Sharp concentration of the number of submaps in random planar triangulations

Zhicheng Gao*
Department of Mathematics and Statistics
Carleton University
Ottawa CANADA K1S 5B6
zgao@math.carleton.ca

Nicholas C. Wormald†
Department of Mathematics and Statistics
University of Melbourne VIC 3010
AUSTRALIA
nick@ms.unimelb.edu.au

Abstract

We show that the maximum vertex degree in a random 3-connected planar triangulation is concentrated in an interval of almost constant width. This is a slightly weaker type of result than our earlier determination of the limiting distribution of the maximum vertex degree in random planar maps and in random triangulations of a (convex) polygon. We also derive sharp concentration results on the number of vertices of given degree in random planar maps of all three types. Some sharp concentration results about general submaps in 3-connected triangulations are also given.

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1 Introduction

For a random graph, the distribution of the number of vertices of given degree is fairly well understood (see Barbour et al. [1], for example). For random maps, or graphs embedded on a surface such as the plane, not much is known. The main results concerning degrees of vertices involve enumeration of maps with given vertex degrees permitted as in [3, 7, 9]. On the other hand, our main result in [8] is that in both random planar maps and random triangulations of a (convex) polygon, the numbers of vertices of given degree close to the maximum behave asymptotically as independent Poisson variables. As a result of this, we obtained the limiting distribution of the maximum vertex degree for both classes of maps, which is concentrated in an interval of width Ω_n for any $\Omega_n \to \infty$. This improved the result of Devroye et al. [6, Theorem 1], who used different methods.

To obtain analogous results for other classes of planar maps is not at all straightforward. Our method is to obtain generating function equations for the finite moments of the random variable which counts vertices of a given degree, and then perform asymptotic analysis of the results. There are two basic steps: formation of generating function equations that do not become excessively complicated for higher moments, and the basic asymptotic analysis. These were achieved for both classes of maps considered in [8]. The present paper concentrates on one class of maps, planar triangulations, for which we have solved the asymptotic analysis problem, but only for the lower moments. This is due to the lack of a simple way to deal with the higher moments as in [8]. As a result, we do not obtain the limiting distribution of the number of vertices of given degree, or maximum degree. However, we do obtain sharp concentration results by the second moment method.

Throughout this paper, a *map* is a connected graph G embedded in the plane with no edge crossings. Loops and multiple edges are allowed in G. A map is *rooted* if an edge is distinguished together with a vertex on the edge and a side of the edge. The distinguished vertex and edge are called the root vertex and the root edge of the map. The face on the distinguished side of the root edge is called the root face. A rooted *near-triangulation* is a rooted 2-connected map, with no loops or multiple edges, so that all nonroot faces are triangles. A rooted *triangulation of an n-gon* is a rooted near-triangulation which has n vertices, all on the root face. A rooted *triangulation* is a rooted *near-triangulation* whose root face is also a triangle. Two rooted maps are considered the same if there is a homeomorphism from the plane to itself which transforms one rooted map to the other and preserves the rooting. Throughout this paper, all probability distributions are uniform over a given family of rooted maps. $\mathbf{P}$, $\mathbf{E}$, and $\mathbf{V}$ are used to denote the probability, expectation, and variance of a random variable, respectively. Δ_n denotes the maximum vertex degree of a random map, and $\zeta_k = \zeta_{k,n}$ denotes the number of vertices of degree k in a random map. We use Ω_n to denote a function which goes to infinity arbitrarily slowly. We consider rooted maps for accessibility by generating function techniques. By the results in [13], any almost sure property of one of the classes of rooted maps in this paper is also an almost sure property of the corresponding unrooted versions.

It is clear that for triangulations, $\zeta_{k,n} = 0$ when $k \le 2$ and $n \ge 2$. So we assume $k > 2$ in the rest of the section. Our main sharp concentration results are as follows.

Theorem 1 *For a random rooted triangulation with $n + 2$ vertices,*

$$\mathbf{E}(\Delta_n) = \frac{\log n - (1/2)\log\log n}{\log(4/3)} + O(1), \quad \mathbf{V}(\Delta_n) = O(\log n),$$

and

$$\mathbf{P}\left(\left|\Delta_n - \frac{\log n - (1/2)\log\log n}{\log(4/3)}\right| < \Omega_n\right) = 1 - O\left(1/\log n + (3/4)^{\Omega_n}\right),$$

for any function $\Omega_n \to \infty$*.*

Theorem 2 *Let*

$$\mu_k = \frac{8(k-2)}{(4k^2-1)}\left(-\frac{3}{4}\right)^k \binom{-3/2}{k}.$$

Then for a random rooted triangulation with $n+2$ *vertices,*

(i)

$$\mathbf{E}(\zeta_k) = n\mu_k\left(1 + O\left(\frac{\log^{20} n}{n}\right)\right), \quad \mathbf{V}(\zeta_k) = n\mu_k + (n\mu_k)^2 O\left(\frac{\log^{20} n}{n}\right),$$

uniformly for all $k = O(\log n)$*.*

(ii)

$$\mathbf{P}\left(\zeta_k = n\mu_k(1+o(1))\right) = 1 - o(1)$$

uniformly for $k < (\log n - (1/2)\log\log n)/\log(4/3) - \Omega_n$*.*

$$\mathbf{P}\left(\zeta_k = 0\right) = 1 - o(1)$$

uniformly for $k > (\log n - (1/2)\log\log n)/\log(4/3) + \Omega_n$*.*

The theory of submaps of a random map was begun in [11] and [12] and extended in a general way in [4], where it is shown that a random rooted map with n edges almost surely contains at least cn copies of any given planar submap for some positive constant c. Let W_k denote the wheel with k spokes, and let

$$f(n) = \frac{\log n - (1/2)\log\log n}{\log(4/3)}.$$

Since triangulations have no loops or multiple edges, a vertex of degree k in a planar triangulation corresponds to a copy of W_k. Hence, Theorem 2(ii) implies that when

$$k < f(n) - \Omega_n,$$

the number of copies of W_k in a random rooted planar triangulation with $n+2$ vertices is sharply concentrated around $n\mu_k$. For such k, $n\mu_k \to \infty$. On the other hand, when

$$k > f(n) + \Omega_n,$$

a random triangulation with $n+2$ vertices almost surely contains no copy of W_k. So it may be said that the function $f(n)$ serves as the threshold function on the variable k for the property that a random triangulation with $n+2$ vertices contains a copy of W_k, although this usage of the term 'threshold' is not standard. It seems reasonable to presume that for many other families of planar submaps, there are similar sharp concentration results and threshold functions for the size of the submap in the family. The following lemma allows us to study general submaps in triangulations.

