

In the present paper we prove that for both $\mathcal{G}(n,p)$ and $\mathcal{G}(n,m)$, under the condition $n^{1/3}(c-1) \to \infty$ (but $c = O(1)$), the vector (Y_{n1}, Y_{n2}, Y_{n3}), centralized and normalized, converges in distribution to a Gaussian vector. Consequently, each of the three variables is asymptotically normal. From the asymptotic expectations and the covariance matrix, we may deduce that the joint distribution of any set of fixed linear combinations of these variables, such as the size of the 2-core and its excess, is also asymptotically Gaussian. It follows that this result subsumes all the distributional central limit results described above, showing asymptotic normality of X_n and Y_{n3}, as well as giving the desired new result of asymptotic normality of Y_{n1}, the size of the 2-core in the giant component. The asymptotic normality of Y_{n2}, the tree mantle size, comes as a bonus. The lower end of the range of c covered by our result also improves most of the above results, and is best possible since it covers all the supercritical phase of the random graph. The key ingredients are Theorem 2 and an enumerational "construction" of the giant component which, in a sense, reverses the steps of the pruning algorithm: first we choose a connected 2-core, and then grow trees rooted at the core vertices, stopping when there are no other outside vertices to be added to a current subgraph.

We also give a local limit theorem for the joint distribution of (Y_{n1}, Y_{n2}, Y_{n3}) in $\mathcal{G}(n,m)$, conditional upon the following event. Let B_n stand for the event "there is a unique component of size between $0.5bn$ and $2bn$, and none larger", where b is defined in (2.14). It follows from Theorem 5 that

$$P(B_n) \to 1 \text{ for } \mathcal{G}(n, m = cn/2) \text{ if } n^{1/3}(c-1) \to \infty. \tag{2.19}$$

The same is true in $\mathcal{G}(n, p = c/n)$ by its well-known relationship with $\mathcal{G}(n, m = cn/2)$; see also Stepanov [17].

Theorem 6 *Suppose that* $\limsup c < \infty$ *and* $n^{1/3}(c-1) \to \infty$. *Let* $\mathbf{b}(c) = (b_1(c), b_2(c), b_3(c))^T$ *where*

$b_1 = (1-t)b$, $b_2 = tb$, $b_3 = b(c+t-2)/2$, *let* K_p *be the symmetric matrix*

$$\frac{t(c-t)}{c^2}\begin{pmatrix} \frac{(c+1-2t)(ct+1-2t)}{(1-t)^2} & \frac{(2t-c)(ct+1-2t)}{(1-t)^2} & \frac{(c+1-2t)(c-1)}{1-t} \\ \frac{(2t-c)(ct+1-2t)}{(1-t)^2} & \frac{ct(c-3-2t)+c+4t^2}{(1-t)^2} & \frac{(2t-c)(c-1)}{1-t} \\ \frac{(c+1-2t)(c-1)}{1-t} & \frac{(2t-c)(c-1)}{1-t} & \frac{c^2}{2t}-\frac{3c}{2}+1 \end{pmatrix},$$

and let

$$K_m = K_p - 2c\frac{d\mathbf{b}(c)}{dc}\cdot\frac{d\mathbf{b}(c)}{dc}^T. \tag{2.20}$$

Then K_p *and* K_m *are positive definite, with* $K_p(1,1)$ *and* $K_m(1,1)$ *being* $\Theta(c-1)$, $K_p(2,2)$ *and* $K_m(2,2)$ *being* $\Theta((c-1)^{-1})$, *and* $K_p(3,3)$ *and* $K_m(3,3)$ *being* $\Theta((c-1)^3)$, *as* $c \to 1$. *Moreover*

(i) (Y_{n1}, Y_{n2}, Y_{n3}) *is in the limit Gaussian with mean vector* $n\mathbf{b}$ *and covariance matrix* nK_p *in the case of* $\mathcal{G}(n,p)$, *and* nK_m *for* $\mathcal{G}(n,m)$,

(ii) with $A_m = K_m^{-1}$, *for* $\mathcal{G}(n,m)$

$$\begin{aligned} \mathbf{P}(Y_{n1}=\nu_1, Y_{n2}=\nu_2, Y_{n3}=\mu_1|B_n) &= (1+o(1))\frac{(\det A_m)^{1/2}}{(2\pi n)^{3/2}}e^{-\frac{1}{2}\mathbf{x}^TA_m\mathbf{x}}, \\ \det A_m &= \frac{2c^6(1-t)^3}{t^4(c-t)^4(1-ct)}, \\ \mathbf{x}^T &= \left(\frac{\nu_1-b_1n}{n^{1/2}}, \frac{\nu_2-b_2n}{n^{1/2}}, \frac{\mu_1-b_3n}{n^{1/2}}\right) \end{aligned} \tag{2.21}$$

uniformly for all (ν_1, ν_2, μ_1) *such that* $(K(1,1)^{-1/2}x_1, K(2,2)^{-1/2}x_2, K(3,3)^{-1/2}x_3)$ *is bounded.*

Note 1 Aside from being the threshold of the supercritical phase, the condition $n^{1/3}(c-1) \to \infty$ is necessary and sufficient for $b_3n \gg \sqrt{K(3,3)n}$, which is certainly necessary for asymptotic normality of $(Y_{n3}-b_3n)/\sqrt{K(3,3)n}$ (as seen by the expansions in Note 4 below).

Note 2 The relation (2.20) implies that $\mathbf{x}^TK_p\mathbf{x} \geq \mathbf{x}^TK_m\mathbf{x}$, with equality only when $\mathbf{x} \perp \mathbf{b}'(c)$. Thus, for $\mathbf{Y}_n = (Y_{n1}, Y_{n2}, Y_{n3})$, we have that $\mathbf{Var}(\mathbf{x}^T\mathbf{Y}_n)$ is larger for $\mathcal{G}(n,p)$ than for $\mathcal{G}(n,m)$, except for $\mathbf{x} \perp \mathbf{b}'(c)$. Loosely speaking, the random fluctuations of $\mathbf{Y}_n$ around $n\mathbf{b}(c)$ are larger in the $G(n,p)$ case.

Note 3 The entries of A_m are calculated in the proof. The matrix K_m is more complicated than K_p, but we may calculate for $\mathcal{G}(n,m)$

$$\begin{aligned} K_m(1,1) &= \frac{t(c-t)(-c^2t+c+4t^2c^2-6t^3c-4ct+c^2-4ct^2+2t^4-c^3t)}{c^3(1-t)^2}, \\ K_m(2,2) &= \frac{t(c-t)(2t^4-6t^3c+4t^3+4t^2c^2-4ct^2+2t^2+c^2t-2ct-c^3t+c^2)}{c^3(1-t)^2}, \\ K_m(3,3) &= \frac{t(c-t)(t^2-3ct+2c)}{2c^3}. \end{aligned}$$

These are asymptotic to $1/n$ times the variances of the limiting normal approximations of Y_{n1}, Y_{n2} and Y_{n3} respectively.

Note 4 Let $\epsilon = c - 1 \to 0$. Then $t = 1 - \epsilon + 2\epsilon^2/3 + O(\epsilon^3)$, $b_1 = 2\epsilon^2 + O(\epsilon^3)$, $b_2 = 2\epsilon + O(\epsilon^2)$ and $b_3 = \frac{2}{3}\epsilon^3 + O(\epsilon^4)$. Moreover, K_m and K_p are both of the form

$$\begin{pmatrix} 12\epsilon + O(\epsilon^2) & 4 + O(\epsilon) & 6\epsilon^2 + O(\epsilon^3) \\ 4 + O(\epsilon) & 2\epsilon^{-1} + O(1) & 2\epsilon + O(\epsilon^2) \\ 6\epsilon^2 + O(\epsilon^3) & 2\epsilon + O(\epsilon^2) & \frac{10}{3}\epsilon^3 + O(\epsilon^4) \end{pmatrix}.$$

Note 5 Since $b_1 + b_2 = b$, we obtain that, whenever $n^{1/3}(c-1) \to \infty$ and $\liminf c < \infty$, the size of the largest component, $X_n = Y_{n1} + Y_{n2}$ is in the limit normal with mean $2nb$ and variance $n(K(1,1) + 2K(1,2) + K(2,2))$. The latter for $K = K_p$ is $(c-t)t/(c^2(1-t)^2 = \sigma_p^2$ as given in (2.16), and for $K = K_m$ it is σ_m^2 as given in (2.17). These are both asymptotic to $2n^{1/2}(c-1)^{-1/2}$ as $c \to 1$. Bollobás and Łuczak obtained the weaker concentration result in Theorem 5.

Similarly, for the same c, all positive linear combinations of Y_1, Y_2 and Y_3 will be asymptotically normal. For instance, the number of edges in the 2-core of the largest component, $Y_{n1} + Y_{n3}$, is in the limit normal with mean $2n(b_1 + b_3) = (c-t)^2/(2c)$ and variance $n(K(1,1) + 2K(1,3) + K(3,3))$. The latter for $K = K_p$ is

$$\frac{(c + 2ct - ct^2 - 2t^2)(c-t)^2}{2c^2(1-t)^2}$$

and for $K = K_m$ it is

$$\frac{t(c-t)^2(-2c^2t + 2ct^2 + 2ct + 2c - t^3 - 2t^2 - t)}{2c^3(1-t)^2}.$$

In the same way, the number of edges $Y_1 + Y_2 + Y_3$ in the giant component is asymptotically normal with mean $n(b_1 + b_2 + b_3)$, and the reader may care to calculate the variance.

Note 6 Our first attempt at this calculation considered only the two variables Y_{n1} and Y_{n2}, which forced us to sum over Y_{n3}. However, Y_{n3} is of interest in its own right, and the resulting formulae are arguably simpler than for (Y_{n1}, Y_{n2}) alone.

Note 7 Our proof uses a result from [15] to transfer the distributional result from $\mathcal{G}(n,m)$ to $\mathcal{G}(n,p)$. By considering $\mathcal{G}(n,p)$ directly, our proof would avoid this and the same method would in principle yield a local limit theorem for $\mathcal{G}(n,p)$ analogous to (ii) for $\mathcal{G}(n,m)$.

Note that for the proof of Theorem 5, we only need the leading term of the asymptotics for $C_2^{(1)}(n,m)$, as given in Theorem 2.

3 Useful results and notation

In this section we give various facts which are required in this paper, mainly from [16] relating to the asymptotic number of 2-cores with n vertices and m edges. By quoting these formulae here we do not need to assume familiarity with [16]. This section will be used heavily for the later sections of the paper except for Section 8.

McKay [14] extended a formula of Bender and Canfield [3] for the asymptotic number of graphs with degree sequence $\vec{d} = (d_1, \ldots, d_n)$ as follows. Here $m = \frac{1}{2}\sum_{j=1}^n d_j$.

Theorem 7 *Let $\vec{d}$ be a function of n such that $m = m(n) \to \infty$ and $d_{\max} := \max_{1\le i\le n} d_i = o(m^{1/4})$ as $n \to \infty$, and m is an integer for all n. Then*

$$g(\vec{d}) = \frac{(2m-1)!!}{\prod_{j=1}^n d_j!} \exp\left(-\frac{\eta(\vec{d})}{2} - \frac{\eta^2(\vec{d})}{4} + O\left(\frac{d_{\max}^4}{m}\right)\right) \tag{3.1}$$

where

$$\eta(\vec{d}) := \frac{1}{2m}\sum_{j=1}^n d_j(d_j-1).$$

All the remaining results of this section come from [16], with references after the statements of results.

In various places (such as the final proof in Section 5 and in Section 7) we use

$$f(\lambda) = e^\lambda - 1 - \lambda. \tag{3.2}$$

For all $n \ge 1$ and $m \ge 1$, we have the following upper bounds on numbers of 2-cores:

$$C_k(n,m) \le (2m-1)!!\frac{f_k(\lambda)^n}{\lambda^{2m}}, \quad \forall\, \lambda > 0, \tag{3.3}$$

and

$$C_k(n,m) \le a(2m-1)!!\frac{f_k(\lambda)^n}{\lambda^{2m}\sqrt{n\lambda}}, \quad \forall\, \lambda > 0, \tag{3.4}$$

where a is an absolute constant [16, Equations (14,15)].

We require a fundamental truncated Poisson random variable, $Y = Y(\lambda)$, such that

$$\mathbf{P}(Y=j) = \mathbf{P}(Y(\lambda)=j) = \begin{cases} \dfrac{\lambda^j}{j!f(\lambda)}, & j \ge 2 \\ 0, & j < 2 \end{cases}, \tag{3.5}$$

and note that (2.5) can be written as

$$\mathbf{E}Y = c := \frac{2m}{n}, \tag{3.6}$$

or

$$\frac{\lambda(e^\lambda - 1)}{f(\lambda)} = c. \tag{3.7}$$

These determine λ as a function of c, so to emphasize when this relationship is in force, we denote λ by λ_c.

By [16, Equation (9)], the function $Q(n,m)$ defined in (2.1) satisfies

$$Q(n,m) = \frac{f(\lambda)^n}{\lambda^{2m}} \mathbf{P}\left(\sum_{j=1}^{n} Y_j = 2m\right) \tag{3.8}$$

where $Y_1, \ldots, Y_n$ are n independent copies of $Y(\lambda_c)$ as in (3.6) and (3.7). Various estimates of this probability are given in [16] (see (3.16) below, for example), and the case $k = 2$ of Theorem 3 in that paper is as follows. Here $r = 2m - 2n$.

Theorem 8 *Fix $\epsilon > 0$. For any $r \geq 0$,*

$$C_2(n,m) = (1 + O(\xi))\frac{(2m-1)!!Q(n,m)}{e^{\bar{\eta}/2+\bar{\eta}^2/4}}, \tag{3.9}$$

where

$$\xi = \min\{e^{-r^{\epsilon}} + r^{1/2}n^{-1+\epsilon}, r^{1/2}n^{-2/3}\}.$$

and

$$\bar{\eta}_c = \frac{\lambda_c e^{\lambda_c}}{e^{\lambda_c} - 1}, \tag{3.10}$$

which is estimated in (3.20).

For comparison, [16, Theorem 2] gives a more explicit result which applies for $r \to \infty$:

$$C_2(n,m) = (1 + O(r^{-1} + r^{1/2}n^{-1+\epsilon}))\frac{(2m-1)!!f(\lambda)^n}{\lambda^{2m}e^{\bar{\eta}/2+\bar{\eta}^2/4}} \frac{1}{\sqrt{2\pi nc(1+\bar{\eta}-c)}} \tag{3.11}$$

We quote the part of [16, Lemma 1] required here:

Lemma 1 *The root λ_c of (3.7) exists uniquely, and*

(a) if $m/n \to 1$ then

$$\lambda_c = 3(c-2) + O((c-2)^2), \tag{3.12}$$

(b) $\lambda_c \leq 2m/n$ always.

From this point, we do not examine the dependence of λ_c on c so heavily, and so we drop the c subscript on λ_c for simplicity.

