A Network Model to Optimise Cost in Underground Mine Design

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Abstract

This paper examines the problem of designing an underground mine so as to optimise the development and haulage costs. It focusses particularly on the costs associated with the ramps and shafts which provide passage to and from the ore zones. This mine optimisation problem is modelled as a weighted network. The controls (variables for the optimisation process) and operational constraints are described. A discussion is given on conditions under which the cost function is convex.

1 Introduction

In underground mines ore is typically extracted by being transported along a network of gently sloping ramps and horizontal drives. These ramps and drives service the ore bodies and, for deep mines, one or more vertical shafts. Typically the gradient of these ramps is no more than 1:8. Both the construction costs of these tunnels and the associated haulage costs are a significant component of the overall mine costs. Hence designing the ramps and drives so that these costs are minimised can have a major impact on the economic efficiency and viability of the mine, but has received almost no systematic study.

Our approach to this mine design problem is to build a mathematical model that incorporates the most significant features of an actual mining network. Finding an optimal design for this model provides an excellent basis for designing the mining network so that it is as cost efficient as possible. The gradient constrained network theory, underlying our approach, is described in [2] and [3] and is still being developed. Another paper [1] outlines our general approach in applying this theory, and presents some actual case studies. In this paper we describe our current model and some of its mathematical properties.

2 Mine Design Optimisation

Reducing the cost of mining operations is an important issue for mine developers and operators faced with an extremely competitive market place for mineral commodities. Efficient optimisation algorithms exist for the design of open pit mines, and have been successfully implemented as commercial software systems. The problem of optimising underground mines is, however, less well understood.

Ultimately optimisation of an underground mining system must take into account

1. the choice and operational efficiencies of mining methods (stoping, caving, room and pillar),
2. the delineation of economic ore zones (e g, stope configuration and boundaries),
3. the underground development needed to provide access to and haulage from the ore zones (usually based around some combination of ramps, drives and vertical shafts),
4. ventilation and other mine services, and
5. the efficiency of the transport operations moving ore, waste and backfill within this infrastructure framework.

While all factors need to be incorporated into the global optimisation, this paper concentrates mainly on (3), which we refer to as the mine network, incorporating (4) and (5) as costs associated with the design. The aim is to minimise some agreed combination of development, services and operations costs to model life-of-mine or other project costing bases.

Lee [5] first discussed the problems of setting up and optimising a three-dimensional network modelling the infrastructure costs of an underground mine layout, and suggested some simple heuristics. Brazil et al [2] analysed the exact properties of such networks for the case where the network is restricted to lying in a vertical plane. Some properties of three-dimensional gradient constrained Steiner trees are described in [3].

In the following section we describe in detail our approach to the mine network problem. Note that with this problem, the function minimised is not merely the length of the network but its cost, where factors other than length also have an influence on the cost.

3 A Mining Network Model

A key requirement in applying mathematics to problems such as mine design is developing a model that embeds the essential characteristics of the design problem but remains mathematically tractable. We model the mine layout problem as a weighted network. In this representation, the problem of minimizing the cost of developing and operating the mine is equivalent to minimizing the cost of the associated weighted network.

3.1 Description of the Model

The objective is to minimise the total variable cost of accessing and hauling ore for a given mining plan, where the cost is assumed to be some nominated combination of infrastructure (access) and haulage costs.

The basic assumptions and costs are as follows:

- Mining operational costs, as distinct from haulage costs, are assumed to be effectively invariant with respect to the alternative mine network layouts.
- The network model assumes that all draw points at the stopes are given, together with the expected tonnage for each draw point.
- It is assumed that the mine may contain a vertical haulage shaft. The surface location of the shaft is fixed; the depth of the shaft is generally a free variable. There may also be intermediate access points on the shaft whose depths may be treated as free variables. Like ramps, we think of the shaft or the sections of shaft between access points as link-components of the mine network.
- Development costs and haulage costs for the link-components are given. Development costs are usually modelled as a fixed cost plus a cost per meter; haulage costs are specified in \$/tonne·m and may vary with gradient.

