

The generalised acyclic edge chromatic number of random regular graphs

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Abstract

The r -acyclic edge chromatic number of a graph is defined to be the minimum number of colours required to produce an edge colouring of the graph such that adjacent edges receive different colours and every cycle C has at least $\min(|C|, r)$ colours. We show that $(r - 2)d$ is asymptotically almost surely (a.a.s.) an upper bound on the r -acyclic edge chromatic number of a random d -regular graph, for all constants $r \geq 4$ and $d \geq 2$.

1 Introduction

An edge colouring of a graph is *proper* if adjacent edges are coloured with different colours. A proper edge colouring of a graph is *acyclic* if each cycle has at least 3

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colours. The *acyclic edge chromatic number* $A'(G)$ of a graph G is the minimum number of colours required for a proper acyclic edge colouring of G . The (vertex) acyclic chromatic number was introduced by Grünbaum [7] in the context of planar graphs, and investigated further for example in [2, 5, 1].

In this paper, we consider the following generalisation of these definitions. For $r \geq 3$ fixed, a proper edge colouring is said to be *r-acyclic* if each cycle C has at least $\min(|C|, r)$ colours. The *r-acyclic edge chromatic number* $A'_r(G)$ of a graph G is the minimum number of colours required for a proper *r-acyclic* edge colouring of G . Hence $A'(G) = A'_3(G)$.

(An alternative definition for an *r-acyclic* edge colouring would be to require that *every* cycle has at least r colours. However, this definition is only of interest in graphs with girth at least r .)

We will be interested in the *r-acyclic* chromatic number of random regular graphs. Throughout, d and r are fixed constants while n , the number of vertices of the random graph, tends to infinity such that n is even if d is odd. All asymptotic statements are with respect to n . We say a property holds *asymptotically almost surely* (a.a.s.) if the probability that it holds tends to 1 as n tends to infinity. (In some parts of the argument we restrict to n even, but this should be clear from the context.)

Clearly at least d colours are required for an acyclic edge colouring of a graph with maximum degree d . It is well-known that for d -regular graphs at least $d + 1$ colours are required (this is a special case of Lemma 1.2 below). Alon, McDiarmid and Reed [1] established a linear upper bound on the acyclic edge chromatic number in terms of maximum degree, namely that

$$A'(G) \leq 64d$$

when G has maximum degree d . (This was later improved to $16d$ in [9].) In fact Alon, Sudakov and Zaks [3] conjectured that

$$A'(G) \leq d + 2$$

when G has maximum degree d . They proved this when G has girth bounded below by $cd \log d$, for some fixed constant c (independent of d). They also proved that a uniformly chosen d -regular graph G on n vertices a.a.s. has

$$A'(G) \leq \begin{cases} d + 1 & \text{if } n \text{ is even,} \\ d + 2 & \text{if } n \text{ is odd.} \end{cases}$$

Nešetřil and Wormald [11] improved this, showing that the acyclic edge-chromatic number of a uniformly chosen d -regular graph is asymptotically almost surely $d + 1$. Their result gives further evidence to support a question raised in [3], namely whether it could be possible that $A'(G) = d + 1$ for all graphs with maximum degree d , with the unique exception of the complete graph on an even number of vertices (which has $A'(G) = d + 2$). To the best of our knowledge nothing was previously known about $A'_r(G)$ when $r \geq 4$.

Nešetřil and Wormald's argument [11] is one of the few that heavily use the known *contiguity* results for various random regular graph spaces. Contiguity, a qualitative asymptotic equivalence between two sequences of probability spaces, is the property that any event which holds a.a.s. in one sequence also holds a.a.s. in the other. One of the probability spaces used in [11] consisted of the superposition of d randomly chosen perfect matchings. (For more information on contiguity, see [14].) In this paper we use this space again for the present problem, though the case $r = 3$ does not follow as a special case of our argument. A major new feature of our work arises from the need to bound the expected number of cycles and paths formed in certain ways from the random matchings. More detail is given below.

The main result of this paper is the following.

Theorem 1.1 *Let $d \geq 2$ and $r \geq 4$ be fixed. The r -acyclic edge chromatic number of a uniformly chosen d -regular graph on n vertices is a.a.s. at most $(r - 2)d$.*

To see why $(r - 2)d$ colours is a natural choice, we give the following heuristic argument. Consider a proper colouring of the edges of a d -regular graph using c colours. Choose any $r - 1$ colours and look at the subgraph consisting of just the edges coloured with these colours. Let v_0, \dots, v_t be a walk in this subgraph. Making various unjustified assumptions including independence, the expected number of ways to extend this walk by adding an edge $\{v_t, v_{t+1}\}$ (with $v_{t+1} \neq v_{t-1}$) is $(r - 2)(d - 1)/c$, since for each of the $d - 1$ neighbours $u \neq v_{t-1}$ of v_t in G , the probability that the edge $\{v_t, u\}$ is in the chosen subgraph is $(r - 2)/c$. Standard branching process arguments suggest that if $c < (r - 2)(d - 1)$ then there would be an enormous number of cycles in the graph with at most $r - 1$ colours. While this does not prohibit an asymptotically almost sure upper bound which is lower than $(r - 2)d$, it suggests that our bound might be close to the truth for large d . Moreover, a similar argument shows that our approach in Section 3 has a problem with less than $(r - 2)d$ colours. We conjecture that rd is asymptotically the right answer, for large r and d .