We make use of the first displayed equation in the proof of Lemma 1 in [16], with $k = 2$:

$$\frac{d}{d\lambda}\frac{\lambda(e^{\lambda}-1)}{e^{\lambda}-1-\lambda} > 0 \tag{3.13}$$

for $\lambda > 0$. We also use [16, Lemma 2]:

Lemma 2 *Uniformly for all $\lambda \in (0,\infty)$,*

$$\mathbf{Var}(Y(\lambda)) = c(1 + \bar{\eta}_c - c) = \Theta(\lambda) = \Theta(c-2). \tag{3.14}$$

Some other facts: ([16, Equation (20)]) for $\lambda \to 0$

$$c(1+\bar{\eta}-c) = c\left(\frac{\lambda_c}{2} - (c-2) + O(\lambda_c^2)\right) \sim c(c-2)/2 \sim c-2, \tag{3.15}$$

[16, Equation (22)]:

$$\mathbf{P}\left(\sum_{j=1}^{n} Y_j = 2m\right) = \frac{1+O(r^{-1})}{\sqrt{2\pi n c(1+\bar{\eta}-c)}} \tag{3.16}$$

for $r \to \infty$, and [16, Equation (27)]:

$$\mathbf{P}(Y \geq j_0) = \sum_{j \geq j_0} \frac{\lambda^j}{j! f(\lambda)} = O(\exp(-j_0/2)) \quad \text{for } j_0 > 2e\lambda. \tag{3.17}$$

With $S = \eta(\vec{Y})/2 = \frac{1}{4m}\sum_{j=1}^{n} Y_i(Y_i - 1)$ we have [16, equation (31)]

$$\mathbf{P}\left(|S - \mathbf{E}S| \geq n^{1/2} m^{-1} \log^8 n\right) \leq \exp(-\Theta(\log^3 n)) \tag{3.18}$$

and, for $r = O(n^{1-\epsilon})$ where $\epsilon > 0$, a later equation in [16] is

$$\mathbf{P}\left(|S - \mathbf{E}S| \geq 2m^{-1} r^{1/2+\epsilon}\right) \leq \exp(-\Theta(\log^3 n)) + O(\exp(-r^{3\epsilon/2})). \tag{3.19}$$

We also have, from near the end of the proof of [16, Theorem 4],

$$\mathbf{E}(Y(Y-1)) = c\bar{\eta} = k(k-1) + 2\lambda k/(k+1) + O(\lambda^2). \tag{3.20}$$

Finally, some notation used occasionally in this paper: $a_n =_b b_n$ denotes that $a_n = O(b_n)$.

4 The kernel configuration model

We define a *cycle component* of a graph to be a connected component which is simply a cycle. A (simple) graph with minimum degree at least 2 and with no cycle components is called a *pre-kernel*. The *kernel* of a pre-kernel G is the pseudograph $\mathcal{H}(G)$ obtained from G by repeatedly choosing a vertex v of degree 2, deleting v and its two incident edges, and inserting an edge joining the two (former) neighbours of v. This operation is called *suppressing* v. It is possible that this creates loops or multiple edges. A loop contributes 2 to the degree of its incident vertex. The condition that G contains no isolated cycle ensures that at each step v is not incident with a loop, so the kernel of such G is always a pseudograph. It is clearly well defined; i.e. it does not depend on the order of suppressing vertices of degree 2. The vertices of $\mathcal{H}(G)$ are just the vertices of G of degree at least 3, and they have the same degree in $\mathcal{H}(G)$ as in G. The kernel has been used to obtain various results in random graphs, such as Łuczak [13, Theorem 12.2] which we use below, and also in [12], but we do rather more precise calculations here. It is even featured on the cover of the book by Janson et al. [9].

For the rest of this section $\vec{d} = (d_1, \ldots, d_n)$ is an integer sequence with even sum $2m > 2n$ and with all $d_i \geq 2$. Let $T = T(\vec{d}) = \{i : d_i \geq 3\}$, and set $2m' = \sum_{i \in T} d_i$, so $m - m'$ is the total number of degree 2 vertices; also set $n' = |T(\vec{d})|$.

We define a *kernel configuration* H for $\vec{d}$ as follows. Take a set of $2m'$ *points* partitioned into *cells* indexed by T, with d_i points in cell i. Let P be a perfect matching of the points into m' pairs. Also take an assignment of $[n] \setminus T$ to the pairs of P, such that for each pair $p \in P$ the numbers assigned to p are given a linear ordering. Denote this assignment, including the linear orderings, by f, and by $f^{-1}(p)$ the numbers assigned to p. Then $H = (P, f)$.

Corresponding to each kernel configuration H there is a pseudograph defined as follows. The cells are regarded as vertices of degree at least 3, and the pairs as paths of vertices of degree 2: a pair (x, y) in P corresponds to a path from i to j in the pseudograph where $x \in i$ and $y \in j$, and the degree 2 vertices along the path are the vertices in $f^{-1}(p)$ in the prescribed order (which can be done canonically by working from i to j where $i < j$). Denote the resulting pseudograph $G(H)$. The model $\mathcal{H}(\vec{d})$ is the probability space resulting from the distribution of the pseudograph $G(H)$ when the pairing P and, conditioned on P, the assignment f, are both chosen u.a.r. The number of assignments f is then the same for every pairing P, and the probability of every feasible pair (P, f) is the same. We call this the *kernel configuration* model. Note that $G(H)$ is a pre-kernel with degree sequence $\vec{d}$, and P alone determines its kernel. The kernel has m' edges and n' vertices.

Lemma 3 *The restriction of $\mathcal{H}(\vec{d})$ to simple graphs is a uniform probability space on the pre-kernels with degree sequence $\vec{d}$.*

Proof. If $G(H)$ is a graph then, in $H = (P, f)$, each pair $p \in P$ is uniquely identified by the two cells (vertices) which it joins, together with the vertices in $[n] \setminus T$ which are assigned to it by f. Hence, any non-identity permutation of the points within the cells induces a different configuration H with the same graph $G(H)$. There are precisely $\prod_{i \in T} d_i!$ such permutations. On the other hand, given G and P, the labels of the degree 2 vertices uniquely determine the accompanying assignment f. Thus each G is produced by precisely $\prod_{i \in T} d_i!$ pairs (P, f). Since P and f are chosen u.a.r., the lemma follows. ∎

Let $G_2(\vec{d})$ denote the set of pre-kernels with degree sequence $\vec{d}$, and let **simple** denote the event in $\mathcal{H}(\vec{d})$ that $G(H)$ is simple.

Corollary 2

$$|G_2(\vec{d})| = \frac{(2m' - 1)!!(m - 1)!\mathbf{P}(\mathbf{simple})}{(m' - 1)! \prod_{i \in T} d_i!}.$$

Proof. The number of pairings is $(2m' - 1)!!$, and number of assignments f is $m'(m' + 1) \cdots (m' + (m - m') - 1) = [m - 1]_{m - m'}$ since the places to insert the ordered sequence of $m - m'$ vertices of degree 2 can be chosen one after the other, there are at first m' places, and each insertion creates one new place. The result now follows from the lemma and its proof. ∎

Note that $m = \frac{1}{2} \sum_{i \in T} d_i + \sum_{i \notin T} 1 = m' + n - n'$ and also $m \geq 3n'/2 + n - n' = n'/2 + n$, whence, denoting $2m - 2n$ by r,

$$n' \leq 2m - 2n = r = 2m' - 2n' < 2m' \leq 3r. \tag{4.1}$$

The kernel model is related to the models previously used for enumerating random graphs with given degrees, as by Békéssy et al. [2], Bender and Canfield [3], Bollobás [5] and Wormald [18]. In particular, the pairing P in the definition is a random pairing in the usual model for random graphs with given degrees d_i for $i \in T$.

5 Counting connected 2-cores of small excess using the kernel configuration

Our main aim in this section is to obtain the number of connected 2-cores in terms of the model $\mathcal{H}(\vec{d})$, for $r = 2m - 2n = o(n)$ and $r \to \infty$. The condition $r \to \infty$ is only required to make the probability of a random kernel configuration being connected tend to 1. For r bounded, one could sum over all connected kernels. This would be similar to Wright's approach for enumeration of connected and 2-connected graphs of small excess [20, 22]. Note that comparison of this result with (3.9) gives the asymptotic probability that a random graph with a given number of vertices and edges is connected, conditional upon having no vertices of degree 0 or 1. We obtain bounds on the error terms, but note that to obtain the leading asymptotic term in our final formula in Theorem 1, only the largest term needs to be retained in any of our formulae. Let $\mathbf{cs} = \mathbf{cs}(\vec{d})$ denote the event in $\mathcal{H}(\vec{d})$ that $G(H)$ is simple and connected. We define $C_2^{(1)}(\vec{d})$ to be the number of connected 2-cores with degree sequence $\vec{d}$. For $m > n$ these graphs are just the connected pre-kernels, and so Corollary 2 and its proof immediately give the following, where of course the probability $\mathbf{P}(\mathbf{cs})$ refers to the space $\mathcal{H}(\vec{d})$.

Lemma 4 *For $m > n$*

$$C_2^{(1)}(\vec{d}) = \frac{(2m'-1)!!(m-1)!\mathbf{P}(\mathbf{cs})}{(m'-1)!\prod_{i\in T} d_i!}. \quad \blacksquare$$

Let $\mathcal{D}_{n,m}$ be the set of all sequences $\vec{d}$ under consideration, that is with $\sum d_i = 2m = cn$. In this section we consider $r \to \infty$ such that $r = o(n)$. We first compute the asymptotic number of connected 2-cores with degree sequence $\vec{d}$ (for the important $\vec{d}$). This is done by counting inside-out, using the kernel configuration to build the connected 2-cores from their kernels. We first obtain the following from Lemma 4 by Stirling's formula with remainder:

$$m! = (1 + O(m^{-1}))\sqrt{2\pi m}(m/e)^m. \tag{5.1}$$

Corollary 3 *For $r \to \infty$ and $r = o(n)$,*

$$C_2^{(1)}(\vec{d}) = \frac{(2m-1)!!\mathbf{P}(\mathbf{cs})\sqrt{m'/m}}{\prod_{1\le i\le n} d_i!}\left(1 - O\left(\frac{1}{r}\right)\right). \quad \blacksquare$$

We concentrate on $\vec{d} \in \tilde{\mathcal{D}}$ where

$$\tilde{\mathcal{D}} = \left\{ \vec{d} \in \mathcal{D}_{n,m} \,:\, \max_i d_i \le 6\log n,\ \sum_{i\in T(\vec{d})} \binom{d_i}{2} < 4r \right\}. \tag{5.2}$$

Lemma 5 *For the kernel configuration $\mathcal{H}(\vec{d})$, we have $\mathbf{P}(\mathbf{cs}) = 1 - O(r^{-1} + rn^{-1})$ uniformly for $\vec{d} \in \tilde{\mathcal{D}}$ with $r \to \infty$, $r = o(n)$.*

Proof. In $\mathcal{H}(\vec{d})$, consider the pairing P on the $2m'$ points, and let X denote the set of pairs in P which are involved in loops or multiple edges of the kernel K of $G(H)$. The probability that two given points are paired in P is $1/(2m'-1)$. Thus, the expected number of loops in K is, for $\vec{d} \in \tilde{\mathcal{D}}$,

$$\frac{1}{2m'-1} \sum_{i \in T(\vec{d})} \binom{d_i}{2} = O(1)$$

by (5.2) and (4.1). Similarly, the expected number of *pairs* of pairs in P which create two parallel non-loop edges in K is

$$\frac{1}{(2m'-1)(2m'-3)} \sum_{i \neq j \in T(\vec{d})} 2\binom{d_i}{2}\binom{d_j}{2} = O(1).$$

Hence $\mathbf{E}|X| = O(1)$.

For any pair in X to become a loop or part of a multiple edge of $G(H)$, it must be assigned less than two numbers by the function f in the kernel configuration. The probability that f assigns none of the $m - m'$ numbers in $[n] \setminus T$ to a given pair is $\frac{[m-2]_{m-m'}}{[m-1]_{m-m'}} = \frac{m'-1}{m-1}$. The probability it assigns exactly one of the numbers is (since the unique number assigned can be chosen in $m - m'$ ways)

$$(m-m')\frac{[m-3]_{m-m'-1}}{[m-1]_{m-m'}} = \frac{(m-m')(m'-1)}{(m-1)(m-2)} < \frac{m'-1}{m-1}.$$

Hence the expected number of pairs involved in loops and multiple edges of $G(H)$ is at most

$$\mathbf{E}|X|\left(\frac{m'-1}{m-1}\right) = O\left(\frac{m'}{m}\right) = O\left(\frac{r}{n}\right)$$

by (4.1). So by the first moment principle, $\mathbf{P}(\mathbf{simple}) = 1 - O(r/n)$.

By the last statement in the proof of Łuczak [13, Lemma 12.1(i)] applied to the pairing P, the kernel of $G(H)$ is connected with probability $1 - O(1/m')$. When the kernel of $G(H)$ is connected, so is $G(H)$. The lemma follows by (4.1). ∎

This theorem easily gives the asymptotic probability of connectedness of a random 2-core in the low density case, complementing the results in [13] for r/n bounded away from 0.

Corollary 4 *If $r \to \infty$ and $r = o(n)$ then, uniformly over $\vec{d} \in \tilde{\mathcal{D}}$, the probability that a graph chosen u.a.r. from $G_2(\vec{d})$ is connected is equal to $e^{3/4}\sqrt{m'/m}\,(1 - O(r^{-1} + rn^{-1}))$.*

Proof. Applying Theorem 7 for $\vec{d} \in \tilde{\mathcal{D}}$, we have $\eta(\vec{d}) = 1 + O(r/n)$ by (5.2), and thus the exponential factor in (3.1) is $e^{-3/4} + O(r/n + n^{-1/2})$. This together with Corollary 3 and Lemma 5 produce the result. ∎

We are now ready to sum over degree sequences to obtain Theorem 1. Recall $C_2^{(1)}(n,m)$ is the number of connected 2-cores with n vertices and m edges, and $Q(n,m)$ is defined in (2.1).

Proof of Theorem 1 Write $\vec{Y} = (Y_1, \dots, Y_n)$ where, as in (3.8), $Y_1, \dots, Y_n$ are n independent copies of $Y(2, \lambda)$. By Corollary 3, with $w(\vec{d}) = \mathbf{P}(\mathbf{cs}(\vec{d}))\sqrt{m'}$ (noting that m' is also a function of $\vec{d}$),

$$\begin{aligned} C_2^{(1)}(n,m) &= \left(1 - O\left(r^{-1}\right)\right) \sum_{\vec{d} \in \mathcal{D}_{n,m}} \frac{(2m-1)!!\, w(\vec{d})}{\sqrt{m} \prod_{1 \le i \le n} d_i!} \\ &= \left(1 - O\left(r^{-1}\right)\right) (2m-1)!!\, Q(n,m) m^{-1/2} \mathbf{E}(w(\vec{Y}) \mid \Sigma) \end{aligned} \tag{5.3}$$

where $(. \mid \Sigma)$ denotes conditioning on the event $\sum_j Y_j = 2m$, i.e. the event $\vec{Y} \in \mathcal{D}_{n,m}$.