Lemma 1 *For each n, let $M = M(n)$ be a planar near-triangulation with external face of degree k and with j internal vertices such that $k \geq 4$ and $k + j = o(n)$. Let $\eta_n(M)$ be the number of copies of M in a random rooted triangulation with $n+2$ vertices. Suppose there are r distinct ways to root the external face of M.*

(i) *If M is 3-connected, then*

$$\mathbf{E}(\eta_n(M)) = r\left(\frac{27}{256}\right)^{j-1}\mathbf{E}(\zeta_{k,n+1-j})(1+o(1)),$$

and furthermore for $k + j = O(1)$, we have

$$\mathbf{E}(\eta_n(M)(\eta_n(M)-1)) = r^2\left(\frac{27}{256}\right)^{2j-2}\mathbf{E}(\zeta_{k,n+2-2j}(\zeta_{k,n+2-2j}-1))(1+o(1)).$$

(ii) *If M is 2-connected, then*

$$\left(\frac{27}{256}\right)^{k+j-1}\mathbf{E}(\zeta_{3,n+1-k-j})(1+o(1)) \leq \mathbf{E}(\eta_n(M)) \leq r\left(\frac{27}{256}\right)^{j-1}\mathbf{E}(\zeta_{k,n+1-j})(1+o(1)),$$

and furthermore for $k + j = O(1)$, we have

$$\mathbf{E}(\eta_n(M)(\eta_n(M)-1)) \leq r^2\left(\frac{27}{256}\right)^{2j-2}\mathbf{E}(\zeta_{k,n+2-2j}(\zeta_{k,n+2-2j}-1))(1+o(1)).$$

Proof: Let $D_n(M)$ be the number of rooted triangulations with $n+2$ vertices and with a copy of M distinguished. Define $D_n(W_k)$ similarly for the wheel W_k. Take a triangulation counted by $D_n(M)$, and replace M by a distinguished W_k. The result is clearly 3-connected and has $n+3-j$ vertices. When M is 3-connected, i.e., there is no internal edge in M joining two external vertices of M, we can reverse this process by removing W_k and insert M back in r different ways. This gives an r to 1 mapping when M is 3-connected and the root face is not in the distiguished M. However, since $k + j = o(n)$, the probability that the root face is in the distinguished copy is trivially $o(1)$. Hence when M is 3-connected

$$\mathbf{E}(\eta_n(M)) = \frac{D_n(M)}{T_n} = r(1+o(1))\frac{D_{n+1-j}(W_k)}{T_{n+1-j}}\frac{T_{n+1-j}}{T_n} = r\left(\frac{27}{256}\right)^{j-1}\mathbf{E}(\zeta_{k,n+1-j})(1+o(1)).$$

When M is 2-connected, the above argument gives the right hand inequality stated in (ii). The left hand inequality of (ii) is obtained by embedding M in a single triangle to obtain a new triangulation $\bar{M}$. Since $\bar{M}$ is 3-connected and has 3 external vertices and $k + j$ internal vertices, we obtain the desired inequality by observing $E(\eta_n(M)) \geq E(\eta_n(\bar{M}))$.

Similarly we can consider triangulations with two distinguished copies of M. When $k+j = O(1)$, the number of copies of M is almost surely at least a constant times n by the results in [12] and [4]. Hence, the probability that the two distinguished copies of M overlap is $o(1)$, and the above argument also gives the desired estimates for $\mathbf{E}\left(\eta_n(M)(\eta_n(M)-1)\right)$. ∎

From Theorem 2, Lemma 1 and Chebyshev's inequality we immediately have the following.

Theorem 3 *Let M be a 3-connected near-triangulation with external face of degree k and with j internal vertices such that there are r distinct ways to root the external face. Let $\eta_n(M)$ be defined as in Lemma 1. Then, for $k \geq 4$ and $k + j = O(1)$,*

$$\mathbf{P}\left(\eta_n(M) = rn\left(\frac{27}{256}\right)^{j-1}\frac{8(k-2)}{(4k^2-1)}\left(-\frac{3}{4}\right)^k\binom{-3/2}{k}(1+o(1))\right) = 1 - o(1). \ ∎$$

The case $k = 3$ deserves a separate treatment, since no two copies of M can overlap when the external face of M has degree 3. The first part of the following result arises from this observation and Lemma 1. The second part comes from applying Chebyshev's inequality in the first case and Markov's in the second.

Theorem 4 *Let M be a 3-connected triangulation with $j+3$ vertices such that $j = o(n)$ and there are r distinct ways to root M. Let $\eta_n(M)$ be defined as in Lemma 1. We have*

(i)

$$\begin{aligned}\mathbf{E}(\eta_n(M)) &= 2rn\left(\frac{27}{256}\right)^j(1+o(1)),\\ \mathbf{E}(\eta_n(M)(\eta_n(M)-1)) &= 4r^2n^2\left(\frac{27}{256}\right)^{2j}(1+o(1)).\end{aligned}$$

(ii) *Let $c \le rj^{-\beta} \le C$ for some constants $c,\ C > 0$ and $\beta \ge 0$. Then*

$$\mathbf{P}\left(\eta_n(M) = 2rn\left(\frac{27}{256}\right)^j(1+o(1))\right) = 1 - o(1)$$

for $j < (\log n + \beta \log\log n)/\log(256/27) - \Omega_n$, and

$$\mathbf{P}\left(\eta_n(M) = 0\right) = 1 - o(1)$$

for $j > (\log n + \beta \log\log n)/\log(256/27) + \Omega_n$. ∎

Madras [10] studied the number of occurrences of a given pattern in large lattice clusters, and he proved a general pattern theorem similar to the general submap density theorem in [4]. He suggested that there should be some kind of law of large numbers for pattern occurrence. That is, given a proper pattern P, there should exist a number $\alpha > 0$, such that almost all clusters of size n contain between $(\alpha-\epsilon)n$ and $(\alpha+\epsilon)n$ copies of P. Our Theorem 4(ii) shows that there is such a law of large numbers for the subtriangulation occurrence in large planar triangulations.

