Linear arboricity and linear k-arboricity of regular graphs

N. Alon * V. J. Teague† N. C. Wormald ‡

Abstract

We find upper bounds on the linear k-arboricity of d-regular graphs using a probabilistic argument. For small k these bounds are new. For large k they blend into the known upper bounds on the linear arboricity of regular graphs.

1 Introduction

A *linear forest* is a forest each of whose components is a path. The *linear arboricity* of a graph G is the minimum number of linear forests required to partition $E(G)$ and is denoted by $\mathrm{la}(G)$. It was shown by Akiyama, Exoo and Harary [1] that $\mathrm{la}(G) = 2$ when G is cubic, and they conjectured that every d-regular graph has linear arboricity exactly $\lceil (d+1)/2 \rceil$. This was shown to be asymptotically correct as $d \to \infty$ in [3], and in [4] the following result is shown.

Theorem 1 *There is an absolute constant $c > 0$ such that for every d-regular graph G*

$$la(G) \leq \frac{d}{2} + cd^{2/3}(\log d)^{1/3}.$$

(Actually a slightly weaker result is proved explicitly there, but it is noted that the same proof with a little more care gives this theorem.)

A *linear k-forest* is a forest consisting of paths of length at most k. The *linear k-arboricity* of G, introduced by Bermond et al. [5], is the minimum number of linear k-forests required to partition $E(G)$, and is denoted by $\mathrm{la}_k(G)$. In [2] it was shown that for cubic G, $\mathrm{la}_k(G) = 2$ for all $k \geq 9$, or 7 in the case of graphs with edge-chromatic number 3. Thomassen has very recently improved this by proving the following, which was a conjecture of [5].

Theorem 2 (Thomassen [8]) *If G is cubic then $la_5(G) = 2$.*

*School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, and Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Email: noga@math.tau.ac.il. Research supported in part by a Sloan Foundation grant 96-6-2, by a State of New Jersey grant and by a USA-Israel BSF grant

†Department of Mathematics and Statistics, University of Melbourne, Parkville, VIC 3052, AUSTRALIA. Email:vjt@ms.unimelb.edu.au

‡Department of Mathematics and Statistics, University of Melbourne, Parkville, VIC 3052, AUSTRALIA. Email:nick@ms.unimelb.edu.au. Research supported by the Australian Research Council

In this paper we obtain improved upper bounds on $\mathrm{la}_k(G)$ for d-regular graphs G when d is fairly large. Note that by simply counting edges, the linear k-arboricity of a d-regular graph must be at least $\frac{(k+1)d}{2k}$.

Theorem 3 *There is an absolute constant $c > 0$ such that for every d-regular graph G and every $\sqrt{d} > k \geq 2$*

$$la_k(G) \leq \frac{(k+1)d}{2k} + c\sqrt{kd\log d}.$$

Section 2 gives a quite short proof of this result. By specialising to $k = d^{1/3}/(\log d)^{1/3}$ (which minimises the upper bound up to a constant factor) we immediately obtain Theorem 1.

Moreover, for sufficiently large d, Theorem 3 gives non-trivial results even when $k = 2$. It is convenient to define

$$\mathrm{la}_k(d) = \max_{G \text{ is } d\text{-regular}} \mathrm{la}_k(G).$$

Immediately we have

$$\mathrm{la}_k(G) \leq \mathrm{la}_k(\Delta(G)) \tag{1}$$

since every graph with maximum degree d occurs as a subgraph (not necessarily spanning) of a d-regular graph, and the restriction of a linear k-forest to a subgraph is again a linear k-forest. In Section 3 we examine for various small k the smallest d for which we obtain an improvement over existing results, using the method of proof of Theorem 3 and also another argument. First, for very small d Vizing's theorem on edge-chromatic number, which gives $la_k(G) \leq d+1$ for every $d \geq 2$ and $k \geq 1$, is better than Theorem 3. This can be improved by 1 for $k \geq 2$ by using the argument of [7, Lemma 2.1] (where the following was proved for cubic graphs).

Lemma 1 *For every $d \geq 2$ and $k \geq 2$*

$$la_k(G) \leq d.$$

Proof. Colour the edges of a d-regular graph G using d colours and such that the minimum number of pairs of edges of the same colour are adjacent. Then by minimality, the colour of an edge x appears only once among the $2d-2$ edges incident with x (since otherwise x could be recoloured). Hence each edge is incident with only one other of the same colour, so each colour class induces a 2-linear forest. ∎

Throughout this paper, $X(n,p)$ denotes a binomial random variable distributed as $\mathrm{Bin}(n,p)$.

2 Proof of Theorem 3

Lemma 2 *Let $k \geq 2$, let $d \geq 4$ be even and define $f(d,k)$ to be the least integer which satisfies*

$$\tfrac{1}{2}e(k+1)(d^2-d+2)\mathbf{P}\left(X(d/2, 1/(k+1)) > f(d,k)\right) < 1.$$

Then

$$\mathrm{la}_k(d) \leq (k+1)f(d,k) + \mathrm{la}_k(2f(d,k)).$$

Proof. Let G be a d-regular graph. Orient its edges along an Euler cycle. Then each vertex has indegree and outdegree $d/2$. Colour the vertices randomly with $k+1$ colours.