For $\vec{Y} \in \mathcal{D}_{n,m}$, we have $m' = m'(\vec{Y}) = O(r)$ by (4.1). Denoting probability in the kernel configuration model by $\mathbf{P}_H$, we have

$$\begin{aligned} \mathbf{E}(w(\vec{Y}) \mid \Sigma) &= \mathbf{E}(\sqrt{m'} \mid \Sigma) - O(\sqrt{r})\Big(1 - \mathbf{E}\Big(\mathbf{P}_H(\mathbf{cs}(\vec{Y})) \mid \Sigma\Big)\Big) \\ &= \mathbf{E}(\sqrt{m'} \mid \Sigma) - O(\sqrt{r})\left(1 - \mathbf{E}(\mathbf{P}_H(\mathbf{cs}(\vec{Y})) \mid \tilde{\mathcal{D}}) + \mathbf{P}\left(\tilde{\mathcal{D}}^c \mid \Sigma\right)\right) \\ &= \mathbf{E}(\sqrt{m'} \mid \Sigma) - O(\sqrt{r})\left(r^{-1} + rn^{-1} + \mathbf{P}\left(\tilde{\mathcal{D}}^c \mid \Sigma\right)\right) \end{aligned} \tag{5.4}$$

by Lemma 5, where X^c denotes the complement of X.

First consider the unconditional probability $\mathbf{P}\left(\tilde{\mathcal{D}}^c\right)$. Noting that $\lambda \to 0$ in (3.17),

$$\mathbf{P}(\max_i Y_i > 6 \log n) = O(n^{-2}).$$

Hence, denoting the second condition in (5.2) by W, (3.16) and (3.15) imply

$$\mathbf{P}\left(\tilde{\mathcal{D}}^c \mid \Sigma\right) \le \frac{\mathbf{P}\left(\tilde{\mathcal{D}}^c\right)}{\mathbf{P}(\mathcal{D}_{n,m})} = O(\sqrt{r})(n^{-2} + \mathbf{P}(W^c)). \tag{5.5}$$

By Lemma 1(a),

$$\lambda = 3r/(2n) + O(r^2 n^{-2}) \tag{5.6}$$

and so by (3.20), $m\mathbf{E}\eta(\vec{d}) = n + 2r + O(r/n)$. So, if W is false,

$$m\eta(\vec{d}) = \sum_i \binom{d_i}{2} \ge 2(n - n') + 4r \ge n + 3r$$

by (4.1), and so $\eta(\vec{d}) - \mathbf{E}\eta(\vec{d}) > r/(2m) \sim r/(2n)$. Thus, if $r = O(n^{1-\epsilon})$ for some $\epsilon > 0$, (3.19) gives $\mathbf{P}(W^c) = O(n^{-1} + r^{-3/2})$, and if $n^{1-\epsilon} < r = o(n)$, then (3.18) gives $\mathbf{P}(W^c) = O(n^{-1})$. Thus from (5.5),

$$\mathbf{P}\left(\tilde{\mathcal{D}}^c \mid \Sigma\right) = O(r^{-1} + rn^{-1}). \tag{5.7}$$

Finally, we may apply (5.6) and the forthcoming Lemma 6 to show that $\mathbf{E}(\sqrt{m'} \mid \Sigma) = \sqrt{3r/2}(1 + O(r^{-1} + rn^{-1}))$. The theorem follows from this and (5.3), (5.4) and (5.7). Note that if one is satisfied with a weaker error bound in the result of this theorem (but still $o(1)$), it is easier to estimate $\mathbf{E}(\sqrt{m'} \mid \Sigma)$ by observing that $n' = n'(\vec{Y})$ is binomially distributed, with expectation $n\mathbf{P}(Y_1 > 2) = 2r + O(r/n)$, and applying Chernoff's bound to show sharp concentration of n'. This yields $\mathbf{E}(\sqrt{m'} \mid \Sigma) = \sqrt{3r/2}(1 + O(r^{-1/2}\log r + rn^{-1}))$. ∎

The result postponed from the end of the previous proof is the following.

Lemma 6 *Let $Y = Y(2,\lambda)$ as in (3.7), such that $\mathbf{E}(Y) = 2m/n$. Let $Y_1, \ldots, Y_n$ be independent copies of Y. Let $m' = \sum_{j=1}^n Y_j\,\mathbf{I}(Y_j \ge 3)$. Suppose that $m - n = o(n)$. Then*

$$\mathbf{E}\Big(\sqrt{m'}\,\Big|\,\sum_j Y_j = 2m\Big) = \sqrt{n\lambda}(1 + O((n\lambda)^{-1})).$$

Proof. As before, we denote the event $\sum_j Y_j = 2m$, by Σ. Begin with some simple algebra: it is easy to verify that for $a \ge 0$, $b > 0$,

$$\sqrt{a} - \sqrt{b} = \frac{a-b}{2\sqrt{b}} - R$$

where

$$0 \le R = \frac{(b-a)^2}{2\sqrt{b}(\sqrt{b}+\sqrt{a})^2} \le \frac{(b-a)^2}{b^{3/2}}.$$

Using these facts for $a = m'$, $b = \mathbf{E}(m' \mid \Sigma)$ and computing the conditional expectation, we have

$$\Big|\mathbf{E}(\sqrt{m'} \mid \Sigma) - \sqrt{\mathbf{E}(m' \mid \Sigma)}\Big| \le \frac{\mathbf{Var}(m' \mid \Sigma)}{(\mathbf{E}(m' \mid \Sigma))^{3/2}}. \tag{5.8}$$

What remains is to show that $\mathbf{E}(m' \mid \Sigma)$ is very close to $n\lambda$, and that $\mathbf{Var}(m' \mid \Sigma)$ is of order $n\lambda$.

Observe first that

$$\mathbf{P}(Y_1 = k \mid \Sigma) = \mathbf{P}(Y = k)\frac{\mathbf{P}(\sum_{j=2}^n Y_j = 2m - k)}{\mathbf{P}(\sum_{j=1}^n Y_j = 2m)}.$$

The ratio of probabilities here is $1 + O((\mathbf{E}Y - k)^2/n\lambda)$ for $k \le \sqrt{n\lambda}$, by [1, equation (6)]. So

$$\mathbf{P}(Y_1 = k \mid \Sigma) = \mathbf{P}(Y_1 = k)\left(1 + O((\mathbf{E}Y - k)^2/n\lambda)\right). \tag{5.9}$$

For all $k \ge 2$,

$$\mathbf{P}(Y_1 = k \mid \Sigma) \le_b \sqrt{n\lambda}\,\mathbf{P}(Y_1 = k). \tag{5.10}$$

Analogously to (5.9) and (5.10), for $k_1, k_2 \le \sqrt{n\lambda}$

$$\mathbf{P}(Y_1 = k_1, Y_2 = k_2 \mid \Sigma) = \mathbf{P}(Y_1 = k_1, Y_2 = k_2)\left(1 + O((2\mathbf{E}Y - k_1 - k_2)^2/n\lambda)\right), \tag{5.11}$$

and for all $k_1, k_2 \ge 2$,

$$\mathbf{P}(Y_1 = k_1, Y_2 = k_2 \mid \Sigma) \le \sqrt{n\lambda}\,\mathbf{P}(Y_1 = k_1)\mathbf{P}(Y_2 = k_2). \tag{5.12}$$

Using (5.9) and (5.10), and denoting for brevity $t = n\lambda$,

$$\begin{aligned}\mathbf{E}(Y_1\,\mathbf{I}(Y_1 \geq 3) \mid \Sigma) &= \sum_{3\leq k\leq t^{1/2}} k\,\mathbf{P}(Y_1 = k) + O\left(t^{-1}\sum_{k=3}^{t^{1/2}} k^3\,\mathbf{P}(Y_1 = k)\right) \\ &\quad + O\left(t^{1/2}\sum_{k\geq t^{1/2}} k\,\mathbf{P}(Y_1 = k)\right). \qquad (5.13)\end{aligned}$$

The first summation here equals

$$\sum_{3\leq k<\infty} \frac{k\lambda^k}{k!f(\lambda)} + O\left(\frac{t^{1/2}\lambda^{t^{1/2}}}{f(\lambda)\left(\frac{t^{1/2}}{e}\right)^{t^{1/2}}}\right) = \lambda(1 + O(t^{-K}))$$

for any $K > 0$, since $f(\lambda) = \sum_{k\geq 2}\lambda^k/k!$. Likewise, the second remainder term in (5.13) is also $O(n^{-K})$ for any $K > 0$. The first remainder term in (5.9) is of order $t^{-1}\lambda^3/f(\lambda) = O(t^{-1}\lambda)$. Therefore

$$\mathbf{E}(Y_1\,\mathbf{I}(Y_1 \geq 3) \mid \Sigma) = \lambda(1 + O(t^{-1})). \qquad (5.14)$$

Analogously, $\mathbf{E}(Y_1^2\,\mathbf{I}(Y_1 \geq 3) \mid \Sigma) = O(\lambda)$ and thus

$$\mathbf{Var}(Y_1\,\mathbf{I}(Y_1 \geq 3) \mid \Sigma) = O(\lambda). \qquad (5.15)$$

Furthermore, using (5.11) and (5.12), we obtain in the same fashion that

$$\begin{aligned}\mathbf{E}(Y_1\,\mathbf{I}(Y_1 \geq 3)Y_2\,\mathbf{I}(Y_2 \geq 3) \mid \Sigma) &= \lambda^2(1 + O(t^{-1})) \\ &\quad + O\left(t^{-1}\sum_{k_1,k_2\geq 3}(k_1^4 + k_2^4)\mathbf{P}(Y_1 = k_1)\mathbf{P}(Y_2 = k_2)\right) \\ &= \lambda^2(1 + O(t^{-1})). \qquad (5.16)\end{aligned}$$

From (5.14) it follows that

$$\mathbf{E}(m' \mid \Sigma) = t(1 + O(t^{-1})), \qquad (5.17)$$

and from (5.14–5.16) that

$$\begin{aligned}\mathbf{Var}(m' \mid \Sigma) &= n\mathbf{Var}(Y_1\mathbf{I}(Y_1 \geq 3) \mid \Sigma) \\ &\quad + n(n-1)\left[\mathbf{E}(Y_1\mathbf{I}(Y_1 \geq 3)Y_2\mathbf{I}(Y_2 \geq 3) \mid \Sigma) - \mathbf{E}^2(Y_1\mathbf{I}(Y_1 \geq 3) \mid \Sigma)\right] \\ &= O(t). \qquad (5.18)\end{aligned}$$

By (5.17) and (5.18), the right-hand side of (5.8) is seen to be of order $t^{-1/2}$, and thus $\mathbf{E}(\sqrt{m'} \mid \Sigma) - t^{1/2} = O(t^{-1/2})$, which proves the lemma. ∎

6 Counting connected 2-cores in the full range of m using inversion

In this section we obtain an asymptotic formula for $C_2^{(1)}(n,m)$ when $m = O(n \log n)$ and $r := 2m - 2n \to \infty$. This extends the range considered in the previous section. The method is quite different, using our enumeration formula for 2-cores.

Proof of Theorem 2

Step 1. First let us determine the limiting probability that a random 2-core has no isolated cycles. Let X_n denote the total number of isolated cycles, and set

$$R_{nm}(a) = \frac{n! C_2(n-a, m-a)}{(n-a)! C_2(n,m)},$$

which, to be non-zero, requires $m - a \le \binom{n-a}{2}$ and hence

$$n - a \ge \sqrt{r}. \tag{6.1}$$

Then, since $\sum_{\ell \ge 3} \frac{x^\ell}{2\ell}$ is the exponential generating function which counts (undirected) cycles,

$$\begin{aligned} \mathbf{E}\left(\binom{X_n}{k}\right) &= \frac{1}{k!} \sum_{a \ge 0} \binom{n}{a} \frac{C_2(n-a, m-a)}{C_2(n,m)} \cdot a! [x^a] \left(\sum_{\ell \ge 3} \frac{x^\ell}{2\ell} \right)^k \\ &= \frac{1}{k!} \sum_{a \ge 0} R_{nm}(a) [x^a] \left(\sum_{\ell \ge 3} \frac{x^\ell}{2\ell} \right)^k, \end{aligned}$$

with square brackets denoting the extraction of the coefficient.

Using the inversion formula (equivalent to inclusion-exclusion)

$$\mathbf{P}(X_n = 0) = \sum_k (-1)^k \mathbf{E}\left(\binom{X_n}{k}\right),$$

and

$$\sum_{\ell \ge 3} \frac{x^\ell}{2\ell} = \log(1-x)^{-1/2} - \frac{x}{2} - \frac{x^2}{4},$$

we obtain

$$\mathbf{P}(X_n = 0) = \sum_{a \ge 0} R_{nm}(a) [x^a] h(x), \tag{6.2}$$

where h is defined in (2.3). We will show that $R_{nm}(a) \sim \sigma^a$ for the dominant values of a, where σ is given by (2.4). One can then expect that

$$\mathbf{P}(X_n = 0) \sim \sum_{a \ge 0} [x^a] h(x) \sigma^a = h(\sigma), \tag{6.3}$$

which is what we require, together with an estimate of the error. It will become apparent that $h(\sigma) \to 0$ if $r = o(n)$, so that (since $R_{nm}(0) = 1$) there is cancellation in (6.3). Thus, obtaining asymptotics is more delicate in this case.

Using (3.11) for $C_2(n,m)$ and (3.4) (with parameter λ) for $C_2(n-a, m-a)$, and using (3.8), (3.16), (3.15) and (3.12), we obtain that $R_{nm}(a)$ is of order

$$\begin{aligned} O(e^{\bar{\eta}^2}) \prod_{j=1}^{a} \frac{n-j+1}{2m-(2j-1)} \sqrt{\frac{n}{n-a}} \cdot \frac{\lambda^{2a}}{f(\lambda)^a} \ &=_b \ e^{\bar{\eta}^2} \left(\frac{n\lambda^2}{2mf(\lambda)} \right)^a \sqrt{\frac{n}{n-a}} \\ &=_b \ e^{\bar{\eta}^2} \sigma^a \sqrt{\frac{n}{n-a}}, \end{aligned} \tag{6.4}$$

by (3.6) and (3.7), uniformly for all feasible a (see (6.1)), where σ is given by (2.4). (Here and elsewhere we write $a_n =_b b_n$ if $a_n = O(b_n)$ and the expression for b_n is not short.) Note from (3.7) that

$$\sigma < n/m = 1 - r/m < e^{-r/m}. \tag{6.5}$$

We also observe that

$$h_a := [x^a]h(x) = O(a^{-3/2}), \tag{6.6}$$

as $h(x)$ is $(1-x)^{1/2}$ times an entire function.

Introduce $A_n = \delta n r^{-1/2}$, where $\delta > 0$ is a fixed small number.