We will now outline our argument. The result for $d = 2$ is trivial. Hence we may assume that $d \geq 3$. Recall that $[n] = \{1, \dots, n\}$ for all $n \in \mathbb{Z}^+$. Let $\mathcal{G}_{n,d}$ be the uniform probability space on the set of all d -regular graphs on the vertex set $[n]$. A deterministic step in the proof of Theorem 1.1 is given in Theorem 2.1. Then it remains to show that the conditions of Theorem 2.1 hold a.a.s. for $G \in \mathcal{G}_{n,d}$. This is established in Section 3 for the case that n is even. We use the fact that for n even and $d \geq 3$, the models $\mathcal{G}_{n,d}$ and $d\mathcal{G}_{n,1}$ are contiguous (shown by combining results of Robinson and Wormald [12] with Janson [8] or Molloy et al. [10]). Here $d\mathcal{G}_{n,1}$ denotes the graph sum of d independent, uniformly chosen perfect matchings on the vertex set $[n]$. (The graph sum is the model obtained by taking the union of the edge sets and conditioning on no repeated edges.) The necessary alterations for the case of n odd are described in Section 4.

To finish this section we give a simple lower bound on $A'_r(G)$ which is linear in d , for d -regular graphs G with girth at least r .

Lemma 1.2 *For every $r \geq 3$, $d \geq 2$ and every d -regular graph G with girth at least r , we have*

$$(r - 1)d/2 < A'_r(G).$$

Proof. Suppose G has an r -acyclic colouring with c colours. Let M_1, \dots, M_c be the colour classes, which form a partition of $E(G)$. The union of any $r - 1$ of these has at most $n - 1$ edges, by acyclicity and the fact that G has girth at least r . Each colour class is involved in exactly $\binom{c-1}{r-2}$ of these $(r - 1)$ -wise unions. Therefore

$$\binom{c - 1}{r - 2} \sum_{i=1}^c |M_i| \leq (n - 1) \binom{c}{r - 1},$$

which implies that $nd/2 = \sum_{i=1}^c |M_i| \leq (n - 1)c/(r - 1)$. Thus

$$A'_r(G) \geq nd(r - 1)/2(n - 1) > (r - 1)d/2,$$

as claimed. □

Noga Alon made a suggestion which led to an improved lower bound greater than $0.6884rd$ for large r and d . But this is undoubtedly not the best possible, so we do not give further details here.

It follows from Lemma 1.2 that every asymptotically almost sure upper bound on $A'_r(G)$ for $G \in \mathcal{G}_{n,d}$ must be at least $(r - 1)d/2$, since the probability that such G has girth at least r is bounded below by a positive constant (see for example [14]). We also comment that the proof of the lemma is easily modified to show that a.a.s. $A'_r(G) \geq (1 + o(1))(r - 1)d/2$, since the total number of edges in cycles of length at most r is bounded in probability.

One might also ask what upper bound on $A'_r(G)$ is valid for *all* d -regular graphs G . The complete bipartite graph $K_{d,d}$ has $A'_4(K_{d,d}) = d^2$, since every edge must receive a different colour to avoid 3-coloured 4-cycles. This is an example of a graph where $A'_4(G)$ is at least quadratic in d (in contrast with $A'(G)$ which is always linear in d). But the situation can be much worse. In particular, Greenhill and Pikhurko [6] constructed a d -regular graph G with $A'_r(G) \geq c_r d^{\lfloor r/2 \rfloor}$, where c_r depends on r but is constant with respect to d . They did this for all $r \geq 6$, and an infinite strictly increasing sequence of values of d depending on r . Matching upper bounds of the order $O(d^{\lfloor r/2 \rfloor})$ were given in [6], proved using the Lovász Local Lemma. (Again the constant implicit in the $O(\cdot)$ depends on r but not on d .)

2 Deterministic step

In this section we prove a deterministic result which gives conditions under which we can find an r -acyclic colouring of a given graph. Then to complete the proof of Theorem 1.1 it remains to show that the conditions of Theorem 2.1 below hold a.a.s. for uniformly chosen d -regular graphs on n vertices. These details are given in the next two sections

for the cases that n is even and n is odd respectively. In order to describe our line of attack, we need some definitions.