2 Some basic equations

Let $T_{n,k}$ be the number of rooted triangulations with $n+2$ vertices and root vertex degree k, and let $T_{n,k,l}$ be the number of rooted triangulations with $n+2$ vertices, root vertex degree k, and with another distinguished vertex of degree l. Then $T_n = \sum_k T_{n,k}$ is the number of all rooted triangulations with $n+2$ vertices. Define generating functions

$$T(x,y) = \sum_{n,k} T_{n,k}x^ny^k \qquad \text{and} \qquad \bar{T}(x,y,z) = \sum_{n,k,l} T_{n,k,l}x^ny^kz^l.$$

Note that

$$T(x,1) = \sum T_nx^n.$$

Deleting a root vertex of degree k of a triangulation gives a near-triangulation with root face degree k (including by convention the one which is just a single edge). A rooting of the near-triangulation can be canonically selected using a suitable convention, to obtain a bijection between rooted triangulations and rooted near-triangulations.

Hence $T(x,y)$ is also the generating function for rooted near-triangulations with x marking the number of nonroot vertices and y marking the root face degree, and $\bar{T}(x,y,z)$ is also the generating function for those rooted near-triangulations which have a distinguished vertex with z marking the distinguished vertex degree. Note that the distinguished vertex in the near-triangulation may coincide with its root vertex. Noting that $T(x,y) = x^2y^3D(x,xy) + xy^2$, where $D(x,y)$ is defined in [5], we obtain from [5, (4.1)–(4.4)] that

$$T(x,y) = \frac{y(1-u)(1+2u)-1}{2} + \frac{1-y(1-u)}{2}(1-4u(1-u)y)^{1/2}, \tag{1}$$

where

$$u = \sum_{j\geq 1} \frac{1}{j}\binom{4j-2}{j-1}x^j \tag{2}$$

satisfies

$$x = u(1-u)^3. \tag{3}$$

Hence

$$T(x,1) = u(1-2u), \tag{4}$$
$$[y^3]T(x,y) = u^2(1-u)^3(1-2u). \tag{5}$$

To derive an expression for $\bar{T}(x,y,z)$, it is convenient to introduce the generating function $\bar{T}_1(x,y,z)$ for those near-triangulations counted by $\bar{T}(x,y,z)$ whose distinguished vertex is on the root face. Then $\bar{T}_2(x,y,z) = \bar{T}(x,y,z) - \bar{T}_1(x,y,z)$ is the generating function for those near-triangulations whose distinguished vertex is not on the root face. It is also convenient to define the generating function $t(x,y,z)$ for those near-triangulations whose distinguished vertex is the same as the root vertex. Throughout this paper, we use t_y to denote the partial derivative of $t(x,y,z)$ with respect to y, etc.

Theorem 5 *Let $u = u(x)$ be as given in (3), and*

$$\begin{aligned} A(x,y) &= y - 1 - 2T(x,y) + x^{-1}y\ [y^3]T(x,y) \\ &= (y(1-u)-1)(1-4u(1-u)y)^{1/2}. \end{aligned} \tag{6}$$

We have

$$t(x,y,z) = \frac{xy^2z(y-z)}{y-z+yT(x,z)-zT(x,y)}, \tag{7}$$
$$\bar{T}_1(x,y,z) = yt_y(x,y,z), \tag{8}$$
$$[y^3]\bar{T}_2(x,y,z) = \frac{x(1-u)}{u}t(x,1/(1-u),z) - x(1-u)^2z - xT(x,z), \tag{9}$$
$$A(x,y)\bar{T}_2(x,y,z) = t(x,y,z) - xy^2z - yT(x,y)T(x,z) - x^{-1}yT(x,y)[y^3]\bar{T}_2(x,y,z). \tag{10}$$

Proof: We first derive the following equations for $t(x,y,z)$ and $\bar{T}_2(x,y,z)$:

$$\begin{aligned} t(x,y,z) &= xy^2z + y^{-1}zT(x,y)t(x,y,z) \\ &\quad + y^{-1}z\left(t(x,y,z) - xy^2z - x^{-1}yz^{-1}t(x,y,z)[y^3]t(x,y,z)\right), \end{aligned} \tag{11}$$

$$\begin{aligned} \bar{T}_2(x,y,z) &= 2y^{-1}T(x,y)\bar{T}_2(x,y,z) \\ &\quad + y^{-1}\left(t(x,y,z) - xy^2z - x^{-1}yT(x,y)[y^3]t(x,y,z)\right) \\ &\quad + y^{-1}\left(\bar{T}_2(x,y,z) - x^{-1}yT(x,y)[y^3]\bar{T}_2(x,y,z) - x^{-1}y\bar{T}_2(x,y,z)[y^3]T(x,y)\right). \end{aligned} \tag{12}$$

Proof of (11) and (12): The argument is standard in map enumerations. The first term on the right side of (11) corresponds to the case where the near-triangulation consists of just a single edge. Let NT be a rooted near-triangulation with root vertex a and root edge ab. Let abc be the triangular face, different from the root face, containing ab. If vertex c is on the root face, then deleting edge ab decomposes T into two near-triangulations. The one containing ac is counted by $t(x, y, z)$, and the one containing bc is counted by $T(x, y)$. If c is not on the root face, then deleting ab gives one near-triangulation NT' in which there is no edge joining a and b. This gives the third term on the right side of (11), where xy^2z inside the parenthesis corresponds to the singe edge case, and the last term corresponds to the case that there is an edge joining a and b in NT' (which can be further decomposed into a near-triangulation NT'' and a triangulation T'').

The proof of (12) is similar. By the definition of $\bar{T}_2(x, y, z)$, a near-triangulation T counted by $\bar{T}_2(x, y, z)$ contains a distinguished vertex d not on the root face, and hence the single edge case does not occur here. The first term on the right side of (12) corresponds to the case where c is on the root face. The coefficient 2 is from the fact that the distinguished vertex may be in either one of the two near-triangulations after the decomposition. When c is not on the root face, we consider two subcases. First $c = d$. In this subcase we choose c to be the root vertex of the new near-triangulation T', and this gives the second term on the right side of (12). The second subcase $c \neq d$ gives the last term on the right side of (12), where the second term inside the parenthesis corresponds to the case that the distinguished vertex d is in T'', and the third term inside the parenthesis corresponds to the case that d is in NT''. This completes the proof of (11) and (12).