Consider first $a \geq A_n$. We have

$$\begin{aligned} \sum_{a \geq A_n} R_{nm}(a) h_a \ &=_b \ n^{1/2} e^{\bar{\eta}^2 - rA_n/m} n^{1/2} \sum_{a \geq A_n} \frac{1}{a^{3/2}(n-a)^{1/2}} \\ &= \ O(e^{-\Theta(nm^{-1/2})}), \end{aligned} \tag{6.7}$$

since

$$\bar{\eta} = O(1+\lambda) = O(r/n) = O(\log n), \tag{6.8}$$

by (3.10), Lemma 1(b) and $m = O(\log n)$. We also notice at once that

$$\sum_{a \geq A_n} \sigma^a |h_a| = O(e^{-rA_n/m}) =_b \exp(-\Theta(r^{1/2} + nr^{-1/2})), \tag{6.9}$$

which we will use at the end of this proof to bound the tail of a similar series, performing a "swap-the-tails" operation.

Turn to $a < A_n$. Uniformly for these a

$$a/n = o(1), \quad a^2 r/n^2 \leq \delta^2, \tag{6.10}$$

the relations we will need shortly. This time we need to use (3.9) for both $C_2(n,m)$ and $C_2(n-a, m-a)$. Notice that usage of (3.11) would have resulted—because of the local probability term—in an extra factor $1 + O(r^{-1})$, which would turn out to be too far from 1 for our purposes, due to the cancellation mentioned above, after (6.3). The idea is to show that, for moderate a, the ratio of the local probabilities is closer to 1 than $1+O(r^{-1})$. To this end, we observe that if $\lambda(a)$ is the parameter for $C_2(m-a,n-a)$, and $Y_j(a)$ are copies of the truncated Poissons $Y(\lambda(a))$ defined in (3.5), then, writing Y_j for $Y_j(\lambda(0))$,

$$\mathbf{P}\left(\sum_{j=1}^{n-a} Y_j(a) = 2(m-a)\right) = \frac{\lambda(a)^{2(m-a)}}{f(\lambda(a))^{n-a}} \cdot \frac{f(\lambda)^{n-a}}{\lambda^{2(m-a)}} \cdot \mathbf{P}\left(\sum_{j=1}^{n-a} Y_j = 2(m-a)\right).$$

Consequently

$$\begin{aligned} R_{nm}(a) &= (1+O(\xi))\frac{\lambda^{2a}}{f(\lambda)^a}\prod_{j=1}^{a}\frac{n-j+1}{2m-(2j-1)} \\ &\times \frac{\zeta(a)}{\zeta(0)} \cdot \frac{\mathbf{P}\left(\sum_{j=1}^{n-a} Y_j = 2(m-a)\right)}{\mathbf{P}\left(\sum_{j=1}^{n} Y_j = 2m\right)}. \end{aligned} \tag{6.11}$$

Here

$$\zeta(a) = \exp(-\bar{\eta}(\bar{c})/2 - \bar{\eta}(\bar{c})^2/4), \quad \bar{c} = \frac{m-a}{n-a}, \tag{6.12}$$

and $\bar{\eta}(c) = \bar{\eta}_c$ defined in (3.10). Using

$$\frac{n-a+1}{2m-(2a-1)} \le \frac{n-j+1}{2m-(2j-1)} \le \frac{n}{2m-1}, \quad (1 \le j \le a),$$

and exponentiating, by the definition of σ in (2.4) we see that the expression in the first line of (6.11) is

$$\sigma^a[1+O(a^2r/n^2 + a/n)]. \tag{6.13}$$

The remainder term here is $O(\delta^2)$, uniformly for $a \le A_n$. To proceeed, $\lambda(a)$ is the minimum point of $f(x)^{n-a}/x^{2(m-a)}$, so that

$$\frac{\lambda(a)(e^{\lambda(a)}-1)}{e^{\lambda(a)}-1-\lambda(a)} = \frac{2(m-a)}{n-a}. \tag{6.14}$$

Via implicit differentiation of (6.14) we find

$$\frac{d\lambda(a)}{da} = 2\frac{r(e^{\lambda(a)}-1-\lambda(a))^2}{(n-a)^2((e^{\lambda(a)}-1)^2-\lambda(a)^2e^{\lambda(a)})} = O(rn^{-2}),$$

as the $\lambda(a)$-dependent fraction is $O(1)$ for all $a < n$. So, by (3.12),

$$\lambda(a) = \lambda + O(arn^{-2}) = \lambda(1+O(an^{-1})). \tag{6.15}$$

Combining (6.15), (3.10) and (6.12), we obtain

$$\frac{\zeta(a)}{\zeta(0)} = 1 + O(arn^{-2} + ar^2n^{-3}). \tag{6.16}$$

As for the ratio of the probabilities in (6.11), the following result will be proved at the end of this section.

Lemma 7 *For $a = o(n)$,*

$$\frac{\mathbf{P}\left(\sum_{j=1}^{n-a} Y_j = 2(m-a)\right)}{\mathbf{P}\left(\sum_{j=1}^{n} Y_j = 2m\right)} = 1 + O(an^{-1} + a^2rn^{-2}). \tag{6.17}$$

Lemma 7 together with (6.13) and (6.16) reduce (6.11) to

$$R_{nm}(a) = (1 + O(\xi + a^2rn^{-2} + ar^2n^{-3} + an^{-1}))\sigma^a. \tag{6.18}$$

So, combining (6.18) and (6.6),

$$\begin{aligned}\left|\sum_{a<A_n} R_{nm}(a)h_a - \sum_{a<A_n} \sigma^a h_a\right| &=_b \sum_{a<A_n} \sigma^a \left(a^{-3/2}\xi + a^{1/2}\frac{r}{n^2} + a^{-1/2}\frac{r^2}{n^3} + a^{-1/2}\frac{1}{n}\right) \\ &=_b \xi + \left(\frac{m}{r}\right)^{3/2}\frac{r}{n^2} + \left(\frac{m}{r}\right)^{1/2}\left(\frac{r^2}{n^3} + \frac{1}{n}\right),\end{aligned} \tag{6.19}$$

using (6.5) to bound σ, and noting that hence for $p > -1$

$$\sum_{a=0}^{\infty} \sigma^a a^p =_b \sum_{a=0}^{\infty} \sigma^a \binom{a+p}{a} = (1-\sigma)^{-p-1} < \left(\frac{m}{r}\right)^{p+1}.$$

Combining (6.19), (6.7) and (6.9), we have

$$\begin{aligned}\mathbf{P}(X_n = 0) &= \sum_{a\ge 0} R_{nm}(a)h_a \\ &= \sum_{a<A_n} R_{nm}(a)h_a + O(e^{-\Theta(nm^{-1/2})}) \\ &= \sum_{a<A_n} \sigma^a h_a + O\left(\xi + (m/r)^{3/2}rn^{-2} + (m/r)^{1/2}(r^2n^{-3} + n^{-1})\right) \\ &= h(\sigma) + O\left(\xi + (m/r)^{3/2}rn^{-2} + (m/r)^{1/2}(r^2n^{-3} + n^{-1})\right) \\ &= h(\sigma) + O\left(\xi + (nr)^{-1/2}\right)\end{aligned} \tag{6.20}$$

since ξ (defined in Theorem 8) dominates the error terms when r/n is bounded away from 0 (noting $m = O(n \log n)$), whilst in the case that $r = O(n)$, we have $m = O(n)$ so that the other error terms are $O((nr)^{-1/2})$. Thus

$$\left|\frac{\mathbf{P}(X_n = 0)}{h(\sigma)} - 1\right| =_b h(\sigma)^{-1}\left(\xi + (nr)^{-1/2}\right). \tag{6.21}$$

Note also that

$$h(\sigma) = O(1) \text{ for all } \sigma > 0, \tag{6.22}$$

and that $h(\sigma) \to 0$ is equivalent to

$$h(\sigma) \sim e^{3/4}(1-\sigma)^{1/2} \sim e^{3/4}\left(\frac{3r}{n}\right)^{1/2}. \tag{6.23}$$

Thus, in general, the estimate (6.20) implies the simple bound

$$\mathbf{P}(X_n = 0) \le A\sqrt{\frac{r}{n}}, \tag{6.24}$$

for some (large enough) absolute constant A.

Step 2. Now consider the event W_n that a random 2-core has no cycle components and is disconnected. Then

$$\frac{C_2^{(1)}(n,m)}{C_2(n,m)} = \mathbf{P}(X_n = 0) - \mathbf{P}(W_n), \tag{6.25}$$

so we turn to bounding $\mathbf{P}(W_n)$.

Clearly $\mathbf{P}(W_n) \le E_n$ where E_n is the expected number of ways to partition $[n]$ into an ordered pair of two disjoint subsets, $\mathcal{S}_1$ and $\mathcal{S}_2$, such that the subgraphs G_1, G_2 induced by $\mathcal{S}_1$, $\mathcal{S}_2$ are disjoint, and neither of them contains a cycle component. Without loss of generality, let $\nu = |\mathcal{S}_1| \le n/2$. Let μ denote the number of edges in G_1. As neither G_1 nor G_2 have no cycle components, $\mu > \nu$ and $m - \mu > n - \nu$. Consequently

$$\mu - r < \nu < \mu, \quad \mu < m - n/2,$$

where $r = m - n$. Let $E_n(\nu,\mu)$ denote the expected number of partitions of the random 2-core in which G_1 has these parameters ν, μ. By (6.24), we see that

$$E_n(\nu,\mu) \le_b \binom{n}{\nu} \frac{\sqrt{\frac{\mu-\nu}{\nu}}C_2(\nu,\mu)\sqrt{\frac{(m-\mu)-(n-\nu)}{n-\nu}}C_2(n-\nu, m-\mu)}{C_2(n,m)}. \tag{6.26}$$

To get a tractable bound for this fraction, we use (3.9) for $C_2(n,m)$, (3.3) for $C_2(\nu,\mu)$, and (3.4) for $C_2(n-\nu, m-\mu)$, all with the parameter $\lambda = \lambda(n,m)$. Invoking also a Stirling-based bound (see (5.1))

$$\frac{(2\mu-1)!!(2(m-\mu)-1)!!}{(2m-1)!!} \le_b \frac{1}{\mu^{1/2}\binom{m}{\mu}},$$

we transform (6.26) into

$$E_n(\nu,\mu) =_b E_n^*(\nu,\mu) := \sqrt{\frac{r}{n}}\,\sqrt{\frac{\mu-\nu}{\mu\nu}}\,\frac{\binom{n}{\nu}}{\binom{m}{\mu}}. \tag{6.27}$$

Now

$$\frac{E_n^*(\nu,\mu+1)}{E_n^*(\nu,\mu)} \le \sqrt{2}\,\frac{\mu+1}{m-\mu} < 1/2,$$

if $\mu \le m/5$ and m is large enough. Therefore

$$\begin{aligned}
\sum_{\{\mu,\nu:\mu\le m/5\}} E_n^*(\nu,\mu) \quad &= \quad {}_b\sum_{\nu} E_n^*(\nu,\nu+1) \\
&=_b \quad \sqrt{\frac{r}{n}}\sum_{\nu}\nu^{-1}\frac{\binom{n}{\nu}}{\binom{m}{\nu+1}} =_b m^{-1}\sqrt{\frac{r}{n}}\sum_{\nu}\frac{\binom{n}{\nu}}{\binom{m}{\nu}} \\
&\le \quad m^{-1}\sqrt{\frac{r}{n}}\sum_{\nu\ge 0}(n/m)^{\nu} = r^{-1}\sqrt{\frac{r}{n}}.
\end{aligned} \tag{6.28}$$

Consider $\mu > m/5$. Using $\mu - \nu \le r$ and

$$\nu^{-1/2}\binom{n}{\nu} =_b \frac{\nu^{1/2}}{n}\binom{n+1}{\nu+1},$$

we have

$$E_n^*(\nu,\mu) =_b E_n^{**}(\nu,\mu) := \frac{r\nu^{1/2}}{n^{3/2}m^{1/2}}\,\frac{\binom{n+1}{\nu+1}}{\binom{m}{\mu}}.$$

As a function of ν, $E_n^{**}(\nu,\mu)$ increases for $\nu \le n/2$. Consider $\mu \le n/2$. We have

$$\begin{aligned}\sum_{\{\nu,\mu: m/5<\mu\le n/2\}} E_n^{**}(\nu,\mu) &=_b \frac{r^2}{nm^{1/2}}\sum_{\mu>m/5}\frac{\binom{n+1}{\mu}}{\binom{m}{\mu}} \le \frac{r^2}{nm^{1/2}}\sum_{\mu>m/5}((n+1)/m)^{\mu} \\ &= \frac{re^{-\Theta(r)}}{nm^{1/2}} \ll r^{-1}\sqrt{\frac{r}{n}}, \qquad (6.29)\end{aligned}$$

as $(m/n)^{1/2} < r$. If $\mu > n/2$ then, using $\mu \le m - n/2$, we have $\binom{m}{\mu} \ge \binom{m}{n/2}$. Therefore

$$\begin{aligned}\sum_{\{\nu,\mu:\mu\ge\max(n/2,m/5)\}} E_n^{**}(\nu,\mu) &=_b \frac{r^2}{n\sqrt{m}}m\,\frac{\binom{n}{n/2}}{\binom{m}{n/2}} = r^2\frac{m^{1/2}}{n}e^{-\Theta(r)} \\ &\ll r^{-1}\sqrt{\frac{r}{n}}. \qquad (6.30)\end{aligned}$$

Combining (6.29), (6.30) and (6.28) we conclude that

$$E_n = \sum_{\nu,\mu} E_n(\nu,\mu) = O(r^{-1}\sqrt{r/n}) = O(r^{-1}\mathbf{P}(X_n = 0)).$$

Therefore, in view of (6.25),

$$\frac{C_2^{(1)}(n,m)}{C_2(n,m)} = (1 + O(r^{-1}))\mathbf{P}(X_n = 0).$$

In combination with (6.21), (6.23) and (6.22), this gives

$$\frac{C_2^{(1)}(n,m)}{C_2(n,m)} = \left(1 + O\left(\xi n^{1/2}r^{-1/2} + r^{-1}\right)\right)h(\sigma)$$

for $r = o(n)$, and otherwise (noting the definition of ξ in Theorem 8) the right hand side can be written as $\left(1 + O\left(n^{-1/2+\epsilon/2}\right)\right)h(\sigma)$ since $r < n^{1+\epsilon}$. It is now easy to check that gives the theorem in all cases, redefining ϵ as $\epsilon/2$. ∎

Proof of Lemma 7. Let $Z = Y - 2$ (see (3.5) and recall that the Y_j are copies of Y) and let $Z_1, \dots, Z_n$ be independent copies of Z. Then

$$\frac{\mathbf{P}\left(\sum_{j=1}^{n-a} Y_j = 2(m-a)\right)}{\mathbf{P}\left(\sum_{j=1}^{n} Y_j = 2m\right)} = \frac{\mathbf{P}\left(\sum_{j=1}^{n-a} Z_j = 2r\right)}{\mathbf{P}\left(\sum_{j=1}^{n} Z_j = 2r\right)} = \frac{P_{na}}{P_n}.$$

Let $\phi(x) = \mathbf{E}(e^{ixZ})$, the characteristic function of Z. Then, by the inversion formula,

$$\begin{aligned} P_{na} &= (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-i2rx} \phi(x)^{n-a}\, dx, \\ P_n &= (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-i2rx} \phi(x)^{n}\, dx. \end{aligned}$$

