Enumeration of rooted cubic planar maps

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Abstract

Although much work has been done on enumerating rooted planar maps since Tutte's poineering works in early 1960s, many classes of maps with no loops or multiple edges are still untreated. In this paper, we enumerate three classes of cubic planar maps with no loops or multiple edges: 1-connected; 2-connected; 3-connected and triangle-free.

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1 Introduction

We consider *planar maps* in this paper. These are connected graphs embedded in the plane. A planar map is *rooted* if an edge is distinguished together with a vertex on the edge and a side of the edge. The face on the distinguished side is called the *root face*. For planar maps, by convention the root face must be the unbounded face. (Thus a rooted planar map is equivalent to a rooted map on the sphere with arbitrary root face.)

Much work has been done on enumerating rooted planar maps since Tutte's poineering works in early 1960s, and simple explicit expressions (in the sense that they are *hypergeometric*) are obtained for many classes of planar maps which allow loops and/or multiple edges. On the other hand, loops and multiple edges are often forbidden in non-topological graph theory studies. However, if loops and multiple edges are not allowed in maps the enumeration usually becomes more involved and there are few simple expressions known. A notable exception is the formula

$$\frac{2(4n-3)!}{n!(3n-1)!}$$

for rooted 3-connected triangulations with $2n$ faces (or dually, 3-connected cubic maps with $2n$ vertices) derived by Tutte [3]; see also Tutte's survey [4].

Our aim in this paper is to provide results on some of the most important and interesting classes of planar maps which have been shunned so far because of the difficulty of obtaining appealing results: rooted cubic maps which do not have loops or multiple edges. This will be done by enumerating their duals, which are triangular maps satisfying certain conditions. We will give both exact and asymptotic results. The exact results are in the form of generating function equations, and produce efficient algorithms for calculation of the numbers. These are of interest in checking programs for generating all such maps, such as that of Brinkmann and McKay [2], whose work was assisted by communication of the results we obtain here.

For the asymptotics, there are some heavy algebraic manipulations involved which will be handled by the computer algebra system Maple. It would be very interesting if another approach could be found which gave much simpler results, in particular the results in Corollary 1. However, it appears that these results are intrinsically much more complicated than for other classes of maps in which the radius of convergence of the generating function is a simple rational number.

We enumerate all cubic maps in Section 3 using a recursive approach; 2-connected cubic maps in Section 4 and 3-connected triangle-free in Section 5 are enumerated using a compositional approach; a table is given in Section 6 for the exact numbers of these three types of cubic maps with relatively small numbers of edges. All these results imply dual results about planar triangulations, as given in Lemma 1.

This section closes with some definitions. Throughout this paper, all maps will be planar, and cubic maps will have no loops or multiple edges. A rooted map is called *near-triangular* if each nonroot face has degree 3. A near-triangular map is called *triangular* if the root face also has degree 3. Triangular maps may have loops or multiple edges. One configuration which can occur in triangular maps and plays a special role in our work we call a *twisted dumbell*, shown in Figure 1. This consists of two faces sharing two parallel edges, the other edges on the faces both being loops, one inside the parallel edges and one outside. The shaded regions contain other vertices, edges and faces. Throughout the paper, a vertex will be called *exterior* if it is in the root face, otherwise it is called *interior*. A twisted dumbell is called exterior if the two parallel edges are in the root face, otherwise it is called interior. Finally, we use K_3 to denote the rooted triangulation with exactly three vertices, and hence one non-root face, and K_4 the rooted 3-connected triangulation with four vertices.

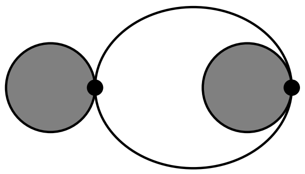

Figure 1: *A twisted dumbell*

2 Preliminary results

The bijections in the following lemma are obtained by taking dual maps. The proof is easy and is omitted.