Noting that $[y^3]t(x, y, z)$ is the generating function for rooted triangulations with x marking the number of nonroot vertices, and z marking the root vertex degree, we have

$$[y^3]t(x, y, z) = xT(x, z).$$

Substituting the above equation into (11) and solving for $t(x, y, z)$, we obtain (7). Collecting the coefficients of $\bar{T}_2(x, y, z)$ in (12), we obtain (10). Setting $y = 1/(1-u)$ in (10) and using (1), we obtain (9). For each rooted near-triangulation with root face degree k, there are k ways to distinguish a vertex on the root face. Therefore

$$\bar{T}_1(x, y, z) = \sum_{k \geq 2} k[y^k]t(x, y, z)y^k = yt_y(x, y, z).$$

This completes the proof. ∎

3 Asymptotic expansions for $T(x, y)$ and $T(x, y, z)$

We need to use some analytic results from [8]. In the present case the situation is a little simpler because we only need to use up to three variables. In the following, ϵ will denote a sufficiently small positive constant, ϕ is a constant satisfying $0 < \phi < \pi/2$, and $\mathbf{y} = (y_1, y_2, \ldots, y_d)$, where d is the dimension of $\mathbf{y}$ which will be either 1 or 2 in this paper. In what follows, $\mathbf{y}$ denotes y when the dimension is 1, and $\mathbf{y}$ denotes (y, z) when the dimension is 2. As in [8], define

$$\begin{aligned} \Delta_x(\epsilon, \phi) &= \{x : \ |x| \leq 1 + \epsilon, \ x \neq 1, \ |\mathrm{Arg}(x-1)| \geq \phi\}, \\ \mathcal{R}(\epsilon, \phi) &= \{(x, \mathbf{y}) : |y_j| < 1, \ 1 \leq j \leq d, x \in \Delta_x(\epsilon, \phi)\}. \end{aligned}$$

Definition 1. We write

$$f(x,\mathbf{y}) = \tilde{O}\left((1-x)^{-\alpha}\prod_{j=1}^{d}(1-y_j)^{-\beta_j}\right)$$

if there are $\epsilon > 0$ and $0 < \phi < \pi/2$ such that in $\mathcal{R}(\epsilon,\phi)$

(i) $f(x,\mathbf{y})$ is analytic, and

$$f(x,\mathbf{y}) = O\left(|1-x|^{-\alpha}\prod_{j=1}^{d}(1-|y_j|)^{-\beta_j}\right)$$

as $(1-x)(1-y_j)^{-p} \to 0$, for $1 \le j \le d$, and some $p \ge 0$.

(ii)

$$f(x,\mathbf{y}) = O\left(|1-x|^{-\alpha'}\prod_{j=1}^{d}(1-|y_j|)^{-q}\right) \qquad \text{for some } q \ge 0 \text{ and some real number } \alpha'.$$

We also use $[x^n\mathbf{y}^{\mathbf{k}}]f(x,\mathbf{y})$ to denote the coefficient of $x^n\mathbf{y}^{\mathbf{k}}$ in the power series expansion of $f(x,\mathbf{y})$. We first derive some asymptotic estimates for $T_{n,k}$ and $T_{n,k,k}$.

Lemma 2 **(i)**

$$T_{n,k} = O\left((4/3)^{-k}(1+\epsilon')^k n^{-5/2}(256/27)^n\right)$$

uniformly for all n, k, and constant $\epsilon' > 0$;

(ii) *Define*

$$X = (1-256x/27)^{1/2}, \quad Y = (1-3y/4)^{1/2}, \quad Z = (1-3z/4)^{1/2}.$$

Then

$$\begin{aligned}
T_{n,k} &= (\sqrt{6}/72)[x^n y^k](1-Y^2)(1-3Y^2+2Y^3)Y^{-3}X^3 \\
&\quad + O\left(n^{-7/2}(\log n)^{20}(3/4)^k(256/27)^n\right), \\
T_{n,k,k} &= (\sqrt{6}/12)[x^n y^k z^k](1-Y^2)(1-3Y^2+2Y^3)Y^{-3}(Z-1)^3Z^{-1}X \\
&\quad + O\left(n^{-5/2}(\log n)^{20}(3/4)^{2k}(256/27)^n\right),
\end{aligned}$$

uniformly for $1 < k = O(\log n)$.

Proof: For convenience, we rescale the variables $x, \mathbf{y}$, and define

$$\begin{aligned}
\Delta'_x(\epsilon,\phi) &= \{x : |x| \le 27/256(1+\epsilon),\ x \ne 27/256,\ |\mathrm{Arg}(x-27/256)| \ge \phi\}, \\
\mathcal{R}'(\epsilon,\phi) &= \{(x,\mathbf{y}) : |y_j| < 4/3,\ 1 \le j \le d, x \in \Delta'_x(\epsilon,\phi)\}.
\end{aligned}$$

We also define

$$f(x,\mathbf{y}) = \tilde{O}\left((1-256x/27)^{-\alpha}\prod_{j=1}^{d}(1-3y_j/4)^{-\beta_j}\right)$$

accordingly. Since u satisfies (3) and has positive coefficients, it is easy to see that $27/256$ is the unique singularity of $u(x)$ on its circle of convergence, and $27/256$ is a branch point of order 2. Hence $u(x)$ is analytic in $\Delta'_x(\epsilon,\phi)$ (recalling that $\epsilon>0$ can be made arbitrarily small), and has a power series expansion in X near $x=1$. Substituting

$$u(x)=\sum_{i\ge 0} c_i X^i, \quad x=27(1-X^2)/256$$

into (3) and equating the coefficients, we obtain

$$u(x)=1/4-(\sqrt{6}/8)X+(1/12)X^2-(31\sqrt{6}/1728)X^3+O(X^4). \tag{13}$$

Hence

$$u(1-u) = (3/16)-(\sqrt{6}/16)X-(5/96)X^2+(41\sqrt{6}/3456)X^3+O(X^4), \tag{14}$$

$$T(x,1) = Q(x)+(\sqrt{6}/24)X^3+O(X^5), \tag{15}$$

where Q is a quadratic polynomial in x.

