Regular graphs with no homomorphisms onto cycles

I. M. Wanless

Christ Church, St Aldates, Oxford OX1 1DP, UK
E-mail: wanless@maths.ox.ac.uk

N. C. Wormald*

Department of Mathematics and Statistics, University of Melbourne VIC 3010, Australia
E-mail: nick@ms.unimelb.edu.au

We prove the existence of d-regular graphs with arbitrarily large girth and no homomorphism onto the cycle C_s, where $(d, s) = (3, 9)$ and $(4, 5)$.

1. INTRODUCTION

A homomorphism from a graph G to a graph H is a mapping of the vertices of G into the vertices of H which sends each edge of G to an edge in H. It is not known if there exists a cubic (3-regular) graph with arbitrarily large girth and with no homomorphism onto the cycle C_5. Note that a graph containing C_i for i odd has no homomorphism onto C_j for any $j > i$. Thus, the large girth condition is required to make the problem interesting. Kostočka, Nešetřil and Smolikova [4] have shown that the graph asked for above does exist if C_5 is replaced by C_{11}, and Kim and Nešetřil [3] have shown it for C_5 if 3-regular is replaced by 4-regular. In this note we give a lemma on constrained optimisation which permits us to improve the former of these results (obtaining it for C_9 rather than C_{11}) and to give a different proof of the latter result. Other questions relating to graph homomorphisms onto odd cycles have also been studied (see Lai [5] for example). The general topic of graph homomorphisms derives most interest from the analogy with graph colouring, which is equivalent to homomorphism onto a complete graph.

We will use a graph G formed from the superposition of d perfect matchings of n vertices (n even), restricted to having no multiple edges. The

* Research supported by the Australian Research Council.

matchings are chosen independently and uniformly at random. We call this probabilistic space $d\mathcal{G}_{n,1}$, as in [8]. Using this space rather than the space $\mathcal{G}_{n,d}$ of uniformly distributed random regular graphs simplifies some calculations. The two spaces (actually sequences of spaces indexed by n) are *contiguous*, represented by

$$d\mathcal{G}_{n,1} \approx \mathcal{G}_{n,d}. \tag{1}$$

By this it is meant that any statement a.a.s. true in one space (sequence) is also a.a.s. true in the other. The contiguity expressed in (1) was proved by Janson [2] and independently by Molloy et al. [7]. See [8] for a survey of this topic.

The probability that d independent random perfect matchings on n vertices form no multiple edges is asymptotic to

$$c_d = e^{-d(d-1)/4} \tag{2}$$

as $n \to \infty$. This can be verified, for example, by calculating the factorial moments of the variable counting multiple edges, and applying Brun's sieve.

2. PRELIMINARY RESULT

In the next section we will encounter a function of many variables. In this section we obtain a useful bound on such a function. Define $f(x,y) = x^{yx}$. In the following result $\bar{x}_c = (x_0, x_1, \ldots, x_{c-1})$ will be a vector in which the subscripts are taken modulo c (so any reference to x_c defaults to x_0). Also, sums and products will be over $i = 0, 1, 2, \ldots, c-1$.

Lemma 2.1. *Let $c \geq 3$, $d_1 \geq 1$ and $d_2 \geq 1$ be integers satisfying $\delta = d_1 - \frac{1}{2}d_2 > 0$, and $b \in \big(0, 1/(2c)\big)$ a real constant. Let M be the supremum of $\prod f(x_i + x_{i+1}, d_1) f(x_i, -d_2)$ taken over all $\bar{x}$ in which $\sum x_i = 1/2$ and each $x_i \geq b$. Then*

$$M \leq 2^{d_2/2} f\big(2b, (c-1)\delta\big) f\big(1 - 2b(c-1), \delta\big).$$

Proof. For $|x| < a$ define $g = g(a,x) = f(2a, d_1) f(a + x, -\frac{1}{2}d_2) f(a - x, -\frac{1}{2}d_2)$. Since

$$\frac{\partial}{\partial x}\log(g) = \frac{1}{2}d_2 \log\left(\frac{a-x}{a+x}\right),$$

we infer that $g(a,x) \leq g(a,0) = f(2a, d_1) f(a, -d_2)$. Hence

$$\prod f(x_i + x_{i+1}, d_1) f(x_i, -d_2) \;=\; \prod f(x_i + x_{i+1}, d_1) f(x_i, -\frac{1}{2}d_2) f(x_{i+1}, -\frac{1}{2}d_2)$$

$$\begin{aligned}&= \prod g\left(\frac{1}{2}(x_i + x_{i+1}), \frac{1}{2}(x_i - x_{i+1})\right)\\ &\leq \prod\left(2^{(x_i+x_{i+1})d_2/2} f(x_i + x_{i+1}, \delta)\right)\\ &= 2^{d_2/2} \prod f(x_i + x_{i+1}, \delta) \qquad (3)\end{aligned}$$

since $\sum x_i = 1/2$. Next consider a problem involving variables $u_0, u_1, \ldots, u_{c-1}$ where each $u_i \geq 2b$ and $\sum u_i = 1$. If the aim is to maximise $\prod f(u_i, \delta)$ then at most one of the variables u_i can differ from $2b$. To see this, note that the convexity of $x\log(x)$ implies that $f(u_i, \delta)f(u_j, \delta) < f(u_i - \epsilon, \delta)f(u_j + \epsilon, \delta)$ whenever $0 < \epsilon < u_i \leq u_j$. Applying this idea to (3) yields the claimed bound. ■