So, using

$$\begin{aligned} \phi^{n-a} - \phi^n &= \phi^{n-a}(1 - \phi^a) \\ &= \phi^{n-a}(1 + ixa\mathbf{E}Z - \phi^a) - ixa\mathbf{E}Z\phi^{n-a}, \end{aligned}$$

we get

$$|P_{na} - P_n| =_b |I_1| + I_2,$$

where

$$\begin{aligned} I_1 &= \int_{-\pi}^{\pi} (-ixa\mathbf{E}Z) e^{-i2rx} \phi(x)^{n-a}\, dx \\ I_2 &= \int_{-\pi} |\phi(x)|^{n-a} |1 + ixa\mathbf{E}Z - \phi(x)^a|\, dx. \end{aligned}$$

Consider I_2 first. Using

$$e^{iz} = 1 + iz + O(z^2), \quad z \in \mathbb{R},$$

we bound

$$\begin{aligned} 1 + ixa\mathbf{E}Z - \phi(x)^a &= \mathbf{E}\left(1 + ix \sum_{j=1}^{a} Z_j - \exp\left(ix \sum_{j=1}^{a} Z_j\right)\right) \\ &=_b x^2 \mathbf{E}\left[\left(\sum_{j=1}^{a} Z_j\right)^2\right] =_b x^2(a\mathbf{Var}Z + a^2\mathbf{E}^2(Z)) \\ &=_b x^2(a(r/n) + a^2(r/n)^2) \end{aligned}$$

since $\mathbf{E}Z = 2r/n$ and $\mathbf{Var}Z = \mathbf{Var}Y = \Theta(r/n)$ by Lemma 2. Also

$$\begin{aligned} |\phi(x)| &= |\mathbf{E}(e^{ix(Y-2)})| = |\mathbf{E}(e^{ixY})| = \frac{|f(\lambda e^{ix})|}{f(\lambda)} \\ &\leq \exp\left(-\lambda \frac{1 - \cos x}{3}\right); \end{aligned}$$

see [15] for the last inequality. Therefore, using $1 - \cos x = \Theta(x^2)$ and $a = o(n)$,

$$|\phi(x)|^{n-a} \leq \exp(-\Theta(n\lambda x^2)) = \exp(-\Theta(rx^2)). \tag{6.31}$$

Hence

$$\begin{aligned} I_2 &=_b (a(r/n) + a^2(r/n)^2) \int_{-\infty}^{\infty} x^2 \exp(-\Theta(rx^2))\, dx \\ &=_b r^{-1/2}(a/n + a^2 r/n^2). \end{aligned} \tag{6.32}$$

Turning to I_1, write $I_1 = I_{11} + I_{12}$, where I_{11} (I_{12} resp.) is the contribution from x such that $|x| \le r^{-\delta}$ ($|x| > r^{-\delta}$ resp.); $\delta \in (1/3, 1/2)$. Using (6.31),

$$\begin{aligned} I_{12} &=_b \frac{ar}{n} \int_{r^{-\delta}}^{\infty} x \exp(-\Theta(rx^2))\, dx \\ &=_b \frac{ar}{n} e^{-\Theta(r^{1-2\delta})}. \end{aligned} \tag{6.33}$$

Consider I_{11}. We know that both $\mathbf{E}Z = (d/d(ix)) \log \phi(x)|_{x=0}$ and $\mathbf{Var}Z = (d^2/d(ix)^2) \log \phi(x)|_{x=0}$ are of order r/n. It is easy to check that this pattern holds for all other derivatives of $\log \phi(x)$ at $x = 0$, i.e. the semiinvariants of $\phi(x)$. So

$$\begin{aligned} \phi(x) &= \exp\left(ix\mathbf{E}Z - \frac{x^2}{2}\mathbf{Var}Z + O(|x|^3 r/n)\right) \\ &= \exp\left(ix\mathbf{E}Z - \frac{x^2}{2}\mathbf{Var}Z\right)(1 + O(|x|^3 r/n)). \end{aligned}$$

Notice that

$$(1 + O(|x|^3 r/n))^{n-a} = \exp(O(|x|^3 r)) = 1 + O(|x|^3 r),$$

as $|x|^3 r = O(r^{-(3\delta-1)}) = o(1)$. Replacing $(1 + O(|x|^3 r/n))$ in the formula for $\phi(x)^{n-a}$ with 1 results, again using $\mathbf{E}Z = 2r/n$, in an error of order

$$\frac{ar^2}{n^2} \int_0^{\infty} x^4 e^{-\Theta(rx^2)}\, dx =_b r^{-1/2} \frac{a}{n^2}. \tag{6.34}$$

So what remains from I_{11} is

$$\begin{aligned} & -i\frac{2ar}{n} \int\limits_{|x| \le r^{-\delta}} x e^{-i2rx} \exp\left(ix(n-a)\mathbf{E}Z - \frac{(n-a)x^2}{2}\mathbf{Var}Z\right) dx \\ &= -i\frac{2ar}{n} \int\limits_{-\infty}^{\infty} x \exp\left(-i\frac{2ra}{n}x - b_{na}(rx^2)\right) dx + D_{na}, \end{aligned} \tag{6.35}$$

where $b_{na} = b + O(a/n)$, $b > 0$, and

$$D_{na} =_b \frac{ar}{n} \int\limits_{|x| \ge r^{-\delta}} |x| e^{-\Theta(rx^2)}\, dx =_b \frac{a}{n} e^{-\Theta(r^{1-2\delta})}. \tag{6.36}$$

Finally, using

$$-\frac{i}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} z e^{-iuz} e^{-z^2/2}\, dz = \frac{d}{du} e^{-u^2/2} = -u e^{-u^2/2},$$

we see that the remaining integral in (6.35) is of order $r^{1/2}(a/n)^2$. Combining the last estimate with (6.32), (6.33), (6.34), and (6.36), we have

$$P_{na} - P_n =_b r^{-1/2} \left(\frac{a}{n} + \frac{a^2 r}{n^2}\right).$$

Since $P_n = \Theta(r^{-1/2})$, the proof of the lemma is complete. ∎

7 Asymptotic enumeration of connected graphs

In this section we prove Theorem 3 and give some related results. Let ν and μ denote the number of vertices and the number of edges, respectively, of the 2-core of a connected graph with n vertices and m edges in total. Note that this 2-core must be connected. Clearly $\mu - \nu = m - n$, and, provided $m \geq n$ the connected graph must have at least one cycle, so $\nu \geq 3$ for n sufficiently large. We often call the 2-core simply the *core*. Given a connected core with ν vertices, there are $\nu n^{n-\nu-1}$ ways to form a forest of ν trees rooted at the core vertices and containing the $n-\nu$ vertices not included in the core. Thus

$$C^{(1)}(n,m) = \sum_{\nu=3}^{n} \binom{n}{\nu} \nu n^{n-\nu-1} C_2^{(1)}(\nu,\mu); \quad \mu = m - n + \nu. \tag{7.1}$$

Our next task is to determine those ν, or equivalently μ, that provide the dominant contribution to the sum. In the following, we will use the following consequence of Stirling's formula (5.1): $k! = (k/e)^k \sqrt{2\pi k}(1 + O((k+1)^{-1}))$ for $k \geq 1$. This is in order to produce estimates of $k!$, and its inverse $1/k!$, correct to a factor $1 + O((k+1)^{-1})$ uniformly for all $k \geq 0$.

Introduce $y = \nu/n$ and $u = \mu/n$; then $c - 1 = u - y$. According to (3.11) and Theorem 2 we can write, for $3 \leq \nu \leq n-1$,

$$\binom{n}{\nu} \nu n^{n-\nu-1} C_2^{(1)}(\nu,\mu) = (1 + O(\beta_1)) n^m F_n(y) \exp(nH(y,\lambda)), \tag{7.2}$$

where

$$H(y,x) = -y \log y - (1-y)\log(1-y) + u \log \frac{2u}{ex^2} + y \log f(x), \tag{7.3}$$

and (recalling $f(\lambda) = e^\lambda - 1 - \lambda$) $\lambda = \lambda(y)$ is the root of

$$\frac{\partial H}{\partial x} = y \frac{f'(x)}{f(x)} - \frac{2u}{x} = 0. \tag{7.4}$$

It is easily verified that, for fixed y, λ minimizes $y \log f(x) - 2u \log x$ and consequently $H(y,x)$. It is also easily verified, using (3.13), that $\lambda(y)$ is decreasing. Furthermore, the factor $F_n(y)$ is given by

$$F_n(y) = \frac{1}{2\pi n} \sqrt{\frac{(1-\sigma)y}{u(1+\bar\eta - 2u/y)(1-y+\rho)}} \exp\left(\frac{1}{2}(\sigma - \bar\eta) + \frac{1}{4}(\sigma^2 - (\bar\eta)^2)\right),$$

where

$$\bar\eta = \frac{\lambda e^\lambda}{e^\lambda - 1}, \quad \sigma = \frac{\lambda}{e^\lambda - 1}, \quad \rho = \begin{cases} 0 & \text{if } y \neq 1 \\ 1/n & \text{if } y = 1, \end{cases},$$

and we have used the approximation for $(n-\nu)!$ as described above. Alternatively, using also (7.4),

$$\begin{aligned} F_n(y) &= \frac{1}{2\pi n \sqrt{1-y+\rho}} \sqrt{\frac{2(e^\lambda - 1 - \lambda)^3}{\lambda(e^\lambda - 1)[(e^\lambda - 1)^2 - \lambda^2 e^\lambda]}} \\ &\quad \times \exp\left(-\frac{\lambda}{2} - \frac{\lambda^2}{4}\coth\frac{\lambda}{2}\right). \end{aligned} \tag{7.5}$$

Finally,

$$\beta_1 = \beta_1(\nu) = \nu^{-a} + (m-n)^{-1} + (n-\nu+1)^{-1} \quad \text{for any } a < 1/2. \tag{7.6}$$

Here $(n-\nu+1)^{-1}$ arises as the error in the approximation of $(n-\nu)!$.

We return to determining ν that dominates the sum in (7.1). Since $\partial^2 H/\partial y^2 < 0$, $H(y,x)$ is concave as a function of y. Since $H(y,\lambda)$ is equal to the minimum of (the concave functions) $H(y,x)$ over all x, it then follows that $\mathcal{H}(y) := H(y,\lambda)$ is also concave. Naturally we aim to find $y \in [0,1]$ that *maximizes* $\mathcal{H}(y)$. Since $\frac{\partial}{\partial x}H(y,x) = 0$ at $x = \lambda$, and $du/dy = 1$, we have

$$\mathcal{H}'(y) = \frac{\partial H(y,x)}{\partial y} = -\log y + \log(1-y) + \log 2u + \log f(\lambda) - 2\log\lambda. \tag{7.7}$$

From this it is easy to see that $\mathcal{H}'(y) \to \infty$ or $-\infty$, respectively, if $y \to 0$ or 1, and hence at a maximum point we have $\mathcal{H}'(y) = 0$, which gives

$$\frac{(1-y)2uf}{y\lambda^2} = 1. \tag{7.8}$$

Combining (7.8) with (7.4) we get

$$1 - y = \frac{y\lambda^2}{2uf} = \frac{\lambda}{e^\lambda - 1}, \quad y = \frac{e^\lambda - 1 - \lambda}{e^\lambda - 1} = \frac{f}{e^\lambda - 1}, \tag{7.9}$$

and, using (7.4) again,

$$u = \frac{\lambda}{2}. \tag{7.10}$$

Plugging these values of $1-y$ and u into $1 - y + u = c$, we obtain an equation for the parameter λ:

$$\frac{\lambda}{2} + \frac{\lambda}{e^\lambda - 1} = c,$$

or equivalently

$$\frac{\lambda}{2}\coth\frac{\lambda}{2} = c. \tag{7.11}$$

From now on we will use $\overline{\lambda}$, $\overline{y}$, and $\overline{u}$ to denote the values determined at the maximum point, given by (7.9), (7.10) and (7.11), and note that $\overline{\lambda} = \overline{\lambda}(c)$, $\overline{y} = \overline{y}(c)$, and $\overline{u} = \overline{u}(c)$.

For later use, note that from (7.4), for $\lambda \to 0$

$$y = (c-1)\left(\frac{6}{\lambda} + O(1)\right). \tag{7.12}$$

This is also helpful in an easy argument, starting with (7.11), showing that if $c \to 1$ (i.e. $m-n = o(n)$), then

$$\overline{\lambda} = \sqrt{12(c-1)} + O(c-1), \quad \overline{y} = \sqrt{3(c-1)} + O(c-1), \tag{7.13}$$

and if $c \to \infty$ $(m/n \to \infty)$, then

$$\overline{\lambda} = 2c + O(ce^{-c}), \quad \overline{y} = 1 - 2ce^{-2c} + O(c^2e^{-3c}). \tag{7.14}$$

If $c-1$ is bounded away from 0 and ∞, then so is $\overline{\lambda}$, and $\overline{y}$ is bounded away from 0 and 1.