Lemma 1 *There are bijections between the following pairs of maps.*

(i) all cubic maps, and all triangular maps which have no vertex of degree 1 or 2 and do not contain a twisted dumbell;

(ii) 2-connected cubic maps, and loopless triangular maps with no vertex of degree 2;

(ii) 3-connected triangle-free cubic maps (i.e. of girth at least four), and triangular maps with no loops or multiple edges and with minimum vertex degree at least four.

The following lemma gives an algebraic explanation why the asymptotic numbers of many families of planar maps grow at the *normal rate*

$$(C + o(1))n^{-5/2}r^{-n}$$

where n is the number of edges and r and C are constants. By "higher order terms" we mean terms with higher total degree.

Lemma 2 *Let $g(x)$ be a power series with nonnegative coefficients. Suppose $g(x)$ has a unique singularity x_0 at its circle of convergence, where $x_0 > 0$, and $g(x)$ converges at x_0 with $g(x_0) = g_0$. Further suppose $g(x)$ is algebraic and satisfies $G(x, g) = 0$ where G is a polynomial in x and g with the following Taylor expansion at (x_0, g_0):*

$$\begin{aligned} G(x,g) &= c_{20}(x-x_0)^2 + c_{11}(x-x_0)(g-g_0) + c_{02}(g-g_0)^2 \\ &\quad + c_{30}(x-x_0)^3 + c_{21}(x-x_0)^2(g-g_0) + c_{12}(x-x_0)(g-g_0)^2 + c_{03}(g-g_0)^3 \\ &\quad + \textit{higher order terms}, \end{aligned} \tag{1}$$

such that

(i) $c_{11}^2 - 4c_{20}c_{02} = 0$,

(ii)

$$c_{02} \neq 0, \qquad \textit{and } \alpha = -\frac{c_{11}}{2c_{02}} > 0,$$

(iii)

$$\beta = \frac{1}{c_{02}}\left(c_{30} + c_{21}\alpha + c_{12}\alpha^2 + c_{03}\alpha^3\right) > 0.$$

Then $g(x)$ has the following asymptotic expansion at x_0 in powers of $(x_0 - x)^{1/2}$:

$$g(x) = g_0 + \alpha(x - x_0) + \beta^{1/2}(x_0 - x)^{3/2} + \cdots$$

and

$$[x^n]g(x) \sim (3/4)\,(\beta/\pi)^{1/2}\, x_0^{3/2} n^{-5/2} x_0^{-n}.$$

Proof: Using the Taylor expansion of $G(x, g)$ and condition (ii), we have

$$\begin{aligned}
&\left((g - g_0) + \frac{c_{11}}{2c_{02}}(x - x_0)\right)^2 \\
&= \; -(1/c_{02})\left(c_{30}(x - x_0)^3 + c_{21}(x - x_0)^2(g - g_0) + c_{12}(x - x_0)(g - g_0)^2 + c_{03}(g - g_0)^3\right) \\
&\quad\; + \text{higher order terms} \qquad (2)
\end{aligned}$$

By conditions (i) and (ii), we know from the quadratic terms in (1) that the expansion of $g(x)$ in $(x - x_0)$ has the linear term $\alpha(x - x_0)$. Hence it follows from (2) that

$$\begin{aligned}
g - g_0 \;&=\; \alpha(x - x_0) + \beta^{1/2}(x_0 - x)^{3/2} \\
&\quad + \text{higher power terms in } (x_0 - x)^{1/2}\;,
\end{aligned}$$

where the positive square root is chosen for β because $g(x)$ has nonnegative coefficients. Now the required asymptotic expansion for $g(x)$ follows by iteration. The asymptotics of $[x^n]g(x)$ follows from the well-known Darboux's Theorem (see, for example, [1]). ∎

3 Enumeration of all cubic maps

Let $C(x)$ be the generating function for all rooted cubic maps with x marking the number of faces. Let $m = m(x)$ be the power series satisfying

$$\begin{aligned}
&16x^{11}m^4 + (-24x^9 + 32x^8 + 72x^7)m^3 + (-15x^7 - 108x^6 - 194x^5 - 92x^4 + x^3)m^2 \\
&+ (-2x^5 - 33x^4 - 70x^3 - 46x^2 + 16x - 1)m - x^2 - 11x + 1 = 0. \qquad (3)
\end{aligned}$$

Equating coefficients in this equation shows that $m(x) = 1 + \cdots$ is uniquely determined.