Controls (Variables for the Optimisation Process) The topology of the mine network and actual locations of any network junctions are allowed to vary. Hence, the main control variables are:

- The topology of the mine network.
- The location of the junctions between link-components in the mine network.

Operational Constraints The network model may incorporate any of the following key operational constraints which are characteristic of underground mining designs:

- Ramps and drives must be navigable and are constrained to a maximum allowable absolute gradient m where m is generally in the range 1:9 through to 1:7 depending on mining equipment specifications.
- The shaft, if included, must be vertical.
- The minimum spatial separation between ramps, drives is specified by geotechnical and safety parameters.

Solution Elements The solution should describe the values of the control variables which minimise the nominated cost objective:

- The topology of the network of shaft/ramps/drives, locations of inserted junctions between link-components and specification of haulage paths and tonnages in the optimal solution.
- The estimated cost of the optimal solution.
- The sensitivity of the design to variations in design specifications, operational data or cost data.

3.2 A Mathematical Formulation of the Model

In general, as more operational constraints are placed on the model it becomes increasingly difficult to fully optimise with respect to all these constraints. In practice, some constraints can usually be shown to be more significant for a particular design problem than others, and one can optimise with respect to one or two constraints and then modify the best solutions to incorporate the less significant constraints.

In a similar way, the controls are not all equally important. There are two distinct classes of control in this problem: the topology control and the control for the location of junctions. The first of these is a discrete and finite control since there are only a finite number of possible topologies for a given set of draw points. The second control is a continuous control since the junctions can be located in any positions in $\mathbf{R}^3$ that do not violate the operational constraints.

It follows that the key problem to focus on in order to develop a finite algorithm is that of optimising the positions of the junctions for a given topology. This is a local optimization problem. Extending this to finding a global optimum can be achieved by running through all possible topologies, or, where this is not practical, can be achieved heuristically via methods such as simulated annealing.

Here we describe a mathematical formulation of the problem where each link-component is assumed to be a ramp or a section of the shaft. We will assume that the only operational constraint on ramps is the gradient constraint, i.e., that all ramps have a maximum allowable absolute gradient of m, where m is a given positive real number. Furthermore, the surface location of the shaft is assumed to be given but the depth of each shaft access point (where a ramp meets the shaft) can vary. The influence on the model of including other link-components and other operational constraints does not significantly alter the mathematical formulation of the problem and will not be considered here.

We specify, for this case, the cost function to be minimised in order to compute the network design of optimal cost. Our approach is to develop expressions for the per meter costs of the ramps and shaft, and build a model for the problem from these expressions.

The control variables are the locations in $\mathbf{R}^3$ of the junctions between link-components. We are given the locations of draw points that have to be interconnected by the mining network and the surface location for the mine exit. We are also given the topology of the mining network. These can be specified in the model as follows:

- Let $N(1),\ldots,N(n+k)$ represent the nodes, including both the given points (or *terminals*) and the junctions, in the network. There are n terminals $N(1),\ldots,N(n)$, whose locations in $\mathbf{R}^3$ are fixed. There are k variable nodes $N(n+1),\ldots,N(n+k)$, whose locations are to be determined, corresponding to shaft access points and to junctions between ramps. For each node $N(i)$ we denote the x,y, and z-coordinates by x_i, y_i and z_i, respectively.

- There are two components to the topology. Firstly, the topology determines the natural number k, which is the number of variable nodes. The second component is the set of directed edges $\mathbf{E} = \{(i,j)\}$ of the underlying graph of the mining network, where each element of $\mathbf{E}$ represents a directed link from $N(i)$ to $N(j)$, where the direction is away from the draw points, i.e., in the direction of haulage. These two components clearly suffice to determine the graph structure of the network. Let $\mathbf{E} = \mathbf{E}_{\mathrm{R}} \cup \mathbf{E}_{\mathrm{S}}$ where $\mathbf{E}_{\mathrm{R}}$ is the set of ramp-links and $\mathbf{E}_{\mathrm{S}}$ is the set of shaft-links, which represent sections of the vertical shaft.