Knowing that $\mathcal{H}'(\overline{y})=0$, let us evaluate $\mathcal{H}(\overline{y})$, $F_n(\overline{y})$ and $\mathcal{H}''(\overline{y})$. Using (7.3), (7.7) and (7.9), we see that

$$\begin{aligned}\mathcal{H}(\overline{y}) &= -\log(1-\overline{y})-\overline{u}+(c-1)\log\overline{\lambda}-2(c-1)\log\overline{\lambda}\\ &= \log\frac{e^{\overline{\lambda}}-1}{(\overline{\lambda})^c}-\frac{\overline{\lambda}}{2}\\ &= \log\frac{e^{\overline{\lambda}/2}-e^{-\overline{\lambda}/2}}{(\overline{\lambda})^c}.\end{aligned}\tag{7.15}$$

Furthermore, from (7.5), (7.9) and (7.11),

$$F_n(\overline{y})=\frac{2^{1/2}(e^{\overline{\lambda}}-1-\overline{\lambda})^{3/2}}{2\pi n\overline{\lambda}\Big((e^{\overline{\lambda}}-1)^2-\overline{\lambda}^2e^{\overline{\lambda}}\Big)^{1/2}}\exp\left(-\frac{(c+1)\overline{\lambda}}{2}\right).\tag{7.16}$$

Computation of $\mathcal{H}''(\overline{y})$ is more technical. First, use (7.7) and $du/dy=1$ to get

$$\mathcal{H}''(y)=-\frac{1}{y}-\frac{1}{1-y}+\frac{1}{u}+\left(\frac{e^{\lambda}-1}{e^{\lambda}-1-\lambda}-\frac{2}{\lambda}\right)\lambda'(y).\tag{7.17}$$

To determine $\lambda'(y)$, differentiate $\frac{\partial}{\partial x}H(y,x)|_{x=\lambda}=0$ with respect to y, to obtain

$$0=\frac{e^{\lambda}-1}{e^{\lambda}-1-\lambda}-\frac{2}{\lambda}+y\left[\frac{e^{\lambda}}{e^{\lambda}-1-\lambda}-\left(\frac{e^{\lambda}-1}{e^{\lambda}-1-\lambda}\right)^2\right]\lambda'(y)\tag{7.18}$$

and solve this equation for λ'. Plugging the solution into (7.17), and using the resulting formula at $\overline{y}$, $\overline{u}$ and $\overline{\lambda}$, we get

$$\begin{aligned}\mathcal{H}''(\overline{y}) &= \frac{N(\overline{\lambda})}{D(\overline{\lambda})};\\ N(\overline{\lambda}): &= -\frac{e^{\overline{\lambda}}(e^{\overline{\lambda}}-1)}{\overline{\lambda}(e^{\overline{\lambda}}-1-\overline{\lambda})}+\frac{(e^{\overline{\lambda}}-1)^3}{\overline{\lambda}(e^{\overline{\lambda}}-1-\overline{\lambda})^2}\\ &- \frac{(e^{\overline{\lambda}}-1)^2}{\overline{\lambda}^2(e^{\overline{\lambda}}-1-\overline{\lambda})}+\frac{2e^{\overline{\lambda}}}{\overline{\lambda}(e^{\overline{\lambda}}-1)}\\ &+ \frac{2(e^{\overline{\lambda}}-1)}{\overline{\lambda}(e^{\overline{\lambda}}-1-\overline{\lambda})}-\frac{2}{\overline{\lambda}^2}-\left(\frac{e^{\overline{\lambda}}-1}{e^{\overline{\lambda}}-1-\overline{\lambda}}\right)^2,\\ D(\overline{\lambda}): &= \frac{e^{\overline{\lambda}}}{e^{\overline{\lambda}}-1}-\frac{e^{\overline{\lambda}}-1}{e^{\overline{\lambda}}-1-\overline{\lambda}}+\frac{1}{(\overline{\lambda})}.\end{aligned}\tag{7.19}$$

To simplify $N(\overline{\lambda})$, we may add its seven summands in the order 2, 7, 1, 5, 3, 6 and 4, simplifying each updated sum, and obtain

$$N(\overline{\lambda})=-\frac{e^{2\overline{\lambda}}-1-2\overline{\lambda}e^{\overline{\lambda}}}{(\overline{\lambda})^2(e^{\overline{\lambda}}-1)}.$$

A much simpler procedure yields

$$D(\overline{\lambda}) = \frac{(e^{\overline{\lambda}}-1)^2-(\overline{\lambda})^2 e^{\overline{\lambda}}}{\overline{\lambda}(e^{\overline{\lambda}}-1)(e^{\overline{\lambda}}-1-\overline{\lambda})},$$

and substituting these expressions back into (7.19) gives

$$\mathcal{H}''(\overline{y}) = -\frac{(e^{2\overline{\lambda}}-1-2\overline{\lambda}e^{\overline{\lambda}})(e^{\overline{\lambda}}-1-\overline{\lambda})}{\overline{\lambda}[(e^{\overline{\lambda}}-1)^2-(\overline{\lambda})^2 e^{\overline{\lambda}}]} < 0, \tag{7.20}$$

as $\sinh x > x$ for $x > 0$, and the denominator is readily shown to be positive by differentiation. (That $\mathcal{H}''(\overline{y}) \le 0$ should be expected, as $\mathcal{H}(y)$ is concave!) In particular, using (7.13) and (7.14),

$$\begin{aligned}
\mathcal{H}''(\overline{y}) &= -2+O(c-1), \text{ if } c\to 1; && (7.21)\\
\mathcal{H}''(\overline{y}) &= -\frac{e^{2c}}{2c}(1+O(c^2e^{-c})), \text{ if } c\to\infty;\\
\lim \mathcal{H}''(\overline{y}) &\in (-\infty,0), \text{ if } \lim c\in(1,\infty), && (7.22)
\end{aligned}$$

where, here and afterwards, lim denotes $n\to\infty$. With the exact computations at $\overline{y}$ complete, it remains to bound $|\mathcal{H}^{(3)}(y)|$ for y in a small neighborhood of $\overline{y}$. We compute $\mathcal{H}^{(3)}(y)$ via (7.17), using (7.18) to find $\lambda''(y)$.

Using concavity, we will only need to consider y close to $\overline{y}$. So pick $\epsilon\in(1/3,1/2)$ and define I as follows:

$$I = \begin{cases} \{y:\ |y-\overline{y}| < (c-1)^{1/2}(m-n)^{-\epsilon}\}, & \text{if } \lim c = 1;\\ \{y:\ |y-\overline{y}| < n^{-\epsilon}(ce^{-2c})^{1-\epsilon}\}, & \text{if } \lim c > 1.\end{cases} \tag{7.23}$$

(The second alternative covers both finite and infinite $\lim c$.) Note that if $y = O(\sqrt{c-1})\to 0$, then from (7.12), $\lambda(y) = 6(c-1)/y + O(c-1)$. On the other hand, if $c\to\infty$ then for $y\in I$

$$1-y \sim 1-\overline{y} \sim 2ce^{-2c}$$

by the assumption $me^{-2m/n}\to\infty$ and (7.14). Then also $\lambda = 2c+O(c^2e^{-c})$. Skipping over the details, here are the bounds:

$$\max\{|\mathcal{H}^{(3)}(y)|:\ y\in I\} = \begin{cases} O((c-1)^{-1/2}), & \text{if } \lim c = 1;\\ O(c^{-2}e^{4c}), & \text{if } \lim c\in(1,\infty].\end{cases} \tag{7.24}$$

Thus, uniformly for $z_1, z_2\in I$ as defined in (7.23),

$$\exp\left(n|\mathcal{H}^{(3)}(z_2)(z_1-\overline{y})^3|\right) = 1+O(\beta_2), \tag{7.25}$$

where

$$\beta_2 = d^{-(3\epsilon-1)}, \qquad d = \begin{cases} m-n, & \text{if } \lim c = 1;\\ me^{-2m/n}, & \text{if } \lim c > 1.\end{cases} \tag{7.26}$$

This estimate explains why the condition $\epsilon > 1/3$ is needed. Also, for y an endpoint of I, by (7.21–7.22) $\mathcal{H}'' < -\Theta(1)$ and moreover

$$\exp\left(\frac{n\mathcal{H}''(\overline{y})}{2}(y-\overline{y})^2\right) = \exp(-O(d^{1-2\epsilon})),$$

so we need the condition $\epsilon < 1/2$ to ensure that $\frac{\nu}{n} \notin I$ will contribute negligibly to the value of $C^{(1)}(n,m)$ in (7.1).

We also need to consider $F_n(y)$. From (7.5), if $\overline{\lambda} = o(1)$, $\overline{y} = o(1)$ and $y/\overline{y} \sim 1 \sim \lambda/\overline{\lambda}$ then

$$\frac{F_n(y)}{F_n(\overline{y})} = 1 + O(y - \overline{y}) + O(\lambda - \overline{\lambda}). \tag{7.27}$$

Solving (7.4) for y gives

$$y(\lambda) = 2(c-1)\frac{e^{\lambda} - 1 - \lambda}{(\lambda - 2)e^{\lambda} + \lambda + 2} \tag{7.28}$$

and differentiating gives

$$\frac{dy}{d\lambda} = \frac{2(c-1)(-e^{2\lambda} + (\lambda^2 + 2)e^{\lambda} - 1)}{(e^{\lambda}(\lambda - 2) + \lambda + 2)^2}. \tag{7.29}$$

Hence if $c \to 1$ and $y \in I$ we have (recalling $\overline{\lambda} = o(1)$ and $\overline{y} = o(1)$ by (7.13)) that $dy/d\lambda = -6(c-1)/\lambda^2 + O(1)$ and thus $|\lambda - \overline{\lambda}| = O(|y - \overline{y}|) = O((c-1)^{1/2}(m-n)^{-\epsilon})$. Now it is easy to obtain

$$\frac{F_n(y)}{F_n(\overline{y})} = 1 + O(\beta_2) \tag{7.30}$$

from (7.27), with β_2 as before. On the other hand, (7.27) also holds if c is bounded away from 1 and ∞ and $y/\overline{y} \sim 1 \sim \lambda/\overline{\lambda}$. Then the numerator in (7.29) is bounded away from 0 (noting that its derivative is clearly negative), and so for such c and $y \in I$, (7.30) again follows. Finally, if $c \to \infty$, a similar argument using $dy/d\lambda = \Theta(-2c/\lambda^2)$ gives (7.30) yet again.

Now expand $\mathcal{H}(y)$ near $y = \overline{y}$ and recall that $\mathcal{H}'(\overline{y}) = 0$. Putting together (7.2), (7.25), (7.26) and (7.30), we arrive at an asymptotic formula: uniformly for $\frac{\nu}{n} \in I$,

$$\begin{aligned}\binom{n}{\nu}\nu n^{n-\nu-1}C_2^{(1)}(\nu,\mu) &= (1 + O(\beta_1 + \beta_2))n^m F_n(\overline{y}) \\ &\quad \times \exp\left(n\mathcal{H}(\overline{y}) + \frac{\mathcal{H}''(\overline{y})(\nu - \overline{y}n)^2}{2n}\right),\end{aligned} \tag{7.31}$$

with $\mathcal{H}(\overline{y})$, $F_n(\overline{y})$, $\mathcal{H}''(\overline{y})$ defined by (7.15), (7.16) and (7.20). Here β_1 was defined in (7.6). Note that by choosing $\epsilon < 1/2$ sufficiently close to 1/2, we may for each fixed $a < 1/2$ assume that

$$\beta_2 < d^{-a}.$$

Let y_1 and y_2 denote the endpoints of I. Using (7.25),

$$\begin{aligned}\exp(n\mathcal{H}(y_i)) &\leq_b \exp\left(n\mathcal{H}(\overline{y}) + \frac{n\mathcal{H}''(\overline{y})}{2}(y - y_1)^2\right) \\ &\leq \exp(n\mathcal{H}(\overline{y}) - \Theta(d^{1-2\epsilon})).\end{aligned} \tag{7.32}$$

To sum (7.31), apply the elementary bound

$$\begin{aligned}\left|\sum_{\nu=\nu_1}^{\nu_2}\Phi(\nu/n) - n\int_{-\infty}^{\infty}\Phi(x)\,dx\right| &\leq_b \int_{-\infty}^{\infty}|\Phi'(x)|\,dx + |\Phi(x_1)| + |\Phi(x_2)| \\ &\quad + n\int_{x\notin[x_1,x_2]}|\Phi(x)|\,dx,\end{aligned} \tag{7.33}$$

$(x_i = \nu_i/n)$, and note that in this application, the first integral on the right hand side is bounded. This produces

$$\begin{aligned}\sum_{\frac{\nu}{n}\in I}\binom{n}{\nu}\nu n^{n-\nu-1}C_2^{(1)}(\nu,\mu) &= (1+O(\max_{y\in I}\beta_1(\nu)+\beta_2))n^m e^{n\mathcal{H}(\overline{y})}F_n(\overline{y})\sqrt{\frac{2\pi n}{-\mathcal{H}''(\overline{y})}}\\ &= (1+O(\beta))\frac{\alpha n^m}{\sqrt{2\pi n}}\left(\frac{2\sinh\overline{\lambda}/2}{\overline{\lambda}^c}\right)^n, \qquad (7.34)\end{aligned}$$

with α and β as defined in the theorem statement. (The error term coming from (7.33) is small enough to be absorbed by $O(\beta_2)$. In verifying this, it is useful to note the well-known estimate $\int_b^\infty e^{-x^2/2}dx = O(b^{-1}e^{-b^2/2})$ for $b\to\infty$.)

It remains to bound the total contribution of $\frac{\nu}{n}\notin I$ to the sum in (7.1). Recall (7.32) and note that, by concavity of $\mathcal{H}(y)$,

$$\mathcal{H}(y) \le \mathcal{H}(y_2) + \mathcal{H}'(y_2)(y-y_2), \qquad (7.35)$$

where $\mathcal{H}'(y_2) < 0$. Also, (as $\mathcal{H}'(\overline{y}) = 0$)

$$\begin{aligned}|\mathcal{H}'(y_2)| &= |\mathcal{H}''(\overline{y})(y_2-\overline{y})| + O(|\mathcal{H}^{(3)}(y)(y_2-\overline{y})^2) \qquad (7.36)\\ &\sim |\mathcal{H}''(\overline{y})(y_2-\overline{y})|, \qquad (7.37)\end{aligned}$$

in view of (7.24), (7.21) and (7.22) (which imply that $\mathcal{H}''$ is bounded above by a negative constant), and $me^{-2m/n}\to\infty$.

Consider $\lim c = 1$. Then $\overline{y}\to 0$ and $\overline{\lambda}\to 0$, and for $\lambda\to 0$ we have from (7.5)

$$F_n(y) \le_b \frac{\overline{\lambda}}{\lambda\sqrt{1-y}}F_n(\overline{y}).$$

For $y < y_1$, we have from (7.28) (see also the following estimates) that $\lambda > \Theta(\overline{\lambda})$. It then follows from (7.2) and (7.32) and the concavity of $\mathcal{H}(y)$ that

$$\begin{aligned}\sum_{\frac{\nu}{n}\le y_1}\binom{n}{\nu}\nu n^{n-\nu-1}C_2^{(1)}(\nu,\mu) &\le_b n^m\exp(n\mathcal{H}(\overline{y}) - \Theta(d^{1-2\epsilon}))F_n(\overline{y})\times(n\overline{y})\\ &\le_b \frac{\alpha n^m}{n^{1/2}}e^{n\mathcal{H}(\overline{y})}d^{1/2}e^{-\Theta(d^{1-2\epsilon})}, \qquad (7.38)\end{aligned}$$

with the last factor approaching 0 faster than d^{-a} for every $a>0$. We need to be a bit more precise for $\nu/n \ge y_2$. By (7.21) and the definition of I, the relation (7.37) becomes

$$|\mathcal{H}'(y_2)| = \Theta(\gamma), \quad \gamma := (c-1)^{1/2}(m-n)^{-\epsilon}.$$

So, according to (7.32), (7.35), for $y\ge y_2$

$$e^{n\mathcal{H}(y)} \le e^{n\mathcal{H}(\overline{y})}e^{-\Theta(d^{1-2\epsilon})}e^{-n\Theta(\gamma)(y-y_2)}.$$

Consequently

$$\sum_{\frac{\nu}{n}\ge y_2}\binom{n}{\nu}\nu n^{n-\nu-1}C_2^{(1)}(\nu,\mu) \le_b \frac{\alpha n^m}{n^{1/2}}e^{n\mathcal{H}(\overline{y})}\cdot e^{-\Theta(d^{1-2\epsilon})}\sum_{\nu\ge ny_2}\frac{\overline{\lambda}e^{-(\nu-ny_2)\Theta(\gamma)}}{\lambda\sqrt{n-\nu+n\rho}}$$

The summation here is $O(1)$ times

$$\frac{\nu}{n^{3/2}\sqrt{c-1}}\sum_{j\ge 0} e^{-j\Theta(\gamma)} + \frac{e^{-\Theta(n\gamma)}}{\sqrt{c-1}}\sum_{j=1}^{n/2} j^{-1/2} \le_b d^{-(1/2-\epsilon)} + e^{-\Theta(n^{1/2}d^{1/2-\epsilon})}\sqrt{\frac{n}{c-1}}$$

and thus

$$\sum_{\frac{\nu}{n}\ge y_2}\binom{n}{\nu}\nu n^{n-\nu-1}C_2^{(1)}(\nu,\mu) \le_b d^{-a}\cdot\frac{\alpha n^m}{n^{1/2}}e^{n\mathcal{H}(\overline{y})}, \tag{7.39}$$

for every $a > 0$. The combination of (7.34), (7.38) and (7.39) completes the proof of the theorem for $\lim c = 1$.