Theorem 1

$$C(x) = (1 - 2x - 4x^2)x^4 m(x) - 2x^8 m^2(x). \qquad (4)$$

Proof: For the purpose of the proof, a *forbidden configuration* is either a vertex of degree 1 or 2, or a twisted dumbell.

Let $M(x, y)$ be the generating function for all rooted near-triangular maps with no interior forbidden configuration, where x marks the total number of vertices, and y marks the root face degree. Let $M(x, y) = \sum_{j \geq 0} M_j(x) y^j$. Then $M_j(x)$ counts the subclass of maps counted by $M(x, y)$ with root face degree j.

We first derive the following equation:

$$\begin{aligned} M(x,y) &= x + y^2 M(x,y)^2 + y^{-1}(M(x,y) - x - yM_1(x)) \\ &\quad -(xyM(x,y) + x(M(x,y) - x) + yM_1(x)M(x,y)). \end{aligned} \tag{5}$$

The proof of this is standard by considering the effect of deleting the root edge of a near-triangular map. The first term on the right corresponds to the single vertex map; the second term corresponds to the case where deleting the root edge gives two disjoint near-triangular maps; the third term corresponds to the case where deleting the root edge gives one near-triangular map with root face degree at least 2. We need to subtract contributions arising from the cases where a forbidden configuration is being created. The three terms in the last bracket correspond to, respectively, the cases where an interior vertex of degree 1, or 2, or an interior twisted dumbell is being created.

Equation (5) is quadratic in $M(x,y)$ and can be solved using the standard "quadratic method" [4]. We first rewrite it as

$$A(x,y)^2 = B(x,y), \tag{6}$$

where

$$A(x,y) = 2y^3 M(x,y) + 1 - y - xy - xy^2 - y^2 M_1(x) \tag{7}$$

$$\begin{aligned} B(x,y) &= 1 - 2y + y^2 - 2xy + 6xy^3 - 4xy^4 + x^2y^2 + 2x^2y^3 - 3x^2y^4 \\ &\quad + \left(-2y^2 + 2y^3 + 4y^4 + 2xy^3 + 2xy^4\right) M_1(x) + y^4 M_1(x)^2 \end{aligned} \tag{8}$$

Let $y = f(x)$ satisfy $A(x,f) = 0$. It is easy to check that such a power series exists. We then have from (6)

$$B(x,f) = B_y(x,f) = 0, \tag{9}$$

where B_y stands for the partial derivative of B with respect to y. We can use Maple to eliminate f from (9) (using the resultant function) and get

$$\begin{aligned} &256(x - M_1)(M_1 + 4 + 3x)(x^7 + 11x^6 + 2x^6M_1 + 33x^5M_1 - x^5 + 15x^4M_1^2 \\ &+ 70x^4M_1 + 46x^3M_1 + 108x^3M_1^2 + 194x^2M_1^2 - 16x^2M_1 + 24x^2M_1^3 + 92xM_1^2 \\ &+ xM_1 - 32xM_1^3 - 16M_1^4 - 72M_1^3 - M_1^2) = 0 \end{aligned}$$

By the combinatorial definition of $M_1(x)$, we know that $M_1(x)$ is a power series in x starting with x^4. Hence $M_1(x)$ is determined by the last factor in the above equation, and we have

$$\begin{aligned} &-16M_1^4 + (24\,x^2 - 32\,x - 72)M_1^3 + (15x^4 + 108x^3 + 194x^2 + 92x - 1)M_1^2 \\ &+ (2x^6 + 33x^5 + 70x^4 + 46x^3 - 16x^2 + x)M_1 + x^7 + 11x^6 - x^5 = 0. \end{aligned}$$

Setting

$$M_1(x) = x^4 m(x) \tag{10}$$

in the above equation, we obtain equation (3).