From (2) and (5), we know that u and $u(1-u)=u^2+T(x,1)$ have nonnegative coefficients in x. It follows from [8, Lemma 4] (after appropriate scaling of the value of the function) that

$$|u|<1/4, \quad |u(1-u)|<3/16, \qquad \text{for } x\in\Delta'_x(\epsilon,\phi). \tag{16}$$

Hence

$$|1-4u(1-u)y|\ge 1-4|u(1-u)y|\ge 1-3|y|/4, \qquad \text{for } x\in\Delta'_x(\epsilon,\phi), \tag{17}$$

and $A(x,y)$ and $T(x,y)$ are analytic in $\mathcal{R}'(\epsilon,\phi)$.

We now claim

$$A(x,y) = -Y^3+(1/12)\left(3-9Y^2+6Y^4\right)Y^{-1}X^2 \tag{18}$$

$$-(\sqrt{6}/36)\left(1-4Y^2+3Y^4\right)Y^{-3}X^3+\tilde{O}\left(X^4Y^{-8}\right), \tag{19}$$

and

$$\begin{aligned} T(x,y) &= 1/4-(3/4)Y^2+(1/2)Y^3-(1/8)\left(1-3Y^2+2Y^4+Y-Y^3\right)X^2Y^{-1} \\ &\quad +(\sqrt{6}/72)\left(1-Y^2\right)\left(1-3Y^2+2Y^3\right)X^3Y^{-3}+\tilde{O}\left(X^4Y^{-8}\right). \end{aligned} \tag{20}$$

We first verify condition (i) in Definition 1 for (19). Using (6), (13), (14), $x=27(1-X^2)/256$, and $y=4(1-Y^2)/3$, we obtain

$$\begin{aligned} A(x,y) &= \left(\frac{4(1-Y^2)}{3}\left(\frac{3}{4}+\frac{\sqrt{6}\,X}{8}-\frac{X^2}{12}+\frac{31\sqrt{6}\,X^3}{1728}+O(X^4)\right)-1\right) \\ &\quad \times Y\left(1+\left(\frac{\sqrt{6}\,X}{4}+\frac{5X^2}{24}-\frac{41\sqrt{6}\,X^3}{864}+O(X^4)\right)(1-Y^2)Y^{-2}\right)^{1/2}. \end{aligned}$$

When $XY^{-2}\to 0$ in $\mathcal{R}'(\epsilon,\phi)$, we can expand the square root by the binomial theorem, and (with the help of Maple) obtain

$$\begin{aligned} A(x,y) &= -Y^3+(1/12)(3-9Y^2+6Y^4)Y^{-1}X^2 \\ &\quad -(\sqrt{6}/36)(1-4Y^2+3Y^4)Y^{-3}X^3+O\left(X^4Y^{-8}\right), \end{aligned}$$

for $(x,y)\in\mathcal{R}'(\epsilon,\phi)$ and $XY^{-2}\to 0$. Hence the error term

$$E(x,y)=A(x,y)-\left(-Y^3+(1/12)(3-9Y^2+6Y^4)Y^{-1}X^2-(\sqrt{6}/36)(1-4Y^2+3Y^4)Y^{-3}X^3\right)$$

in the expansion of $A(x,y)$ satisfies

$$E(x,y)=\tilde{O}\left(X^4Y^{-8}\right),\ \text{for } (x,y)\in\mathcal{R}'(\epsilon,\phi) \text{ and } XY^{-2}\to 0.$$

It is clear that in $\mathcal{R}'(\epsilon,\phi)$, $E(x,y)$ is analytic and

$$E(x,y)=O\left((1-|3y/4|)^{-3}\right),$$

which establishes (19). The expansion (20) follows immediately by observing

$$T(x,y)=\frac{y(1-u)(1+2u)-1}{2}-A(x,y)/2.$$

Now the asymptotics for $T_{n,k}$ stated in Lemma 2 follow from (20) and [8, Lemma 2], with a rescaling of the variables.

In order to derive asymptotics for $\bar{T}(x,y,z)$, we first verify that $t(x,y,z)$ is analytic in $\mathcal{R}'(\epsilon,\phi)$. By (7), we have

$$t(x,y,z)=\frac{xy^2z}{1-(yT(x,z)-zT(x,y))/(z-y)}.$$

It suffices to show that the denominator in the above expression is never zero for $(x,y,z)\in\mathcal{R}'(\epsilon,\phi)$. Note

$$T(x,y)=\sum_{n\ge 1}x^n\sum_{2\le k\le n+1}T_{n,k}y^k,$$

where $T_{1,2}=1$, and $T_{n,k}=D_{n-k+1,k-3}>0$ with $D_{n,m}$ defined in [5, (4.7)]. Hence

$$\frac{yT(x,z)-zT(x,y)}{z-y}=\sum_{n\ge 1}x^n\sum_{2\le k\le n+1}yzT_{n,k}\frac{z^{k-1}-y^{k-1}}{z-y},\tag{21}$$

which is clearly a power series in x,y and z with nonnegative coefficients. Therefore, for $|y|\le 4/3,\ |z|\le 4/3$ and $|x|\le 27/256$,

$$\left|\frac{yT(x,z)-zT(x,y)}{z-y}\right|\le\lim_{y\to 4/3,z\to 4/3}\frac{yT(27/256,z)-zT(27/256,y)}{z-y}=1/2.$$

In view of (21), $(yT(x,z)-zT(x,y))/(z-y)$ is continuous in $\mathcal{R}'(\epsilon,\phi)$, and so its absolute value is strictly less than 1 in $\mathcal{R}'(\epsilon,\phi)$ (choosing ϵ sufficiently small). Hence $t(x,y,z)$ is analytic in $\mathcal{R}'(\epsilon,\phi)$. By (16), we have

$$|1/(1-u)|\le\sum_{k\ge 0}|u|^k<\sum_{k\ge 0}(1/4)^k=4/3,$$

for $x\in\Delta'_x(\epsilon,\phi)$, and hence $t(x,1/(1-u),z)$ is analytic in $\mathcal{R}'(\epsilon,\phi)$. It follows from (9) and (10) that $[y^3]\bar{T}_2(x,y,z)$ and $\bar{T}_2(x,y,z)$ are analytic in $\mathcal{R}'(\epsilon,\phi)$. (It is important to note that $y=1/(1-u)$ does not cause a singularity in $\bar{T}_2(x,y,z)$.)