The case $\lim c > 1$ can be handled in a very similar way, so we omit the details. ∎

We next investigate what happens when the condition $me^{-2m/n} \to \infty$ in Theorem 3 begins to fail. At this frontier, m is asymptotically $\frac{1}{2}n\log n$, but we will treat $m = O(n\log n)$. Note that one of the easily established and well known results on the random graph $\mathcal{G}(n,m)$ is that the number of isolated vertices goes to infinity if $ne^{-2m/n} \to \infty$, and 0 if $ne^{-2m/n} \to 0$. It is also a fundamental result that in the latter case, the graph is a.a.s. connected, and this also follows from the main results in [4]. So the comparison of the number of connected graphs with the total number of graphs starts to become less interesting when $m \to \infty$ and $me^{-2m/n} = O(1)$, which of course implies $ne^{-2m/n} \to 0$. Our argument compares with the total number of 2-cores instead, but we nevertheless don't give a full proof.

Theorem 9 *Let $m, n \to \infty$ in such a way that $me^{-2m/n} = O(1)$ and $m = O(n\log n)$. Then for all $a < 1/2$,*

$$C^{(1)}(n,m) = (1+O(n^{-a}))e^{2me^{-2m/n}}C_2^{(1)}(n,m). \tag{7.40}$$

Sketch of the proof. In this case the terms that dominate the sum in (7.1) are those for $j := n-\nu \ll n$. For $j \le n^{\epsilon}$, where $0 < \epsilon < 1/4$ is fixed, we have

$$\binom{n}{\nu}\nu n^{n-\nu-1}C_2^{(1)}(\nu,\mu) = \exp(-\Theta(j^2/n + j/n))\frac{n^{2j}}{j!}C_2^{(1)}(n-j,m-j).$$

Applying Theorem 2 to both $C_2^{(1)}(n,m)$ and $C_2^{(1)}(n-j,m-j)$ and using $j \le n^{\epsilon}$, we obtain after a little work that, for $a < 1/2$,

$$\frac{C_2^{(1)}(n-j,m-j)}{C_2^{(1)}(n,m)} = (1+O(m^{-a}))\left(\frac{\lambda^2}{2mf(\lambda)}\right)^j,$$

where f is defined in (3.2),

$$\frac{\lambda(e^{\lambda}-1)}{e^{\lambda}-1-\lambda} = \frac{2m}{n}$$

and so $\lambda = \frac{2m}{n} + O((m/n)^2 e^{-2m/n})$. Therefore

$$\binom{n}{\nu}\nu n^{n-\nu-1}C_2^{(1)}(\nu,\mu) = \exp(-\Theta(j^2/n))(1+O(m^{-a}))\left(\frac{\lambda^2 n^2}{2mf(\lambda)}\right)^j \frac{1}{j!},$$

and

$$\frac{\lambda^2 n^2}{2mf(\lambda)} = n\frac{\lambda}{e^\lambda - 1} = 2me^{-2m/n} + O(n^{-1}(me^{-2m/n})^2).$$

Summing over $j \le n^\epsilon$, and then bounding the contribution of $j > n^\epsilon$, we obtain (7.40). ∎

Theorem 4 follows easily from the proof of Theorem 3. Here $\sigma^2 = -1/\mathcal{H}''(\bar{\lambda})$. Similarly, from Theorem 9 we obtain the following.

Theorem 10 *Under the condition of Theorem 9, $n - X_{mn}$ is in the limit Poisson with parameter $2me^{-2m/n}$. In addition, a.a.s. all the vertices outside the 2-core are mutually nonadjacent, each joined to its own vertex in the core.* ∎

8 Distribution of core and tree-mantle size in random graphs

This section is entirely devoted to the proof of Theorem 6. We first consider $\mathcal{G}(n,m)$, from which we derive the result for $\mathcal{G}(n,p)$. Recall the definition of Y_{ni}, c, b and B_n from Section 2.3.

Given ν_1, ν_2 and μ_1, let $N = N(\nu_1,\nu_2,\mu_1)$ denote the number of components of $\mathcal{G}(n,m)$ whose 2-core has ν_1 vertices and $\mu := \nu_1 + \mu_1$ edges (so that that μ_1 is the excess of the 2-core), and whose tree mantle size is ν_2 (recalling that this does not count the root vertices in the 2-core). Clearly $\mu \le \min\{\binom{\nu_1}{2}, m\}$. For $\nu := \nu_1 + \nu_2 \in [0.5bn, 2bn]$, $N \in \{0,1\}$ on the event B_n. Therefore

$$\begin{aligned}\mathbf{P}(Y_{n1} = \nu_1, Y_{n2} = \nu_2, Y_{n3} = \mu_1 | B_n) &= \mathbf{P}(N = 1|B_n)\\ &= \mathbf{E}(N|B_n)\\ &\sim \mathbf{E}(NI_{B_n})\end{aligned} \tag{8.1}$$

by (2.19). Recall that the total number of forests of k rooted trees with given root vertices and with ℓ vertices overall is $k\ell^{\ell-k-1}$. Also, conditional upon a given component of ν vertices and $\nu_1 + \mu$ edges, the remainder of the graph is distributed as $\mathcal{G}(n-\nu, m-\mu-\nu_2)$. Thus, similar to the derivation of (7.1), we have

$$\mathbf{E}(NI_{B_n}) = R(n,m,\nu_1,\nu_2,\mu)\mathbf{P}(D_{n-\nu,m-\mu-\nu_2}) \tag{8.2}$$

where

$$R(n,m,\nu_1,\nu_2,\mu) = \binom{n}{\nu}\binom{\nu}{\nu_1}\nu_1\nu^{\nu_2-1}C_2^{(1)}(\nu_1,\mu)\frac{\binom{\binom{n-\nu}{2}}{m-\mu-\nu_2}}{\binom{\binom{n}{2}}{m}} \tag{8.3}$$

and D_{ik} is the event "$\mathcal{G}(i,k)$ has no component of size $0.5bn$ or larger". We will later observe that $\mathbf{P}(D_{n-\nu,m-\mu-\nu_2}) \sim 1$ for the significant values of the parameters. So we concentrate on estimating $R(n,m,\nu_1,\nu_2,\mu)$.

Here is our plan. First we write the major factors in $R(n, m, \nu_1, \nu_2, \mu)$ asymptotically in the form

$$\exp(ng(\nu_1/n, \nu_2/n, \mu/n)), \tag{8.4}$$

leaving minor (sub-exponential) factors aside. (Actually, g is also a function of m, or c, but we take c fixed at first, and consider g as a function of the other variables.) We determine a stationary point $(\overline{y}, \overline{z}, \overline{u})$ of $g(y, z, u)$ which turns out to be a local maximum point. (Working with μ rather than μ_1 gives simpler expressions up to this point.) Setting $G(y, z, U) = g(y, z, y+u)$ (since $\mu = \nu_1 + \mu_1$) and approximating $G(y, z, U)$ by the second order Taylor polynomial in the vicinity of $(\overline{y}, \overline{z}, \overline{U} = \overline{y} - \overline{u})$, we will get a limiting expression for $\mathbf{E}(NI_{B_n})$, whence for $\mathbf{P}(Y_{n1} = \nu_1, Y_{n2} = \nu_2, Y_{n3} = \mu_1 | B_n)$, under the conditions imposed on ν_1, ν_2 and μ_1 in the theorem. The sum of this probability over the values near the maximum point will be seen to be asymptotic to 1, showing that the contribution from other values is negligible.

Recall that $\mu \leq m = cn/2$. Consulting the asymptotic formulae in Theorem 2 and (3.11) for $C_2^{(1)}(\nu_1, \mu)$, and using the rough Stirling approximation for $(2\mu - 1)!! = (2\mu)!/(2^\mu \mu!)$ and for the other factorials in (8.3), we see that the major (exponential) factor in the expression (8.3) can be written as $\exp(nh(\nu_1/n, \nu_2/n, \mu/n, \lambda))$, where

$$\begin{aligned} h(y, z, u, x) &= (u + z)\log 2 + (c - 2u - 2z)\log(1 - y - z) - u - z \\ &+ \frac{c}{2}\log\frac{c}{2} - \left(\frac{c}{2} - u - z\right)\log\left(\frac{c}{2} - u - z\right) \\ &- (1 - y - z)\log(1 - y - z) - y\log y - z\log z + z\log(y + z) \\ &+ u\log\frac{2u}{e} + y\log f(x) - 2u\log x. \end{aligned} \tag{8.5}$$

Here, by (3.7), $\lambda = \lambda(y, u)$ satisfies

$$\frac{\lambda(e^\lambda - 1)}{e^\lambda - 1 - \lambda} = \frac{2u}{y}. \tag{8.6}$$

We define $g(y, z, u) = h(y, z, u, \lambda(y, u))$. Our task is to determine a stationary point of g, that is a root of $g_y = g_z = g_u = 0$. Since (8.6) means that

$$h_x(y, z, u, \lambda(y, u)) = 0, \tag{8.7}$$

the required root is a solution of

$$h_y(y, z, u, x)|_{x=\lambda} = h_z(y, z, u, x)|_{x=\lambda} = h_u(y, z, u, x)|_{x=\lambda} = 0.$$

Explicitly,

$$\begin{aligned} -\frac{c - 2u - 2z}{1 - y - z} + \frac{z}{y + z} + \log\frac{(1 - y - z)f(\lambda)}{y} &= 0, \\ -\frac{c - 2u - 2z}{1 - y - z} + \frac{z}{y + z} + \log\frac{(c - 2u - 2z)(y + z)}{z(1 - y - z)} &= 0, \\ \log\frac{2u(c - 2u - 2z)}{\lambda^2(1 - y - z)^2} &= 0, \end{aligned} \tag{8.8}$$

and we need to solve this system jointly with equation (8.6). This might have been a formidable task, had we not expected (c.f. (2.14), (2.15), (2.18)) that

$$\begin{aligned} \overline{y} &= (1-t)\left(1-\frac{t}{c}\right) = b_1, \\ \overline{z} &= t\left(1-\frac{t}{c}\right) = b_2, \\ te^{-t} &= ce^{-c}, \quad t \in (0,1). \end{aligned} \tag{8.9}$$

With a bit of extra effort, one obtains a full solution by equating to zero the non-logarithmic expression common to first two equations in (8.8), the result being

$$\overline{\lambda} = c - t, \quad \overline{u} = \frac{(c-t)^2}{2c} = b_1 + b_3. \tag{8.10}$$

Later, using (8.21), we will show that any other solutions, if they exist, are immaterial.

If our analysis is to succeed, we must have $g(\overline{y}, \overline{z}, \overline{u}) = 0$. (This number must be nonnegative, of course, given that we have located the true maximum, and a positive value would be of no use to us, implying that the expected number of (ν_1, ν_2, μ)-components—with $(\nu_1, \nu_2, \mu) \approx n(\overline{y}, \overline{z}, \overline{u})$—is exponentially large!) To this end, first we use (8.5) to write

$$\begin{aligned} g(\overline{y}, \overline{z}, \overline{u}) &= \overline{y} \log \frac{f(\overline{\lambda})(1-\overline{y}-\overline{z})}{\overline{y}} + \overline{z}\left(\log \frac{(c-2\overline{u}-2\overline{z})(\overline{y}+\overline{z})}{(1-\overline{y}-\overline{z})\overline{z}} - 1\right) \\ &+ \overline{u}\left(\log \frac{(c-2\overline{u}-2\overline{z})2\overline{u}}{(1-\overline{y}-\overline{z})^2\overline{\lambda}^2} - 2\right) + (c-1)\log(1-\overline{y}-\overline{z}) \\ &+ \frac{c}{2}\log\frac{c}{2} - \frac{c}{2}\log\left(\frac{c}{2} - \overline{u} - \overline{z}\right). \end{aligned}$$

Now, using the fact that the logarithmic terms (8.8) are all zero, we simplify the above expression and use (8.9), (8.10) to obtain

$$\begin{aligned} g(\overline{y}, \overline{z}, \overline{u}) &= -\overline{z} - 2\overline{u} - \log(1-\overline{y}-\overline{z}) + \frac{c}{2}\log\frac{c(1-\overline{y}-\overline{z})^2}{c-2\overline{u}-2\overline{z}} \\ &= -(c-t) - \log\frac{t}{c} + \frac{c}{2}\log 1 \\ &= -(c-t) + (c-t) = 0. \end{aligned} \tag{8.11}$$

Next we wish to evaluate all six second order derivatives of $G(y,z,U) = g(y,z,y+U)$ at $(\overline{y}, \overline{z}, \overline{U}) = (b_1, b_2, b_3)$. Define $H(y,z,U,x) = h(y,z,y+U,x)$ and note from (8.7) that $H_x(y,z,U,\Lambda(y,U)) = 0$ where $\Lambda(y,U) = \lambda(y, y+U)$. So we have

$$\begin{aligned} G_y &= H_y + H_x\lambda_y = H_y(y,z,u,\Lambda(y,U)), \\ G_z &= H_z + H_x\lambda_z = H_z(y,z,u,\Lambda(y,U)), \\ G_U &= H_U + H_x\Lambda_U = H_U(y,z,u,\Lambda(y,U)). \end{aligned} \tag{8.12}$$

To determine the second order derivatives of G, we need Λ_y and Λ_U (noting $\Lambda_z = 0$). Differentiating the equation $H_x(y, z, U, \Lambda(y, U)) = 0$ with respect to y and to U, we find

$$\Lambda_y = -\frac{H_{xy}(y, z, U, \Lambda)}{H_{xx}(y, z, U, \Lambda)}, \quad \Lambda_U = -\frac{H_{xU}(y, z, U, \Lambda)}{H_{xx}(y, z, U, \Lambda)}.$$