By equating the coefficients of y^2 and y^3, respectively, in equation (5), we obtain

$$\begin{aligned} M_2 - x^2 - 2xM_1 - M_1 &= 0, \\ M_3 + x^2 - xM_1 - M_2 - xM_2 - M_1^2 &= 0 \end{aligned}$$

Hence

$$M_2 = x^2 + (2x+1)M_1 \tag{11}$$
$$M_3 = M_1^2 + (2x^2+4x+1)M_1 + x^3 \tag{12}$$

Now we consider triangular maps with exterior forbidden configurations. There are three cases.

Case 1. There is a vertex of degree 1 in the root face. Such maps are counted by $3xM_1(x)$. (See Figure 2, and note that in the figures, the root edge is represented by drawing an arrow away from the root vertex.)

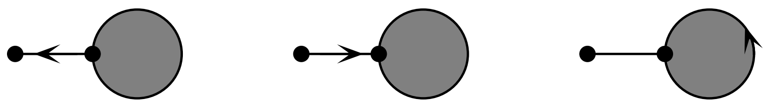

Figure 2: *Vertex of degree 1 in the root face*

Case 2. There is a vertex of degree 2 in the root face. Note that except for K_3, there is exactly one vertex of degree 2 in such maps. Hence (by (11)) such maps are counted by

$$x^3 + 3x(M_2(x) - x^2) = x^3 + 3x(2x+1)M_1(x).$$

(See Figure 3.)

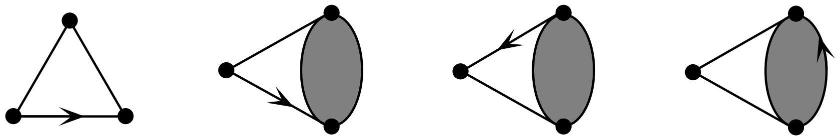

Figure 3: *Vertex of degree 2 in the root face*

Case 3. There is an exterior twisted dumbell. Such maps are counted by $3M_1^2(x)$. (See Figure 4.)

Hence the generating function for all triangular maps with no forbidden configurations is given by

$$M_3(x) - 3xM_1(x) - x^3 - 3x(2x+1)M_1(x) - 3M_1^2(x).$$

Now Theorem 1 follows from Lemma 1 and equations (10) and (12). ∎

Corollary 1 *Let* x_0 *(*≈ 0.0962*) be the smallest positive root of the equation*

$$27x^6 + 216x^5 + 171x^4 - 208x^3 - 339x^2 + 24x + 1 = 0, \tag{13}$$

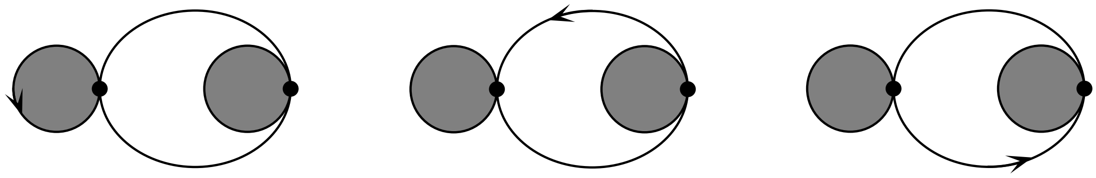

Figure 4: *Exterior twisted dumbell*

$$\begin{aligned}\beta &= (52488/41)(5885227510331778x_0^5 + 6060220775085708x_0^4 - 4968238819400096x_0^3 \\ &\quad -10708104401235624x_0^2 + 746032066360833x_0 + 31271603983952)/ \\ &\quad (4454220494165086093553874 72x_0^5 + 458666341257721035600265101x_0^4 \\ &\quad -376019974869749212390633264x_0^3 - 810440340722629146458439666x_0^2 \\ &\quad +564632599303276914765142 08x_0 + 236678392758470862036 7957) \\ &\approx .3378 \times 10^7\end{aligned}$$

and

$$\begin{aligned} c &= \frac{1348}{41}x_0^5 + \frac{3130}{123}x_0^4 - \frac{13936}{369}x_0^3 - \frac{48751}{1107}x_0^2 + \frac{3536}{1107}x_0 + \frac{146}{1107} \\ &\approx .6607 \times 10^{-4}.\end{aligned}$$