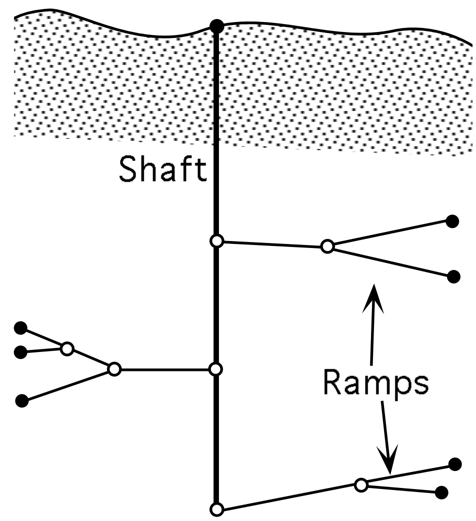

Figure 1: A section view of a simple mine network with 8 terminals (i.e., 7 draw points and the top of the shaft). The network contains 11 ramp-links and 3 shaft-links. In the optimisation, the 4 junctions between ramp-links are free to move anywhere, whereas the 3 junctions on the shaft (the shaft access points) can only be moved up and down, not laterally.

The above is illustrated in Figure 1.

In an optimised mine network we can also assume that the following property holds:

Property **(P)** *Each ramp in the mine network is a straight line if the gradient between its two endpoints is no more than m. Otherwise it is a monotone increasing (or decreasing) piecewise differentiable curve such that the gradient at each differentiable point is m.*

It is clear that there exists a minimum cost mine network satisfying Property **(P)** when optimising with respect to development costs alone. A proof that there exists a minimum cost mine network satisfying Property **(P)** when optimising both development and haulage costs appears in [4].

Our model of the optimization problem can now be constructed as follows. In the mining network we place no restrictions on the locations of the junctions between ramps, but the way of measuring the cost of each ramp-link in $\mathbf{E}_{\mathrm{R}}$ varies according to the gradient between its endpoints. By Property **(P)**, we assume that each ramp has constant gradient between its endpoints. If a given ramp-link has absolute gradient no greater than

m then this is the gradient of the corresponding ramp. If, on the other hand, a ramp-link has absolute gradient greater than m, then the gradient of the corresponding ramp is restricted to m or $-m$ and the length of the link is recalculated accordingly.

This can be represented mathematically as follows. For any $(i,j) \in \mathbf{E}_\mathrm{R}$, let $G_T(i,j)$ be the true gradient of the link between $N(i)$ and $N(j)$. That is,

$$G_T(i,j) = \frac{z_j - z_i}{\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}}.$$

The gradient of this link $G(i,j)$, is defined as

$$G(i,j) = \begin{cases} m & \text{if } G_T(i,j) > m \\ -m & \text{if } G_T(i,j) < -m \\ G_T(i,j) & \text{otherwise.} \end{cases}$$

If the link corresponds to a straight ramp than its length is simply Euclidean length, $L_\mathrm{E}(i,j)$. If, however, $|G_T(i,j)| > m$ then the ramp can be thought of as a zigzag or spiral curve in space satisfying Property **(P)**. Since the gradient is constant (either m or $-m$ in this case) it follows that the length of this curve is a function only of the vertical displacement between $N(i)$ and $N(j)$, and is independent of the shape of the curve. Hence the length of a link, $L(i,j)$, can be defined as

$$L(i,j) = \begin{cases} L_\mathrm{E}(i,j) & \text{if } |G_T(i,j)| \le m \\ |z_j - z_i|\sqrt{1 + m^{-2}} & \text{otherwise.} \end{cases}$$

We now consider how to determine the cost of a ramp-link in terms of these functions. There are two major costs associated with each ramp-link over the life of the mine. The first is the development cost of building the ramp. For each $(i,j) \in \mathbf{E}_\mathrm{R}$, this per meter cost can be modelled as a constant d_{ij}. The values of the d_{ij} may vary for different ramp-links depending on the envisaged usage.

The other principal cost associated with each ramp-link over the life of the mine, is the haulage cost. By Property **(P)**, each ramp has constant gradient between its endpoints. Consequently, the total cost per meter of hauling ore in a given ramp will be proportional to the estimated quantity of ore to be transported along the ramp during the life of the mine. Let t_{ij} represent this quantity measured in tonnes. The constant of proportionality will clearly be a function $f_{ij}(G(i,j))$ of the gradient of the ramp (since, for example, a steeper ramp requires more work and hence more fuel or electricity to haul a given quantity of ore).