3. TWO THEOREMS

THEOREM 3.1. *For every $k > 0$ there exists a 3-regular graph with girth at least k and with no homomorphism onto C_9.*

Proof. For a graph G in a probability space of n-vertex graphs, let $X(G)$ denote the number of homomorphisms of G onto C_9 and let A denote the event that the largest independent set of a graph has cardinality less than $0.4554n$. McKay [6] showed that A holds a.a.s. for $G \in \mathcal{G}_{n,3}$, so by (1),

$$A \text{ holds a.a.s. for } G \in 3\mathcal{G}_{n,1}. \qquad (4)$$

Let $\mathbf{E}$ denote expectation in the space $3\mathcal{G}_{n,1}$. We will bound $\mathbf{E}(XI_A)$, where I_A is the indicator variable for the event A. Let $[n]$ denote $\{1, 2, \ldots, n\}$, the vertex set of graphs in $3\mathcal{G}_{n,1}$.

We take C_9 with vertex set being the congruence classes modulo 9, and edges joining i to $i+1$, where in such notation the integers represent their congruence classes. For a graph G in $3\mathcal{G}_{n,1}$ satisfying A and given a particular homomorphism α of G onto C_9, let n_i denote the number of vertices in $\alpha^{-1}(i)$ for each integer i. Note that

$$\mathbf{E}(XI_A) = \frac{1}{|\mathcal{T}|} \sum_{T \in \mathcal{T}} X(T) I_A(T)$$

where $\mathcal{T}$ is the set of ordered triples of perfect matchings of $[n]$ which have no duplicate edges, and the functions X and I_A are the same for T as for the 3-regular graph which is the superposition of the three matchings in T. This can be rewritten as a double summation where the inner sum is over homomorphisms. Reordering the summations and then rewriting gives

$$\mathbf{E}(XI_A) = \frac{1}{|\mathcal{T}|} \sum_{h\in\mathcal{H}} t(h) \tag{5}$$

where $\mathcal{H}$ is the set of mappings from $[n]$ to $\{0, 1, \ldots, 8\}$ and $t(h)$ is the number of elements T of $\mathcal{T}$ satisfying property A for which h is a homomorphism of each of the three perfect matchings in T.

Let us focus on a particular matching M in a triple T which contributes to $t(h)$ for a homomorphism h. For each i, let m_i be the number of edges between $h^{-1}(i-1)$ and $h^{-1}(i)$ in M. Then $h^{-1}(i+1) \cup h^{-1}(i+3) \cup h^{-1}(i+5) \cup h^{-1}(i+7)$ is an independent set in G of cardinality

$$n_{i+1} + n_{i+3} + n_{i+5} + n_{i+7} = \sum_{j=i+1}^{i+8} m_i = \frac{n}{2} - m_i \leq 0.4554n$$

since property A is satisfied. This gives the last inequality in the following list, which must therefore all be satisfied when $t(h) > 0$:

$$\begin{aligned} \sum_{i=0}^{8} n_i &= n, & \\ n_i &\geq 0 & (0 \leq i \leq 8), \\ m_i + m_{i+1} &= n_i & (0 \leq i \leq 8), \\ m_i &\geq 0.0446n & (0 \leq i \leq 8). \end{aligned} \tag{6}$$

Hence by (5)

$$\mathbf{E}(XI_A) \leq \frac{1}{|\mathcal{T}|} \sum_{n_0,\ldots,n_8} \frac{n!}{n_0! \cdots n_8!} \left(\sum_{m_0,\ldots,m_8} \prod_{i=0}^{8} \frac{(m_i + m_{i+1})!}{m_i!} \right)^3, \tag{7}$$

with the summations restricted by the conditions in (6). Here the first factor in the outer summation chooses the vertices in each pre-image. The factor in the product can be rewritten as

$$\binom{n_i}{m_{i+1}} m_{i+1}!,$$

accounting for the choice of which m_{i+1} vertices in $h^{-1}(i)$ are matched with vertices in $h^{-1}(i+1)$, and how this matching is arranged.