Then

$$[x^n]C(x) \sim (4/3)cx_0^{3/2}(\beta/\pi)^{1/2}n^{-5/2}x_0^{-n}.$$

Proof: By Theorem 1, the singularities of $C(x)$ are the same as, or a subset of, those of $m(x)$. From the combinatorial meaning of $m(x)$ given in (10), it is clear that the radius of convergence of m is at least as large as that of C. Hence the smallest positive singularity of m is the same as that of C. By the implicit function theorem, such a singularity x and the corresponding value $m(x)$ must satisfy (3) and the equation obtained by differentiating both sides of (3) with respect to m. Applying the Maple function "gbasis" to the two resulting polynomials, we find that the singularities of $m(x)$ satisfy

$$(5 + 8x + 4x^2)(27x^6 + 216x^5 + 171x^4 - 208x^3 - 339x^2 + 24x + 1)^3 = 0 \tag{14}$$

and the corresponding values of $m(x)$ are given by a polynomial whose degree is 19. Since $m(x)$ by its combinatorial definition (10) has nonnegative coefficients, its radius of convergence, denoted by x_0, must be a singularity, and $0 < x_0 < 1$. Using Maple we find that equation (13) has a unique root in this range, so this must be x_0, and

$$x_0 \approx 0.09626074081.$$

Using the Maple function "normalf", we can reduce $m(x_0)$ with respect to (13) to

$$\begin{aligned} m(x_0) &= -(1683101/82)x_0^5 - (26830149/164)x_0^4 - (15394870/123)x_0^3 \\ &\quad +(716543995/4428)x_0^2 + (279991921/1107)x_0 - 56939401/2214 \end{aligned} \tag{15}$$

We apply the Maple function "mtaylor" to expand equation (3) at $(x_0, m(x_0))$. It is then routine to verify that the conditions of Lemma 2 are all satisfied (we also make use of the Maple function "normalf" with respect to equation (13).). We find that $m(x)$ has the following asymptotic expansion in powers of $(x-x_0)^{1/2}$:

$$m(x) \quad = \quad m(x_0) + \alpha(x - x_0) + \beta(x - x_0)^{3/2} + \cdots$$

where

$$\alpha = \frac{1}{82}\frac{24813x_0^5 + 202743x_0^4 + 191790x_0^3 - 205354x_0^2 - 313807x_0 + 32279}{1845x_0^5 + 2136x_0^4 - 638x_0^3 - 2784x_0^2 + 189x_0 + 8},$$

and β is as given in Corollary 1. (Warning: obtaining the approximate value of β requires a large number of significant digits, due to the high degree of cancellation.) By Theorem 1 and Lemma 2, we see that

$$[x^n]C(x) \sim \left((1 - 2x_0 - 4x_0^2)x_0^4 - 2x_0^8 2m(x_0)\right) \beta^{1/2} [x^n](x_0 - x)^{3/2}$$

Now Corollary 1 follows from equations (13) and (15). ∎

4 Enumeration of 2-connected cubic maps

Let $C_2(x)$ be the generating function for all 2-connected rooted cubic maps with x marking the number of faces. Let $f = f(x)$ be the power series satisfying

$$\begin{aligned} &16x^2f^3 + (8x^4 + 24x^3 + 72x^2 + 8x)f^2 \\ &+(x^6 + 6x^5 - 5x^4 - 40x^3 + 3x^2 - 14x + 1)f - x^4 - 3x^3 + 13x^2 - x = 0. \end{aligned} \tag{16}$$

Equating coefficients in this equation shows that $f(x) = x + \cdots$ is uniquely determined.