Let $A^+_{v,i}$ ($A^-_{v,i}$) be the event that the number of vertices of colour i in the out-neighbourhood (in-neighbourhood) of vertex v is strictly greater than $f(d,k)$. Every event $A^-_{v,i}$ is independent of every other except the events $A^-_{v,j}$ for $j \neq i$ (there are k of these), $A^-_{w,j}$ where $w \neq v$ and there is a vertex u such that (u,v) and (u,w) are in G (there are at most $(k+1)(d/2)(d/2-1)$ of these), and events $A^+_{w,j}$ where there is some vertex u such that (u,v) and (w,u) are in G (there are at most $(k+1)(d/2)^2$ of these). Hence every event is independent of all except at most $(k+1)d(d-1)/2+k$ of the others.

Since $k+1>1$,

$$\mathbf{P}(A^\pm_{v,i}) = \mathbf{P}\left(X(d/2, 1/(k+1)) > f(d,k)\right)$$

and so by the assumption of the lemma

$$e((k+1)d(d-1)/2+k+1)Pr(A^\pm_{v,i}) < 1.$$

Hence, by the Lovász Local Lemma, (c.f., e.g., [4], Chapter 5, Corollary 1.2), it is possible to colour the vertices such that none of the events $A^\pm_{v,i}$ occurs. The set of all edges from a vertex of colour i to one of colour j, where $i \neq j$ forms a bipartite graph of degree at most $f(d,k)$. This can be covered by $f(d,k)$ matchings. (For example, see corollary 5.2 of Hall's theorem in Bondy and Murty [6].) Call a matching from colour i to colour j a matching of type (i,j).

Consider the complete multi-graph MK_{k+1} on $k+1$ vertices, where each vertex represents a colour and each pair of vertices is connected by two parallel edges (which we call the *first* and *second* edge). This can be covered by $k+1$ paths of length k (described modulo $k+1$ by $t, (t-1), (t+1), (t-2), (t+2), \ldots$). Denote one of these paths by $(i_1, i_2, \ldots, i_{k+1})$. Assign to each edge $\{i_j, i_{j+1}\}$ of this path a matching of type $(min\{i_j, i_{j+1}\}, max\{i_j, i_{j+1}\})$ if the above edge is the first edge of MK_{k+1} connecting i_j and i_{j+1}, and a matching of type $(max\{i_j, i_{j+1}\}, min\{i_j, i_{j+1}\})$ if it is the second edge. Note that the union of any k matchings assigned in such a way to the k edges of the path is a linear k-forest in G. Since there are at most $f(d,k)$ matchings of each type, and since the $k+1$ paths cover MK_{k+1}, all edges joining vertices of distinct colours can be covered with $(k+1)f(d,k)$ linear k-forests. The edges of G remaining uncovered, joining vertices of the same colour, induce a subgraph with in- and out-degrees at most $f(d,k)$, and so can be covered by $\mathrm{la}_k(2f(d,k))$ linear k-forests. The lemma follows. ∎

We now prove Theorem 3. Clearly we can assume d is even. By [4, Theorem A.11] there is an absolute constant c_0 such that for all $2 \leq k \leq d^{2/3}$ (a range chosen just for convenience),

$$f(d,k) < \frac{d}{2(k+1)} + c_0\sqrt{\frac{d\log d}{k}}. \tag{2}$$

Put

$$h(d) = \mathrm{la}_k(d) - \frac{(k+1)d}{2k}.$$

Applying Lemma 2 to (2) gives

$$h(d) \leq \frac{c_0(k+1)^2}{k}\sqrt{\frac{d\log d}{k}} + h(2f(d,k)) \tag{3}$$

for $d \geq k^{3/2}$. We can assume d is sufficiently large to ensure that (2) implies, say, $f(d,k) < \frac{2d}{2(k+1)}$. Thus by induction/iteration on d starting with $h(d) \leq d$ for $d \leq k^{3/2}$, (3) gives that

$$h(d) \leq \frac{c_0(k+1)^2}{k^{3/2}}[\sqrt{d_0 \log d_0} + \sqrt{d_1 \log d_1} + \cdots + \sqrt{d_{q-1} \log d_{q-1}}\ ] + h(d_q),$$

where $d_0 = d, d_i = 2f(d_{i-1}, k)$ for all i and $d_q \leq k^{3/2}$. The sum in the square brackets is easily seen to be bounded by $O(\sqrt{d \log d})$ and the desired result follows. ∎

3 Small d and k

We first have a type of monotonicity result which appears essentially in [2]. (It was presented there in a special case, but the general argument given here is identical.)