In summary, we have the following formulation:

- Let d_{ij} be the development cost per meter of the ramp (i,j);
- Let t_{ij} be the estimated number of tonnes of ore to be transported along the ramp (i,j);
- Let f_{ij} be a function of the gradient $G(i,j)$ representing the haulage cost (per tonne·meter) of moving ore in ramp (i,j) in the direction from $N(i)$ to $N(j)$.

Then the cost per meter of each ramp (i,j) is $d_{ij} + f_{ij}(G(i,j))t_{ij}$.

Next we consider the costs of the shaft-links. The sum of these costs will give the total variable cost associated with the vertical shaft. In this case the haulage cost of transporting a given amount of ore up the shaft can be modelled as a linear function of the vertical distance hauled. Hence, we have the following formulation:

- Let d'_{ij} be the development cost per meter of the shaft-section (i,j);
- Let t_{ij} be the estimated number of tonnes of ore to be transported up the shaft section (i,j);
- Let a_1 and a_2 be constants such that the haulage cost of transporting ore up the shaft is $a_1 + a_2 L_\mathrm{E}(i,j)$ per tonne. As before, $L_\mathrm{E}(i,j)$ represents the Euclidean length of the shaft-section (i,j).

Here, the cost associated with each shaft section is

$$\begin{aligned} &d'_{ij} L_\mathrm{E}(i,j) + (a_1 + a_2 L_\mathrm{E}(i,j))t_{ij} \\ &= a_1 t_{ij} + (d'_{ij} + a_2 t_{ij}) L_\mathrm{E}(i,j). \end{aligned}$$

It follows that for a given topology, $T = (k, \mathbf{E})$, the objective function, which represents the total cost of the mine network, is

$$\begin{aligned} C(T) &= \sum_{(i,j)\in \mathbf{E}_\mathrm{R}} (d_{ij} + f_{ij}(G(i,j))t_{ij}) L(i,j) \\ &+ \sum_{(i,j)\in \mathbf{E}_\mathrm{S}} a_1 t_{ij} + (d'_{ij} + a_2 t_{ij}) L_\mathrm{E}(i,j). \quad (1) \end{aligned}$$

Optimising the development and haulage costs over the life of the mine now corresponds to minimising the above objective function $C(T)$ over all possible topologies T.

4 Optimisation of the Model

The above optimisation problem is closely related to the three-dimensional Steiner network problem. This problem asks us to find a minimum length network in

Euclidean 3-space interconnecting a given set of points (but with no gradient constraint). Like the mine network, the Steiner network may contain extra nodes whose locations are to be determined. The Steiner network problem is known to be NP hard. Furthermore, it has been shown (in, e.g., [6]) that it is not generally possible to find an exact solution to the Steiner network problem in 3-space, even for small values of n. These properties carry over to the mining network problem. Hence it is necessary to use approximation techniques to solve the above optimisation problem.

The authors have developed a novel and highly efficient descent algorithm for solving the mining problem. The key to being able to effectively use such a descent algorithm is to show that the cost function is convex. In this section we will examine the convexity of the objective function (1). In particular, we will show that although the objective function itself it is not always convex, we can obtain convexity by imposing conditions on the functions $f_{ij}(G(i,j))$. In our experience, these conditions hold in all practical applications.

First note that, since a positive linear combination of convex functions is also convex, it suffices to show that the cost function associated with a single edge is convex under the appropriate conditions. For each $(i,j) \in \mathbf{E}_{\mathrm{S}}$, d'_{ij} and t_{ij} are constants, so the cost function for that shaft-link is a linear function of $L_{\mathrm{E}}(i,j)$, and hence is convex. Similarly, for each $(i,j) \in \mathbf{E}_{\mathrm{R}}$, d_{ij} and t_{ij} are constants, so the cost function for that ramp-link is a constant multiple of

$$(1 + af(G))L \qquad (2)$$

where $f = f_{ij}$, $G = G(i,j)$ and $L = L(i,j)$, and where a is a constant. So, it suffices to show that the above expression is convex. Hence, for the remainder of this paper we will only consider ramp-links (i,j).