Theorem 2

$$C_2(x) = x^2(f(x) - x)(1 - 2x)/(1 + x).$$

Proof: Let $f(x)$ be the generating function for rooted loopless triangular maps with no interior vertices of degree 2 and let $T(x)$ be the generating function for rooted triangular maps with no loops or multiple edges, where x marks the total number of vertices minus 2. The following parametric expression for $T(x)$ is given in [3].

$$T(x) \quad = \quad s(1 - 2s), \tag{17}$$
$$x \quad = \quad s(1 - s)^3. \tag{18}$$

Note that, except for K_3, every triangulation can have at most one exterior vertex of degree 2. For convenience, we call a triangular map counted by $f(x)$ of *type* f etc. Let $f_2(x)$ be the generating function for triangular maps of type f which have more than three vertices and have root vertex degree 2. It is easy to see from Figure 5 (where the shaded region is a map of type $f - x - f_2$, appropriately rooted) that

$$f_2(x) = x(f(x) - x - f_2(x)),$$

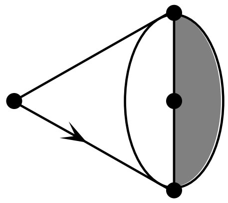

Figure 5: *Shape of f_2-type maps*

and hence

$$f_2(x) = x(f(x) - x)/(1 + x). \tag{19}$$

Now we use the composition approach to derive a functional relation between f and T. This approach is also standard in enumerative map theory (see, e.g. [3]). We describe it briefly here, since it will also be used in the next section. We first note that any T-type triangular map is also an f-type triangular map, but an f-type triangular map may contain multiple edges. Let Δ be an f-type triangular map with more than three vertices. A *digon* in Δ is the region inside (and including) two multiple edges. A digon is *maximal* if it is not contained in another digon. It is clear that each maximal digon (if there is any) in Δ can be uniquely identified, and closing each maximal digon into an edge gives a T-type triangulation. This operation is also reversible: given any T-type triangular map, we can turn any subset of the edges into digons by opening them into faces of valency 2 and inserting rooted triangular maps. The manner of insertion of such a map is to glue the edge opposite the root vertex to one of the inner walls of the face of valency 2. Thus the inserted rooted triangular map must have root vertex degree at least 3. This excludes the f_2-type maps, and also K_3. So each edge of the original map yields a factor $1 + f(x) - x - f_2(x)$, where the term 1 stands for no insertion. Hence we have

$$f(x) = T(x(1 + f(x) - x - f_2(x))^3).$$

It follows from (19) that

$$f(x) = T\left(x(1 + f(x))^3/(1 + x)^3\right).$$

Setting $x(f + 1)^3/(1 + x)^3 = s(1 - s)^3$ and using (18), we obtain

$$\begin{aligned} x(1 + 2s)^3 &= s(1 + x)^3 \\ f(x) &= s(1 - 2s). \end{aligned}$$

Eliminating s from the two equations above, we obtain (16). Using (19) and Lemma 1, we obtain $C_2(x)/x^2 = (f(x) - x) - 3f_2(x) = (f(x) - x)(1 - 2x)/(1 + x)$. ∎

Corollary 2 *Let* $x_0 = \frac{3\sqrt{3}-5}{2} \approx 0.0981$, *and*

$$\beta = \frac{8(520x_0 - 51)}{3(27030x_0 - 2651)} \approx 138.6.$$

Then

$$[x^n]C_2(x) \sim \frac{4(1 - 2x_0)}{3(1 + x_0)} x_0^{7/2} (\beta/\pi)^{1/2} n^{-5/2} x_0^{-n}.$$