Lemma 3 *For every $k \geq 2$ and $d \geq 2$*

$$\mathrm{la}_k(d) \leq la_k(d-1) + 1.$$

Proof. Let G be d-regular and let M be a maximum matching in G. Then $G - M$ has vertices of degrees $d-1$ and d, and the vertices of degree d form an independent set, B, say. The subgraph of $G - M$ induced by the edges incident with B is bipartite and, by Hall's theorem and considerations of degrees, contains a matching, M', which covers the vertices in B. Thus $G' = G - (M \cup M')$ has maximum degree at most $d-1$. By maximality of M, $M \cup M'$ is a linear forest with maximum path length at most 2. The lemma now follows from (1). ∎

A corollary of Thomassen's result is therefore that $\mathrm{la}_5(G) \leq d-1$ for any d-regular graph G, $d \geq 3$. But for $d \geq 8$ this can be improved as in the following.

Corollary 1 *For $d \geq 3$, $\mathrm{la}_5(d) \leq \min\{d-1, \lceil\frac{2d+2}{3}\rceil\}$.*

Proof. The bound $d-1$ is explained above. For the other bound, cover the edges of a d-regular graph by $d+1$ matchings by Vizing's theorem, split these matchings into groups of three to obtain $\lceil\frac{d+1}{3}\rceil$ graphs with maximum degree 3, and find two linear 5-forests in each using (1) and Theorem 2. This gives the upper bound $2\lceil\frac{d+1}{3}\rceil$, which can be reduced by 1 whenever $d \equiv 0 \pmod 3$, (by taking one of the matchings as a linear forest), yielding $\lceil\frac{2d+2}{3}\rceil$.
∎

We can improve on this result for any k if d is sufficiently large using Lemma 2.

Theorem 4 *For $k \geq 2$ and d even,*

$$\mathrm{la}_k(d) \leq (k+3)f(d,k),$$

where $f(d,k)$ is as in Lemma 2.

Proof. Just apply Lemma 1 to estimate $\mathrm{la}_k(2f(d,k))$ in Lemma 2. ∎

It turns out that this gives a better result for the small values of d which we will next consider than iterating the bound in Lemma 2 as in the proof of Theorem 3.

By calculating $f(d,k)$ in Theorem 4 for various values of d and $k < 5$ using the exact values in the appropriate binomial distribution, we obtain the following table. (For $k \geq 5$ we would have to go to much larger d to obtain an improvement over Corollary 1.) The entry for given k and $d-j$ gives the least value of d for which we obtain the bound $\mathrm{la}_k(d) \leq d-j$. Computations were arbitrarily terminated at $j = 10$. For any $d' > d$, it follows that $\mathrm{la}_k(d') \leq d' - j$ by Lemma 3. This fills in bounds on $\mathrm{la}_k(d)$ for values of d in between the ones appearing in the table. Lemma 1 covers all $j \leq 0$.

$d-j$	$k=2$	$k=3$	$k=4$
$d-1$	3026	1580	1282
$d-2$	3042	1580	1290
$d-3$	3058	1600	1298
$d-4$	3074	1600	1306
$d-5$	3080	1620	1314
$d-6$	3096	1620	1322
$d-7$	3112	1634	1330
$d-8$	3128	1634	1338
$d-9$	3134	1654	1346
$d-10$	3150	1654	1354

Table 1. Values of d for which $\mathrm{la}_k(d) \leq d-j$, $j \leq 10$, by Theorem 4.

Note that if the table were extended to the right, Theorem 2 gives 3 for the entries in the first row for $k \geq 5$, and Corollary 1 gives 8 for the row $d-2$, 11 for $d-3$, and so on. The results from Theorem 4 in a given row appear to increase with k for all $k \geq 6$, at least for $d-i \geq d-10$. Nevertheless, asymptotically the larger values of k will give better results if the method is iterated as in Theorem 3, but not necessarily from Theorem 4, so that for k sufficiently large the bound drops even below $2d/3$.

Acknowledgment We would like to thank an anonymous referee for extremely helpful comments.

References

[1] J. Akiyama, G. Exoo and F. Harary, Covering and packing in graphs III, cyclic and acyclic invariants, *Math. Slovaca* **30** (1980), 405-417.

[2] R.E.L. Aldred and N.C. Wormald, Linear k-arboricity of regular graphs, *Australasian Journal of Combinatorics* **18** (1998), 97–104.

[3] N. Alon, The linear arboricity of graphs, *Israel Journal of Mathematics*, 62, No. 3, (1988), 311-325.

[4] N. Alon and J.H. Spencer, *The Probabilistic Method.* Wiley, New York, 1992.

[5] J.C. Bermond, J.L. Fouquet, M. Habib and B. Peroche, On linear k-arboricity, *Discrete Math.* **52** (1984), 123-132.

[6] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications.* Elsevier, New York, 1976.

[7] M.N. Ellingham and N.C. Wormald, Isomorphic factorization of regular graphs and 3-regular multigraphs, *J. London Math. Soc. (2)* **37** (1988), 14–24.

[8] C. Thomassen, Two-colouring the edges of a cubic graph such that each monochromatic component is a path of length at most 5, *J. Combinatorial Theory (Series B)* 75 (1999), 100-109.