Next we note from (7) and (9) that $t(x,y,z)$ and $[y^3]\bar{T}_2(x,y,z)$ are bounded in $\mathcal{R}'(\epsilon,\phi)$, and

$$|T_y(x,y)| = O\left((1-3|y|/4)^{-1/2}\right),$$
$$|t_y(x,y,z)| = O\left((1-3|y|/4)^{-1/2}\right).$$

It follows from (10) that

$$\begin{aligned}\bar{T}_2(x,y,z) &= O\left((1-3|y|/4)^{-1/2}|t_y(x,y,z)+T_y(x,y)|\right)\\ &= O\left((1-3|y|/4)^{-1}\right) \text{ for } (x,y,z)\in\mathcal{R}'(\epsilon,\phi).\end{aligned}$$

Now we can use Maple to obtain asymptotic expansions for $t(x,y,z)$, $t(x,1/(1-u),z)$, $[y^3]\bar{T}_2(x,y,z)$, and $\bar{T}_2(x,y,z)$ in the same way as we did for $A(x,y)$ and $T(x,y)$. For

$$(x,y,z)\in\mathcal{R}'(\epsilon,\phi), \text{ and } XY^{-6}\to 0 \text{ and } XZ^{-6}\to 0,$$

we obtain, using (7)–(10), (13), (19), and (20),

$$\begin{aligned}t(x,y,z) &= p(x,Y,Z)+\tilde{O}\left(X^3Y^{-6}Z^{-6}\right),\\ \bar{T}_1(x,y,z) &= p(x,Y,Z)+\tilde{O}\left(X^3Y^{-8}Z^{-6}\right),\\ [y^3]\bar{T}_2(x,y,z) &= (27\sqrt{6}/1024)(Z-1)^3Z^{-1}X+p(x,Y,Z)+\tilde{O}\left(X^3Z^{-6}\right),\\ \bar{T}_2(x,y,z) &= (\sqrt{6}/12)(1-Y^2)(1-3Y^2+2Y^3)Y^{-3}(Z-1)^3Z^{-1}X\\ &\quad +p(x,Y,Z)+\tilde{O}\left(X^3Y^{-9}Z^{-6}\right)\end{aligned}$$

where $p(x,Y,Z)$ denotes an expression in x,Y,Z which is linear in x (not necessarily the same at each occurrence). Hence

$$\begin{aligned}\bar{T}(x,y,z) &= \bar{T}_1(x,yz)+\bar{T}_2(x,y,z)\\ &= (\sqrt{6}/12)(1-Y^2)(1-3Y^2+2Y^3)Y^{-3}(Z-1)^3Z^{-1}X\\ &\quad +p(x,Y,Z)+\tilde{O}\left(X^3Y^{-9}Z^{-6}\right). \qquad (22)\end{aligned}$$

Now the asymptotics for $T_{n,k,k}$ stated in Lemma 2 follows from [8, Lemma 2]. This completes the proof of Lemma 2. ∎

4 Proof of Theorems 1 and 2

The following lemma relates the first two moments of ζ_k with $T_{n,k}$ and $T_{n,k,k}$; its proof is the same as that of [8, Lemma 1 (ii)].

Lemma 3 *For $k\geq 1$, we have*

$$\begin{aligned}E(\zeta_k) &= \frac{6n}{k}\frac{T_{n,k}}{T_n},\\ E(\zeta_k(\zeta_k-1)) &= \frac{6n}{k}\frac{T_{n,k,k}}{T_n}.\end{aligned}$$

Lemma 4 *Let ϵ' be any constant satisfying $0 < \epsilon' < 4/3$, and let $k > 2$. For a random rooted triangulation with $n+2$ vertices, we have*

(i)

$$E(\zeta_k) = O\left(n(4/3-\epsilon')^{-k}\right) \quad \textit{uniformly for all } k,\ n;$$

(ii)

$$\begin{aligned} E(\zeta_k) &= \frac{8n(k-2)}{4k^2-1}\left(-\frac{3}{4}\right)^k \binom{-3/2}{k}\left(1+O\left(\frac{\log^{20} n}{n}\right)\right) \\ &= \frac{4}{\sqrt{\pi}} n k^{-1/2}(4/3)^{-k}(1+O(1/k)), \\ E(\zeta_k(\zeta_k-1)) &= (E(\zeta_k))^2\left(1+O\left(\frac{\log^{20} n}{n}\right)\right) \end{aligned}$$

uniformly for all $k = O(\log n)$.

Proof: Applying Darboux's Theorem (see [2]), we obtain from (15)

$$T_n = (\sqrt{6}/(32\sqrt{\pi}))n^{-5/2}(256/27)^n(1+O(1/n)). \tag{23}$$

Now Lemma 4(i) follows from Lemma 2(i) and Lemma 3. Using binomial formula and some algebra, we obtain, for $k \geq 2$,

$$\begin{aligned} [y^k](1-Y^2)(1-3Y^2+2Y^3)Y^{-3} &= [y^k](2-2Y^2+3Y-4Y^{-1}+Y^{-3}) \\ &= \frac{4k(k-2)}{4k^2-1}\binom{-3/2}{k}\left(-\frac{3}{4}\right)^k, \end{aligned}$$

and

$$\begin{aligned} [z^k](Z-1)^3Z^{-1} &= [z^k](-3Z-Z^{-1}) \\ &= \frac{2(2-k)}{4k^2-1}\binom{-3/2}{k}\left(-\frac{3}{4}\right)^k. \end{aligned}$$