Without loss of generality we can transform and rescale the link (i,j) so that one endpoint is fixed at the origin and the other moving endpoint (x,y,z) is initially at the point $\mathbf{p} = (1,0,w)$. Furthermore, we can assume that $w \geq 0$ and that the free endpoint moves such that $z \geq 0$. If either of these conditions are not satisfied then a similar argument to the one below will apply. We can think of G and L as being functions of (x,y,z) with $z \geq 0$. To further simplify the analysis, we introduce the variable $r = \sqrt{x^2+y^2}$.

Notice that for any ramp-link (i,j) the configuration space of its pair of endpoints $N(i), N(j)$ is a Cartesian product of two copies of $\mathbf{R}^3$. Keeping one endpoint fixed at the origin corresponds to projecting onto $\mathbf{R}^3$ by mapping $(N(i), N(j))$ to $N(i) - N(j)$. Let $\widetilde{C}(T)$ denote the induced cost function on $\mathbf{R}^3$ under this map. Now it is easy to see that $C(T)$ is convex on $\mathbf{R}^3 \times \mathbf{R}^3$ if and only if $\widetilde{C}(T)$ is convex on $\mathbf{R}^3$. The same method works if we have one endpoint, say $N(j)$, representing a shaft access point. For in this case, $N(j)$ is free to move along a vertical line, i.e., a copy of $\mathbf{R}^1$. So the configuration of pairs of endpoints is $\mathbf{R}^3 \times \mathbf{R}^1$ and the same projection method works, allowing us to keep the end at the shaft fixed. This relies on the fact that the endpoint $N(j)$ is restricted to a linear subspace of $\mathbf{R}^3$.

In order to investigate the convexity of expression (2), and hence of $C(T)$, it is convenient to separately consider the cases where $f = 0$ and f is linear. Convexity conditions for more general functions f will appear in [4].

4.1 Case 1: $f_{ij} = 0$

The first case, where each $f_{ij} = 0$, corresponds to optimising the development costs for a given topology. Here we will show that $C(T)$ is convex by studying the properties of L. We begin with two elementary results about L, the first of which is immediate.

Lemma 4.1 *Let* $L_1 = \sqrt{r^2+z^2}$ *and* $L_2 = z\sqrt{1+m^{-2}}$. *Then* $L = \max\{L_1, L_2\}$.

Lemma 4.2 $L = L(x,y,z)$ *is convex.*

Proof. Since L is the maximum of two functions, it suffices to show that each of the functions are convex. The second function L_2 is simply a constant multiple of z, and hence is convex. So it remains to show that L_1 is convex. Using δ to denote infinitesimals, note that $\delta r = (x\delta x + y\delta y)/r$ and $\delta^2 r = (x\delta y - y\delta x)^2/r^3$ since all terms involving second variation of the coordinates are zero when perturbing along a straight line vector. Next, observe that $\delta L_1 = (r\delta r + z\delta z)/L_1$. After simplifying, we obtain

$$\delta^2 L_1 = \frac{(r\delta z - z\delta r)^2}{L_1^3} + \frac{r\delta^2 r}{L_1} \qquad (3)$$

which is non-negative when $L_1 \neq 0$, since $\delta^2 r$ is non-negative and $L_1 > 0$. This proves convexity, since L_1 is smooth everywhere except at $(0,0)$, at which point it has a local minimum. ∎

The following theorem now follows from Lemma 4.2.

Theorem 4.3 *The cost function* $C(T)$ *is convex if* $f_{ij} = 0$ *for each* $(i,j) \in \mathbf{E}_R$.

4.2 Case 2: f_{ij} linear

In practical mine design problems, the costs associated with haulage can usually be assumed to be a linear function of the gradient (per tonne·meter), with very little loss of accuracy. Suppose $f(G) = b_1 + b_2 G$ where

b_1 and b_2 are non-negative constants. Then expression (2) becomes

$$(1 + a(b_1 + b_2G))L = (1 + ab_1)(1 + AG)L \qquad (4)$$

where $A = ab_2/(1 + ab_1)$ is a non-negative constant.