Now $|\mathcal{T}| \sim c_3 |\mathcal{G}_{n,1}|^3$ where c_3 is given by (2) and $|\mathcal{G}_{n,1}| = n!\, 2^{-n/2}/(n/2)!$. So (7) implies

$$\mathbf{E}(XI_A) \;=\; O(n^{18}) \frac{((n/2)!)^3 2^{3n/2}}{(n!)^3} \max_{n_0,\ldots,n_8} \frac{n!}{n_0! \cdots n_8!} \left(\max_{m_0,\ldots,m_8} \prod_{i=0}^{8} \frac{(m_i + m_{i+1})!}{m_i!} \right)^3$$

$$\leq (M + o(1))^n \tag{8}$$

where

$$M = \max_{x_0,\dots,x_8} \prod_{i=0}^{8} f(x_i + x_{i+1}, 2) f(x_i, -3) \tag{9}$$

with the maximum taken over $x_i \geq 0.0446$ for each i, and $\sum x_i = \frac{1}{2}$. Here x_i stands for m_i/n. Applying Lemma 2.1 with $c = 9$, $d_1 = 2$, $d_2 = 3$ and $b = 0.0446$ we obtain $M < 0.9984$. Hence $\mathbf{E}(XI_A) = o(1)$ and so in the space $3\mathcal{G}_{n,1}$,

$$\mathbf{P}(X > 0) \leq \mathbf{E}(XI_A) + \mathbf{P}(\overline{A}) = o(1)$$

in view of (4). Thus by (1) $X = 0$ a.a.s. in $\mathcal{G}_{n,3}$. However, for $G \in \mathcal{G}_{n,3}$ the probability that the girth is at least k tends to a non-zero constant for fixed k (see for example the book by Bollobás [1, Cor.19 p.53], or [8, Section 2.3]). The theorem follows. ■

We note that the proof cannot be simplified to the extent of omitting the use of independent sets, for the following reason. If we replace the lower bound on x_i in (9) by $x_i \geq 0$ then $M > 1$. (In either case, the exact value or location of the maximum is not obvious.) Although we used only upper bounds, a similar argument shows that the expected number of homomorphisms of $G \in 3\mathcal{G}_{n,1}$ onto C_9 goes to infinity as $n \to \infty$. It is readily checked that the same property also holds for $G \in \mathcal{G}_{n,3}$, by using the pairing model for random regular graphs as discussed in [1] and in [8].

The next theorem is also proved in [3]. We give a different proof here.

THEOREM 3.2 (Kim and Nešetřil). *For every $k > 0$ there exists a 4-regular graph with girth at least k and with no homomorphism onto C_5.*

Proof. Let $X(G)$ denote the number of homomorphisms of G onto C_5 and let A denote the event that the largest independent set of a graph has cardinality less than $0.4164n$. McKay [6] showed that A holds a.a.s. for $G \in \mathcal{G}_{n,4}$. Then with $\mathbf{E}$ denoting expectation in the space $4\mathcal{G}_{n,1}$, the argument as in the proof of Theorem 3.1 gives $\mathbf{E}(XI_A) \leq (M + o(1))^n$ where

$$M = \max_{x_0,\dots,x_4} \prod_{i=0}^{4} f(x_i + x_{i+1}, 3) f(x_i, -4) \tag{10}$$

with the maximum taken over $x_i \geq 0.0836$ for each i, and $\sum x_i = \frac{1}{2}$. Applying Lemma 2.1 with $c = 5$, $d_1 = 3$, $d_2 = 4$ and $b = 0.0836$ we obtain

$M < 0.84$. The rest of the argument in the proof of Theorem 3.1 now applies with little modification. ■

Acknowledgement. We would like to thank J. Nešetřil for communications on this problem area.

REFERENCES

1. B. Bollobás, *Random graphs*, Academic Press, London, 1985.
2. S. Janson, Random regular graphs: Asymptotic distributions and contiguity, *Combinatorics, Probability and Computing* **4** (1995), 369–405.
3. J.H. Kim and J. Nešetřil, Colorings of Graphs with Small Degrees (manuscript).
4. A.V. Kostočka, J. Nešetřil and P. Smolikova, Complexity of coloring of bounded degree graphs, *Discrete Math.* (to appear).
5. H.-J. Lai, Unique graph homomorphisms onto odd cycles, II, *J. Combinatorial Theory, Ser. B* **46** (1989), 363–376.
6. B.D. McKay, Independent sets in regular graphs of high girth, Proc. Australia-Singapore Joint Conference on Information Processing and Combinatorial Mathematics (Singapore, 1986), *Ars Combinatoria* **23A** (1987), 179–185.
7. M. Molloy, H. Robalewska, R.W. Robinson and N.C. Wormald, 1-factorisations of random regular graphs, *Random Structures & Algorithms* **10** (1997), 305–321.
8. N.C. Wormald, Models of random regular graphs, In *Surveys in Combinatorics, 1999*, J.D. Lamb and D.A. Preece, eds, pp. 239–298.