Proof: The singularities of $f(x)$ satisfy (16) and the equation obtained by differentiating (16) with respect to f. Applying the Maple function "gbasis" to these two equations, we obtain

$$(x-2)^3(x+1)^3(2x^2+10x-1)^3 = 0. \tag{20}$$

Hence the possible singularities of $f(x)$ are

$$-1, \quad -(3\sqrt{3}+5)/2, \quad (3\sqrt{3}-5)/2, \quad 2.$$

By the combinatorial definition of $f(x)$, the dominant singularity x_0 of $f(x)$ must be positive and less than 1. Hence $x_0 = (3\sqrt{3}-5)/2$. Using (16), we find $f(x_0) = 1/8$. As in previous section, we apply the Maple function "mtaylor" to expand equation (16) at $(x_0, f(x_0))$ and to verify that the conditions of Lemma 2 are all satisfied. We find that the expansion of f in powers of $(x-x_0)^{1/2}$ is

$$f(x) \;=\; 1/8 + \alpha(x-x_0) + \beta(x-x_0)^{3/2} + \cdots$$

where

$$\alpha = \frac{8x_0-1}{4(10x_0-1)},$$

and β is as given in Corollary 2. Since the radii of all other roots of (20) are bigger than x_0, Corollary 2 follows from Theorem 2 and Lemma 2. ∎

5 Counting 3-connected triangle-free cubic maps

Let $C_3(x)$ be the generating function for all 3-connected triangle-free rooted cubic maps with x marking the number of faces. Let $g = g(x)$ be the power series with lowest term x^4 and satisfying

$$\begin{aligned} 0 \;=\; & (x^3-3x^2+3x-1)g^4 + (4x^4-12x^3+9x^2+2x-3)g^3 \\ & +(6x^5-10x^4-15x^3+36x^2-14x-3)g^2 \\ & +(4x^6+4x^5-45x^4+82x^3-59x^2+14x-1)g + x^7+5x^6-8x^5+x^4 \end{aligned} \tag{21}$$

Theorem 3

$$C_3(x) = x^2(1-3x)g(x).$$

Proof: Let $g(x)$ be the generating function for rooted 3-connected triangular maps with no interior vertices of degree 3 (excluding K_3) and let $T_4(x)$ be the generating function for rooted 4-connected triangular maps (excluding K_3 and including K_4), where x marks the total number of vertices minus 2 in both cases. The following parametric expression for $T_4(x)$ is an immediate consequence of results in [3].

$$\begin{aligned} T_4(z) &= z + t(t-1)/(1+t)^2, \\ z &= t(1-t)^2. \end{aligned} \tag{22}$$

Note that any g-type triangular map can be generated by replacing some interior faces of a T_4-type triangulation with a g-type triangular map. Applying this procedure in every possible way with appropriate canonicity conventions on inserting rooted maps produces each rooted

g-type map exactly once, and also K_4. Thus, using the composition technique similar to that used in the previous section, we obtain

$$g(x) + x^2 = T_4\left(x(1 + g(x)/x)^2\right)/(1 + g(x)/x).$$

Setting

$$x(1 + g(x)/x)^2 = t(1-t)^2, \tag{23}$$

and using (22), we obtain

$$g(x) + x^2 = (t(1-t)^2 + t(t-1)/(1+t)^2)/(1 + g(x)/x). \tag{24}$$

Eliminating t from (23) and (24), we obtain (21). To complete the proof, we must consider triangular maps with an exterior vertex of degree 3. Let $\bar{g}(x)$ be the generating function of g-type triangular maps with root vertex degree 3. Since deleting the root vertex of a $\bar{g}$-type triangular map yields a g-type triangular map, we have $\bar{g}(x) = xg(x)$. By Lemma 1, we have

$$C_3(x)/x^2 = g(x) - 3\bar{g}(x) = (1-3x)g(x) \qquad \blacksquare$$

Corollary 3 *Let*

$$\theta = \frac{1}{3}\arctan\left(\frac{27}{217}\sqrt{295}\right), \ x_0 = (2 - \cos\theta - \sqrt{3}\sin\theta)/3,$$