Using

$$\begin{aligned} [x^n]X^3 &= \frac{3}{4\sqrt{\pi}}n^{-5/2}(256/27)^n(1+O(1/n)), \\ [x^n](-X) &= \frac{1}{2\sqrt{\pi}}n^{-5/2}(256/27)^n(1+O(1/n)), \end{aligned}$$

and Lemma 2(ii), we obtain

$$\begin{aligned} T_{n,k} &= \frac{\sqrt{6}}{24\sqrt{\pi}}\frac{k(k-2)}{4k^2-1}\binom{-3/2}{k}\left(-\frac{3}{4}\right)^k \\ &\quad \times n^{-5/2}(256/27)^n\left(1+O\left(\frac{\log^{20} n}{n}\right)\right), \\ T_{n,k,k} &= \frac{k\sqrt{6}}{3\sqrt{\pi}}\left(\frac{k-2}{4k^2-1}\right)^2\binom{-3/2}{k}^2\left(-\frac{3}{4}\right)^{2k} \\ &\quad \times n^{-3/2}(256/27)^n\left(1+O\left(\frac{\log^{20} n}{n}\right)\right), \end{aligned}$$

uniformly for $k = O(\log n)$. Using (23), we obtain

$$\begin{aligned}
\frac{T_{n,k}}{T_n} &= \frac{4k}{3}\frac{(k-2)}{4k^2-1}\binom{-3/2}{k}\left(-\frac{3}{4}\right)^k\left(1+O\left(\frac{\log^{20} n}{n}\right)\right), \\
\frac{T_{n,k,k}}{T_n} &= \frac{32kn}{3}\left(\frac{k-2}{4k^2-1}\right)^2\binom{-3/2}{k}^2\left(-\frac{3}{4}\right)^{2k}\left(1+O\left(\frac{\log^{20} n}{n}\right)\right),
\end{aligned}$$

uniformly for $k = O(\log n)$. Now Lemma 4 follows from Lemma 3 and the simple fact

$$(-1)^k\binom{-3/2}{k} = \frac{2}{\sqrt{\pi}}k^{1/2}(1+O(1/k)). \blacksquare$$

Theorem 1 follows from Lemma 4 and [8, Lemma 7]. Theorem 2(i) follows from Lemma 4, Theorem 2(ii) follows from Theorem 1 and Chebyshev's inequality by observing $n\mu_k \to \infty$ when $2 < k < (\log n - (1/2)\log\log n)/\log(4/3) - \Omega_n$.

5 Triangulations of polygons and all maps

In this final section, we discuss similar sharp concentration results for degree counts in triangulations of polygons and in all maps. The main analyses were done in [8], except that the emphasis was on Δ_n and hence k was assumed near $\log n$ in that paper. Let $P(x,y)$ and $\bar{P}(x,y,z)$ be the generating functions defined in [8] for rooted triangulations of polygons, and $M(x,y)$ and $\bar{M}(x,y,z)$ for all rooted maps. As before, let $p(x,y,z)$ ($p(x,y)$) denote an expression which is linear in x. A close look at [8, Section 3] gives

$$\begin{aligned}
P(x/4,2y) &= \frac{-1}{8}\frac{y^2}{(1-y)^2}(1-x)^{1/2} \\
&\quad +p(x,y)+\tilde{O}\left((1-x)^{3/2}(1-y)^{-4}\right), \\
\bar{P}(x/4,2y,2z) &= \frac{1}{16}\frac{y^2z^2}{(1-y)^2(1-z)^2}(1-4x)^{-1/2} \\
&\quad +p(x,y,z)+\tilde{O}\left((1-x)^{1/2}(1-y)^{-2}(1-z)^{-2}\right), \\
M(x/12,5y/6) &= \frac{4y}{15}(1+3y/5)^{-1/2}(1-y)^{-3/2}(1-x)^{3/2} \\
&\quad +p(x,y)+\tilde{O}\left((1-x)^{5/2}(1-y)^{-4}\right),
\end{aligned}$$

and

$$\begin{aligned}
\bar{M}(x/12,5y/6,5z/6) &= \frac{-y}{10}(1+3y/5)^{-1/2}(1-y)^{-3/2}\left((1+3z/5)^{1/2}(1-z)^{-1/2}-1\right) \\
&\quad \times(1-x)^{1/2}+p(x,y,z)+\tilde{O}\left((1-x)^{3/2}(1-y)^{-4}(1-z)^{-4}\right).
\end{aligned}$$

Therefore by [8, Lemma 2],

$$P_{n,k} = [x^ny^k]P(x,y) = \frac{k-1}{16\sqrt{\pi}}n^{-3/2}(1/2)^k4^n\left(1+O\left(\frac{\log^{20} n}{n}\right)\right), \tag{24}$$

$$P_{n,k,k} = [x^n y^k z^k]\bar{P}(x,y,z) = \frac{(k-1)^2}{16\sqrt{\pi}} n^{-1/2}(1/2)^{2k}4^n \left(1 + O\left(\frac{\log^{20} n}{n}\right)\right), \quad (25)$$

$$M_{n,k} = [x^n y^k]M(x,y) = \frac{b_k}{5\sqrt{\pi}} n^{-5/2}(5/6)^k 12^n \left(1 + O\left(\frac{\log^{20} n}{n}\right)\right), \quad (26)$$

$$M_{n,k,k} = [x^n y^k z^k]\bar{M}(x,y,z) = \frac{a_k b_k}{20\sqrt{\pi}} n^{-3/2}(5/6)^{2k}12^n \left(1 + O\left(\frac{\log^{20} n}{n}\right)\right), \quad (27)$$

where

$$a_k = [z^k]\left\{(1+3z/5)^{1/2}(1-z)^{-1/2}\right\}, \quad (28)$$

$$b_k = [y^k]\left\{y(1+3y/5)^{-1/2}(1-y)^{-3/2}\right\}. \quad (29)$$

Since (see [8])

$$P_n = \sum_k P_{n,k} = \frac{1}{16\sqrt{\pi}} n^{-3/2}4^n(1+O(1/n)), \quad \text{and}$$

$$M_n = \sum_k M_{n,k} = \frac{2}{\sqrt{\pi}} n^{-5/2}12^n(1+O(1/n)),$$

it follows from [8, Lemma 1] that

$$\mathbf{E}(\zeta_k) = \frac{nP_{n,k}}{P_n} = (k-1)(1/2)^k n \left(1 + O\left(\frac{\log^{20} n}{n}\right)\right), \text{ and} \quad (30)$$

$$\mathbf{E}(\zeta_k(\zeta_k - 1)) = \frac{nP_{n,k,k}}{P_n} = (k-1)^2(1/2)^{2k}n^2 \left(1 + O\left(\frac{\log^{20} n}{n}\right)\right), \quad (31)$$

for triangulations of polygons, and

$$\mathbf{E}(\zeta_k) = \frac{2nM_{n,k}}{kM_n} = \frac{b_k}{5k}(5/6)^k n \left(1 + O\left(\frac{\log^{20} n}{n}\right)\right), \text{ and}$$

$$\mathbf{E}(\zeta_k(\zeta_k - 1)) = \frac{2nM_{n,k,k}}{kM_n} = \frac{a_k b_k}{20k}(5/6)^{2k}n^2 \left(1 + O\left(\frac{\log^{20} n}{n}\right)\right),$$

for all rooted maps.