We will show that although expression (4) is not always convex, it is convex if A lies below a given upper bound, namely m^{-3}.

Lemma 4.4 *Let* $F_1 = (1 + A\frac{z}{r})\sqrt{r^2 + z^2}$ *and* $F_2 = (1 + Am)z\sqrt{1 + m^{-2}}$. *Then, for any given* A *such that* $0 \leq A \leq m^{-3}$, *the function* $(1 + AG)L = \max\{F_1, F_2\}$.

Proof. First consider the case where $z/r < m$. It suffices to show that $F_1(\mathbf{p}) - F_2(\mathbf{p}) \geq 0$ at $\mathbf{p} = (1, 0, w)$ where $0 \leq w \leq m$. In other words, we require that $wF \geq 0$ where $F = ((1 + Aw)\sqrt{1 + w^{-2}} - (1 + Am)\sqrt{1 + m^{-2}})$. Since F is a decreasing function of A, we investigate the behaviour of A at $F = 0$. Solving for $F = 0$, we obtain:

$$A = A(w) = \frac{\sqrt{1 + w^{-2}} - \sqrt{1 + m^{-2}}}{\sqrt{1 + m^2} - \sqrt{1 + w^2}}.$$

The function A above is a decreasing function of w, hence

$$\inf A = \lim_{w \to m} A(w) = 1/m^3$$

by L'Hôpital's Rule. The lemma now follows for $z/r < m$.

The case where $z/r \geq m$ can be proved by a similar argument. ∎

Lemma 4.5 *The function* $GL = GL(x, y, z)$ *is not convex. However, the function* $(1 + AG)L$ *is convex if* $0 \leq A \leq m^{-3}$.

Proof. To prove the first statement, consider the case where $z/r < m$. Then $G = \frac{z}{r}$, $\delta G = \frac{r\delta z - z\delta r}{r^2}$ and

$$\delta^2 G = \frac{2\delta r}{r^3}(z\delta r - r\delta z) - \frac{2\delta^2 r}{r^2}$$

(again deleting terms involving second variation of the coordinates). Combining this with the expressions for the first and second variations of $L_1 = \sqrt{r^2 + z^2}$ in the proof of Lemma 4.2, we obtain

$$\begin{aligned} \delta^2(GL) &= L_1\delta^2 G + 2\delta L_1 \delta G + G\delta^2 L_1 \\ &= \frac{z(r\delta z - z\delta r)^2(2L_1^2 + r^2) - z^2 r L^2 \delta^2 r}{r^3 L_1^3}. \end{aligned}$$

Applying initial conditions at $\mathbf{p} = (1, 0, w)$, gives

$$\delta^2(GL) = \frac{w(\delta z - w\delta x)^2(2w^2 + 3) - w^3(1 + w^2)(\delta y)^2}{(1 + w^2)^{3/2}} \qquad (5)$$

which is negative, for example, when $\delta x = \delta z = 0$. Hence, GL is not convex.

For the second statement, first note that the function F_2 in Lemma 4.4 is clearly convex. So by Lemma 4.4, it suffices to show that F_1 is convex when $z/r < m$. Combining equations (3) and (5), and simplifying, we can write $\delta^2((1 + AG)L)$ as

$$\frac{(\delta z - w\delta x)^2(2Aw^3 + 3Aw + 1) + (1 - Aw^3)(1 + w^2)(\delta y)^2}{(1 + w^2)^{3/2}}.$$

Hence, $\delta^2((1 + AG)L) \geq 0$ when $0 \leq A \leq \inf w^{-3} = m^{-3}$. The result follows. ∎

The following theorem is an immediate corollary of the above lemma.

Theorem 4.6 *The cost function* $C(T)$ *is convex if the cost of each link* $(i, j) \in \mathbf{E}_R$ *is of the form* $B(1 + AG)L$ *where* A *and* B *are non-negative constants and* $A \leq m^{-3}$.

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