$$\beta = \frac{2048(13 + 217x_0 - 473x_0^2)}{59049(1-x_0)}$$

Then

$$[x^n]C_3(x) \sim (4/3)(1-3x_0)x_0^{7/2}(\beta/\pi)^{1/2}n^{-5/2}x_0^{-n}.$$

Proof: The singularities of $g(x)$ must satisfy equation (21) and the equation obtained by differentiating (21) with respect to g. Applying the Maple function "gbasis" to these two equations, we obtain the equations

$$256x^3 - 512x^2 + 256x - 27 = 0 \tag{25}$$

and

$$\begin{aligned} g(x) \quad &= \quad (1/59049)x(-96957 + 1752084x - 11696128x^2 + 37910528x^3 - 67250176x^4 \\ &\quad +65536000x^5 - 32768000x^6 + 6553600x^7). \end{aligned}$$

Using Maple, we find that the roots of (25) are:

$$2(1+\cos\theta)/3, \quad (2 - \cos\theta - \sqrt{3}\sin\theta)/3, \quad (2 - \cos\theta + \sqrt{3}\sin\theta)/3,$$

where

$$\theta = \frac{1}{3}\arctan\left(\frac{27}{217}\sqrt{295}\right).$$

Hence the smallest positive singularity of $g(x)$ is

$$x_0 = (2 - \cos\theta - \sqrt{3}\sin\theta)/3 \approx 0.1439069937.$$

# faces	# 1-connected	# 2-connected	# 3-connected and Δ-free
4	1	1	0
5	3	3	0
6	19	19	1
7	143	128	3
8	1089	909	12
9	8564	6737	59
10	69075	51683	313
11	569469	407802	1713
12	4783377	3293497	9559
13	40829748	27122967	54189
14	353395155	227095683	311460
15	3096104105	1928656876	1812281
16	27415923905	16582719509	10661303
17	245069538465	144125955717	63336873
18	2209155012387	1264625068163	379601353
19	20064713628389	11190598332502	2293205687
20	183478258249569	99776445196977	13953099573

Table 1: A table for the number of rooted cubic planar maps

Using the Maple function "normalf" with respect to (25), we obtain

$$g(x_0) = (5/27)x_0 - (32/27)x_0^2. \tag{26}$$

As in previous sections, we apply the Maple function "mtaylor" to expand equation (21) at $(x_0, g(x_0))$ and to verify that the conditions of Lemma 2 are all satisfied. We find that expansion of g in powers of $(x - x_0)^{1/2}$ is

$$g(x) = 1/8 + \alpha(x - x_0) + \beta(x - x_0)^{3/2} + \cdots$$

where

$$\alpha = 7(3 - 16x_0)/27$$

and β is as given in Corollary 2. Since the other two roots of (25) are bigger than x_0, x_0 is the unique singularity of $C_3(x)$ in its circle of convergence. Hence Corollary 3 follows from Theorem 3 and Lemma 2. ∎

6 A table

The first few terms in power series $C(x)$, $C_2(x)$ and $C_3(x)$ can be easily obtained using Theorems 1, 2, and 3 and Maple. They are given in Table 1. These numbers up to 19 faces have been verified independently by Brinkmann and McKay [2]. These numbers are not given by hypergeometric expressions, because the radii of convergence of the generating functions (x_0 in each case) are irrational.

References

[1] E.A. Bender, Asymptotic methods in enumeration, *SIAM Rev.* **16** (1974), 485–515.

[2] G. Brinkmann and B.D. McKay, Fast generation of non-isomorphic planar cubic graphs (in preparation).

[3] W.T. Tutte, A census of planar triangulations, *Canad. J. Math.* **14** (1962), 21–38.

[4] W.T. Tutte, The enumerative theory of planar maps, in *A Survey of Combinatorial Theory* (J. N. Srivastava et al. eds.), pp. 437–448, North-Holland, Amsterdam, 1973.