Once Λ_y, Λ_U are determined, we use (8.12), to obtain

$$G_{yy} = H_{yy} + H_{yx}\Lambda_y = H_{yy} - \frac{H_{xy}^2(y, z, U, \Lambda)}{H_{xx}(y, z, U, \Lambda)},$$

and the analogous formulas for the remaining five second derivatives of G. So we need to evaluate the corresponding derivatives of H at $(\overline{y}, \overline{z}, \overline{U}, \overline{\lambda})$ (noting $\Lambda(\overline{y}, \overline{U}) = \lambda(\overline{y}, \overline{u}) = \overline{\lambda}$) and plug the results into the expressions for G_{yy} and the rest. Note that the term in H containing both y and x is $yF(x)$ where

$$F(x) = \log(f(x)/x^2). \tag{8.13}$$

For simplification, we use (8.6), which implies that $e^{c-t} = c/t$ from (8.9), as well as $f'(x)/f(x) \mid_{x=\Lambda} = 2(y + U)/(\Lambda y)$ and hence

$$F'(x) \mid_{x=\Lambda} = \frac{2(y + U)}{\Lambda y} - \frac{2}{\Lambda},$$

and calculate the second derivative (which appears in H_{xx}) using

$$F''(x) = \frac{1}{f(x)} - \frac{x(f'(x)/f(x))}{f(x)} + \frac{2}{x^2}$$

where

$$f(x) \mid_{x=\Lambda} = \frac{(e^\Lambda - 1)\Lambda y}{2(y + U)}.$$

At the maximum point $(y, z, U, x) = (\overline{y}, \overline{z}, \overline{U}, \overline{\lambda})$, we find that $e^\lambda = c/t$ by (8.9) and (8.10),

$$H_{xx} = \frac{1 - ct}{c(1 - t)}$$

and eventually

$$\begin{aligned}
G_{yy} &= -\frac{c(1-t)(2-t)}{t^2} - \frac{c(c+t-2)}{(1-t)(c-t)^2} - \frac{tc}{c-t} - \frac{c(c+t-2)^2}{(1-ct)(c-t)^2(1-t)}, \\
G_{zz} &= -\frac{c(1-t)(2c-ct-t)}{t^2(c-t)}, \\
G_{UU} &= -\frac{2c}{t^2} + \frac{2c(2t-1-ct)}{(c-t)^2(1-ct)}, \\
G_{yz} &= \frac{c(1-t)(2t+ct-2c)}{t^2(c-t)}, \\
G_{yU} &= -\frac{2c(1-t)}{t^2} - \frac{2c(ct+1-c-t)}{(c-t)^2(1-ct)}, \\
G_{zU} &= -\frac{2c(1-t)}{t^2}.
\end{aligned}$$

Denote the negative Hessian matrix by

$$A_m = -\begin{pmatrix} G_{yy} & G_{yz} & G_{yU} \\ G_{yz} & G_{zz} & G_{zU} \\ G_{yU} & G_{zU} & G_{UU} \end{pmatrix}.$$

For the point $(\overline{y}, \overline{z}, \overline{U})$ to be shown to be a (local) maximum of $G(y, z, U)$, it is enough to check that A_m is positive definite, or that

$$G_{yy} < 0, \quad \det \begin{pmatrix} G_{yy} & G_{yz} \\ G_{yz} & G_{zz} \end{pmatrix} > 0, \quad \det A_m > 0.$$

From the last equation in (8.9) it follows that for $t = t(c)$ we have

$$dt/dc = \frac{t(1-c)}{c(1-t)}.$$

It is then straightforward to show that $ct < 1$ and $c + t > 2$. Thus $G_{yy} < 0$. The other two determinants are (using Maple)

$$\frac{c^3(1-t)^3(t^2+2c-3ct)}{t^3(1-ct)(c-t)^3}$$

and

$$\det A_m = \frac{2(1-t)^3c^6}{t^4(c-t)^4(1-ct)}.$$

Using $c + t > 2$ and $ct < 1$ we have $t^2 + 2c - 3ct = t(t + c) + 2c - 4ct > 2(t + c) - 4ct > 0$ and so both determinants are clearly positive, as required.

Let us evaluate asymptotically the overall minor factor, which was omitted from the above discussion, at points (y, z, U) close to $(\overline{y}, \overline{z}, \overline{U})$. The minor factor coming from the first two binomials and powers of ν and ν_1 in (8.3) is asymptotic to

$$\begin{aligned} \frac{\sqrt{2\pi n}}{\sqrt{2\pi(n-\nu)}\sqrt{2\pi\nu_1}\sqrt{2\pi\nu_2}}\frac{\nu_1}{\nu} &= \frac{1}{2\pi n}\sqrt{\frac{y}{(1-y-z)z}}\frac{1}{y+z} \\ &\sim \frac{1}{2\pi n}\sqrt{\frac{c^3(1-t)}{t^2}}(c-t)^{-1}. \end{aligned} \tag{8.14}$$

Denote $\overline{\sigma}$ and $\overline{\eta}$ the parameters that correspond to $(\overline{y}, \overline{z}, \overline{U})$ when using Theorem 2 and (3.11) to evaluate $C_2^{(1)}(\nu_1, \mu)$. It is easy to check, using (8.9) and (8.10), that $\overline{\sigma} = t$ and $\overline{\eta} = c$, and so the minor factor here is asymptotic to

$$\sqrt{2}\frac{e^{-c/2-c^2/4}e^{t/2+t^2/4}}{\sqrt{2\pi n 2\overline{u}(1+c-2\overline{u}/\overline{y})}}(1-t)^{1/2} = \frac{e^{-c/2-c^2/4+t/2+t^2/4}(1-t)}{\sqrt{\pi n}(c-t)}\sqrt{\frac{c}{1-ct}}. \tag{8.15}$$

Finally, the minor factor coming from the ratio of the binomial coefficients in (8.3) is asymptotic to

$$e^{c/2}e^{-\frac{c/2-\overline{u}-\overline{z}}{1-\overline{y}-\overline{z}}}\sqrt{\frac{c}{c-2\overline{u}-2\overline{z}}}e^{-\frac{(c/2-\overline{u}-\overline{z})^2}{(1-\overline{y}-\overline{z})^2}}e^{m^2/n^2} = e^{c/2-t/2}e^{c^2/4-t^2/4}\frac{c}{t}. \tag{8.16}$$

Multiplying the estimates in (8.14), (8.15) and (8.16), we obtain that the overall minor factor is asymptotic to

$$(2\pi n)^{-3/2}\sqrt{\frac{2c^6(1-t)^3}{t^4(c-t)^4(1-ct)}}$$

from above. Thus, referring back to (8.4) and (8.11) and the determinant calculated above, but ignoring third order derivatives,

$$R(n,m,\nu_1,\nu_2,\mu) \sim (2\pi n)^{-3/2}e^{-\frac{1}{2}\mathbf{x}^T A\mathbf{x}}\sqrt{\det A_m} \tag{8.17}$$

for (y,z,U) near $(\overline{y},\overline{z},\overline{U})$, where $\mathbf{x}^T = \left(\frac{\nu_1-b_1n}{n^{1/2}}, \frac{\nu_2-b_2n}{n^{1/2}}, \frac{\mu_1-\overline{U}n}{n^{1/2}}\right)$. From this, we can obtain (2.21) after showing that the error in the approximation is small.

Let $\epsilon = c-1$. It will be shown that (8.17) holds uniformly for $(\nu_1/n, \nu_2/n, \mu_1/n)$ in any box of the form

$$|y-\overline{y}| = O((\epsilon/n)^{1/2}), \quad |z-\overline{z}| = O((\epsilon n)^{-1/2}), \quad |U-\overline{U}| = O((\epsilon^3/n)^{1/2}), \tag{8.18}$$

these values being the important ones in view of the diagonal entries in the matrix in Note 4, which gives the variances of the approximating limiting variables. To establish this, we need to show that the (mixed) third derivatives of G lead to negligible error in using Taylor's theorem for (y,z,U) satisfying (8.18). In view of the factor n in (8.4), we require $G_{yyy}|y-\overline{y}|^3 = o(n^{-1})$, and so on; to be precise, the requirements are

$$\begin{aligned}&\epsilon^{3/2}|G_{yyy}| + \epsilon^{1/2}|G_{yyz}| + \epsilon^{5/2}|G_{yyU}| + \epsilon^{-1/2}|G_{yzz}| + \epsilon^{3/2}|G_{yzU}| \\ &+\epsilon^{7/2}|G_{yUU}| + \epsilon^{-3/2}|G_{zzz}| + \epsilon^{1/2}|G_{zzU}| + \epsilon^{5/2}|G_{zUU}| + \epsilon^{9/2}|G_{UUU}| = o(n^{1/2}).\end{aligned} \tag{8.19}$$

This is for ϵ bounded, but also satisfying the basic assumption of the theorem that $n\epsilon^3 \to \infty$.

Here are some relevant details. First, with $\Lambda = \Lambda(y,U) = \lambda(y, y+U)$ we have from (8.12)

$$G_{yyy} = H_{yyy} + H_{yyx}\Lambda_y + H_{yxx}\Lambda_y^2 + H_{yx}\Lambda_{yy} \tag{8.20}$$

and analogous equations for the other third order partial derivatives of G. Here of course the partials of H are evaluated with $x = \Lambda(y,U)$. The only difficult case is $\epsilon = c-1 \to 0$; after verifying this case, it is easily seen that for ϵ bounded away from 0 (but bounded), the third derivatives are all bounded and (8.19) follows. So we now assume $\epsilon = o(1)$.

The case of G_{yyy} is examined in more detail, because all of the difficulties are essentially encountered in its analysis. We will show that $G_{yyy} = O(\epsilon^{-3})$, as required for (8.19). Considering (2.14), (2.15), (8.6), (8.9) and (8.10), we obtain (using $n\epsilon^3 \to \infty$)

$$y \sim \overline{y} \sim 2\epsilon^2, \quad z \sim \overline{z} \sim 2\epsilon, \quad U \sim \overline{U} \sim 2\epsilon^3/3, \quad \Lambda(y,U) \sim \overline{\lambda} \sim 2\epsilon$$

for (y,z,U) satisfying (8.18). We use these estimates throughout the following.

Take the first term in (8.20), H_{yyy}. This is easily seen to be

$$O(1) + \frac{1}{y^2} + \frac{2z}{(y+z)^3} - \frac{1}{(U+y)^2}.$$

The first and third terms sum to $U(U+2y)y^{-2}(U+y)^{-2} = O(\epsilon^{-3})$.

The next terms in (8.20) involve $H_{yyx} = 0$, $H_{yxx} = F''(x)$ and $H_{yx} = F'(x)$ where F is as in (8.13). Note that $f(x)/x^2$ is analytic and non-zero near $x = 0$, and so $F'(x)$ and $F''(x)$ are bounded. To deal with the partial derivatives of Λ, note that $\Lambda = \Lambda(y, U)$ is defined by

$$q(\Lambda) = 1 + \frac{U}{y}$$

where

$$q(\Lambda) = \frac{\Lambda e^{\Lambda}}{e^{\Lambda} - 1 - \Lambda}.$$

The derivatives of q are all $\Theta(1)$ near 0. Hence by implicit differentiation, we find that $\Lambda_y = O(-U/y^2) = O(\epsilon^{-1})$, $\Lambda_{yy} = O(\epsilon^{-3})$, and, for G_{uuu}, $\Lambda_U = O(\epsilon^{-2})$, $\Lambda_{UU} = O(\epsilon^{-4})$. It now follows from (8.20) that $G_{yyy} = O(\epsilon^{-3})$ as required.

Similarly we find $G_{yyU} = O(\epsilon^{-4})$, $G_{yUU} = O(\epsilon^{-4})$, $G_{UUU} = O(\epsilon^{-6})$, $G_{yyz} = O(\epsilon^{-2})$, $G_{yzz} = O(\epsilon^{-1})$, and G_{zzz}, G_{zzU}, G_{zUU}, G_{yzU} are all $O(1)$. Thus (8.19) follows, and we conclude (8.17) provided (8.18) holds.

We next return to (8.2) for consideration of $\mathbf{P}(D_{n-\nu,m-\nu_2-\mu})$. For $(\nu_1, \nu_2, \mu) = n(\overline{y}, \overline{z}, \overline{u})$, it is easy to calculate that $(m - \nu_2 - \mu)/(n - \nu) = t$, and thus we retrieve the well known fact that for the supercritical $\mathcal{G}(n, m)$ with parameter c, the part lying outside the giant component is a subcritical $\mathcal{G}(n, m)$ with parameter approximately t. Since $(c-1)n^{1/3} \to \infty$, it follows that $(1-t)n^{1/3} \to \infty$. It is well known that in such a graph the largest component has size $o(n^{2/3})$ (by [9, Theorem 5.5] for instance). This applies for all (ν_1, ν_2, μ) presently under discussion (see (8.18)) and thus $\mathbf{P}(D_{n-\nu,m-\nu_2-\mu}) \to 1$. From this and (8.2), $\mathbf{E}(NI_{B_n})$ is given asymptotically by (8.17), and hence (2.21) follows from (8.1).

Note that

$$\sum_{\nu_1,\nu_2,\mu_1} (\det A_m)^{1/2} (2\pi n)^{-3/2} e^{-\frac{1}{2}\mathbf{x}^T A_m \mathbf{x}} \sim 1, \tag{8.21}$$

and that these are the point probabilities in a Gaussian distribution with covariance matrix $nA_m^{-1} = nK_m$ as analysed in Note 4. Since the box (8.18) can have dimensions arbitrarily large multiples of the standard deviations of the respective variables, it thus follows from (8.17) that the distribution of (Y_{n1}, Y_{n2}, Y_{n3}), conditional upon B_n, is concentrated near the local maximum $n\mathbf{b}$ and is asymptotically Gaussian as stated in the theorem. (It is only now that any other potential solutions of (8.6) and (8.8) are finally shown to be irrelevant.) From (2.19), this distributional statement also holds in $\mathcal{G}(n, m)$ unconditionally, as required for conclusion (i) of the theorem. The matrix K_m seems to be rather unwieldy and is not displayed here.

This leaves only the central limit theorem in (i) for $G(n, p = c/n)$. In [15] was proven the following reductive lemma:

Let $k \ge 1$ be fixed and $Y(\cdot)$ be a k-dimensional vector-valued function on the set $\mathcal{G}_n$ of all graphs on $[n]$, $Y : \mathcal{G}_n \to \mathbb{R}^k$. Suppose that for some $\mathbf{b} : (0, \infty) \to \mathbb{R}^k$ and a $k \times k$ symmetric matrix function $K = K(x)$, $x \in (0, \infty)$, $[Y(G(n, m = cn/2)) - nb(c)]n^{-1/2}$ is asymptotically Gaussian with zero mean vector and covariance matrix K. Suppose that $\mathbf{b}(x)$ is continuously differentiable, and that $K(x)$ is continuous. Then $[Y(G(n, p = c/n)) - nb(c)]n^{-1/2}$ is also asymptotically zero-mean Gaussian, with the covariance matrix

$$\mathcal{K}(c) = K(c) + 2c\mathbf{b}'(c)\mathbf{b}'(c)^T.$$

We apply this for $K = K_m$ and $\mathcal{K} = K_p$. There is one technical complication though, since $\mathbf{b}(\cdot)$, $K_p(\cdot)$ and $K_m(\cdot)$ are all defined for $c > 1$ only. However, the proof of the lemma in [15] can be easily adapted to our case, and the key technical point is that we consider $c > 1$ such that $n^{1/3}(c-1) \to \infty$, and thus $n^{1/3}(2m/n - 1) \to \infty$ as well, if $|m - cn/2| = O(n^{1/2})$, $n^{1/2}$ being the standard deviation order for the number of edges in $G(n, p = c/n)$. Thus we obtain (i) for $\mathcal{G}(n,p)$, with K_p defined from K_m via (2.20). ∎

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