It is clear that for triangulations of an n-gon with $n \geq 3$, $\zeta_k = 0$ when $k \leq 1$. So we assume $k > 1$ in the following theorem which is implied immediately from (30) and (31).

Theorem 6 *Let $k > 1$ and*

$$\mu_k = (k-1)(1/2)^k.$$

Then, for a random rooted triangulation of an n-gon,

(i)

$$\mathbf{E}(\zeta_k) = n\mu_k \left(1 + O\left(\frac{\log^{20} n}{n}\right)\right), \quad \mathbf{V}(\zeta_k) = n\mu_k + (n\mu_k)^2 O\left(\frac{\log^{20} n}{n}\right),$$

uniformly for all $k = O(\log n)$.

(ii)

$$\mathbf{P}\left(\zeta_k = \mu_k(1+o(1))\right) = 1 - o(1)$$

uniformly for $k < (\log n + \log\log n)/\log 2 - \Omega_n$*.*

$$\mathbf{P}\left(\zeta_k = 0\right) = 1 - o(1)$$

uniformly for $k > (\log n + \log\log n)/\log 2 + \Omega_n$*.*

Noting

$$\frac{d}{dz}\left\{(1+3z/5)^{1/2}(1-z)^{-1/2}\right\} = \frac{4}{5}(1+3z/5)^{-1/2}(1-z)^{-3/2},$$

we obtain, by (28) and (29), $b_k = (5/4)ka_k$, and hence

$$\frac{a_k b_k}{20k} = \left(\frac{b_k}{5k}\right)^2.$$

This implies that

$$\mathbf{E}(\zeta_k(\zeta_k - 1)) = (\mathbf{E}(\zeta_k))^2\left(1 + O\left(\frac{\log^{20} n}{n}\right)\right)$$

for random rooted maps. Hence we obtain the following result.

Theorem 7 *Let* $k > 0$ *and*

$$b_k = [y^k]\left\{y(1+3y/5)^{-1/2}(1-y)^{-3/2}\right\}, \text{ and } \mu_k = \frac{b_k}{5k}(5/6)^k.$$

Then, for a random rooted map with n *edges,*

(i)

$$\mathbf{E}(\zeta_k) = n\mu_k\left(1 + O\left(\frac{\log^{20} n}{n}\right)\right), \quad \mathbf{V}(\zeta_k) = n\mu_k + (n\mu_k)^2 O\left(\frac{\log^{20} n}{n}\right),$$

uniformly for all $k = O(\log n)$*.*

(ii)

$$\mathbf{P}\left(\zeta_k = \mu_k(1+o(1))\right) = 1 - o(1)$$

uniformly for $k < (\log n - (1/2)\log\log n)/\log(6/5) - \Omega_n$*.*

$$\mathbf{P}\left(\zeta_k = 0\right) = 1 - o(1)$$

uniformly for $k > (\log n - (1/2)\log\log n)/\log(6/5) + \Omega_n$*.*

In conclusion, the major part of the present paper was involved in finding the first two moments of ζ_k in the case of rooted triangulations. This led to sharp concentration of the maximum vertex degree and of the number of vertices of degree k. The requisite moment calculations were already in [8] for triangulations of polygons and for rooted maps. Other classes of maps could be considered, if the moment calculations can be carried out; we have now covered some of the most interesting classes of maps. In another paper yet to appear, we use a different technique to show that the number of copies of a fixed triangulation in a random triangulation is asymptotically normal. It would be of interest to do the same for the number of copies of an arbitrary submap.

References

[1] A.D. Barbour, L. Holst and S. Janson, *Poisson Approximation*, Clarendon Press, Oxford, 1992.

[2] E.A. Bender, Asymptotic methods in enumeration, *SIAM Rev.* , **16** (1974), 485–515.

[3] E.A. Bender and E.R. Canfield, Face Sizes of 3-polytopes, *J. Combin. Theory, Ser. B* **46** (1989), 58–65.

[4] E.A. Bender, Z.C. Gao and L.B. Richmond, Submaps of maps I: General 0-1 laws, *J. Combin. Theory Ser. B* **55** (1992) 104–117.

[5] W.G. Brown, Enumeration of triangulations of the disk, *Proc. London Math. Soc.* **14**(3) (1964), 746–768.

[6] L. Devroye, P. Flajolet, F. Hurtado, M. Noy and W. Steiger, Properties of random triangulations and trees, *Discrete Comput. Geom.* **22** (1999), 105–117.

[7] Z.C. Gao and L.B. Richmond, Root vertex valency distributions of rooted maps and rooted triangulations, *Europ. J. Combin.* **15** (1994), 483–490.

[8] Z.C. Gao and N.C. Wormald, The distribution of the maximum vertex degree in random planar maps, *J. Combin. Theory Ser. A*, **89** (2000), 201–230.

[9] V.A. Liskovets, A pattern of asymptotic vertex valency distributions in planar maps, *J. Combin. Theory Ser. B* **75** (1999), 116–133.

[10] N. Madras, A pattern theorem for lattice clusters, *Ann. Comb.* **3** (1999), 357–384.

[11] L.B. Richmond, R.W. Robinson and N.C. Wormald, On Hamilton cycles in 3-connected cubic maps, in "Cycles in Graphs" (B. Alspach and C.D. Godsil, eds.), *Annals of Discrete Mathematics* **27** (1985), 141–150.

[12] L.B. Richmond and N.C. Wormald, Random triangulations of the plane, *European J. Combinatorics* **9** (1988), 61–71.

[13] L.B. Richmond and N.C. Wormald, Almost all maps are asymmetric, *J. Combin. Theory, Ser. B* **63** (1995), 1–7.