A k -cycle (respectively, k -path) is a cycle (respectively, path) of length k (where the length equals the number of edges). Consider a proper edge colouring of a (multi)graph G . Say that a k -cycle is *undercoloured* if it is coloured with fewer than $\min(k, r)$ colours. (Note that an undercoloured cycle has length at least four, since the colouring is proper.) An undercoloured cycle is *short* if it has at most $\log^5 n$ edges, otherwise it is *long*.

We will say that a path is *undercoloured* if it is coloured with fewer than r colours. Note that the definition of undercoloured is different for paths and cycles. In particular k -paths with $k < r$ are always undercoloured, whereas a k -cycle with $k < r$ is only undercoloured if it contains two edges with the same colour.

If C is an undercoloured k -cycle, then a *block* of C is an interval of $\min(k - 1, r - 1)$ consecutive vertices of C together with the edges of C incident with them. (If $k \leq r$, then the block contains all the edges of C .) We want to show that it is possible to choose a block for each undercoloured cycle, such that after recolouring certain edges in each block no cycle is undercoloured. It turns out that we have to be fairly fussy when choosing a block from a long undercoloured cycle, in order to make sure that we do not create any new undercoloured cycles. To explain further we need some more definitions.

A path $P = (v, w_1, w_2, \dots, w_s)$ from a vertex $v \in C$ is said to be *initially disjoint* from C if $\{v, w_1\} \notin E(C)$. A vertex v of C is *good* (with respect to the given colouring) if all paths $P = (v, w_1, w_2, \dots, w_r)$ of length r in G which are initially disjoint from C have $\{w_1, \dots, w_r\} \cap C = \emptyset$ and are coloured with r distinct colours. A block is called *good* if all its vertices are good. A vertex or block is *bad* if it is not good. Finally, a long undercoloured cycle C is called *bad* if it has a subgraph P which is a path of length at least $r(r - 1)(d - 1)^r \log^2 n$ containing no good block with respect to C . Otherwise C is *good*.

The deterministic result for recolouring blocks is as follows.

Theorem 2.1 *Let $d \geq 3$ and $r \geq 4$ be fixed. Let G be a d -regular graph on n vertices which is properly edge coloured with $(r - 2)d$ colours in such a way that the following conditions hold:*

- (i) *There are at most $\log^2 n$ undercoloured cycles in G .*
- (ii) *No two distinct vertices which lie in short undercoloured cycles in G are joined by an undercoloured path (except in the case that both vertices lie in a short undercoloured cycle C and the path is a subgraph of C). In particular, no two short undercoloured cycles in G intersect in just one vertex.*
- (iii) *G has no long bad undercoloured cycles.*

If n is sufficiently large, then we can find a set S of edges such that recolouring some edges in S gives an r -acyclic $(r - 2)d$ -colouring of G .

Proof. Label the undercoloured cycles C_1, \dots, C_T , where for some t , C_1, \dots, C_t are the short undercoloured cycles and C_{t+1}, \dots, C_T are the long undercoloured cycles. We will

process the undercoloured cycles in the given order, from each cycle choosing a block and adding the block to an initially empty set of blocks S . Later the blocks in S will be recoloured in such a way that forces the resulting edge colouring of G to be proper and r -acyclic. The details are given below.

Initially set $S = \emptyset$. Assume that the cycles C_1, \dots, C_{i-1} have been processed and we are about to process cycle C_i . If C_i is a short (undercoloured) cycle then we choose any block of C_i and add it to S . If C_i is long, subdivide C_i into at least $\log^3 n / (r(r-1)(d-1)^r)$ disjoint intervals of at least $r(r-1)(d-1)^r \log^2 n$ consecutive edges. From these intervals, choose one such that all of its vertices are at distance at least $r+1$ from edges of blocks already in S . (This is always possible since at most $O(T) = O(\log^2 n)$ of these intervals contain vertices which are at distance at most r from edges of blocks already in S , by assumption (i).) By assumption (iii) there is a good block in the chosen interval. Choose one such good block and add it to S . This completes the description of how to form the set of blocks S .

Now suppose that all blocks in S have been recoloured in such a way that no block in S contains two edges of the same colour and the resulting colouring Y is a proper edge colouring of G . (We describe how to perform this recolouring below.) Say that a cycle is Y -undercoloured if it is undercoloured with respect to Y . Clearly, none of the original undercoloured cycles C_1, \dots, C_T is Y -undercoloured since they all contain a block in S . Any other cycle C' which is Y -undercoloured was not undercoloured in the original edge colouring. Hence C' contains some edges of blocks in S which have been recoloured. If the recoloured edges in C' only involve edges from C_1, \dots, C_t , then this contradicts assumption (ii). Therefore C' contains a recoloured edge e from a chosen block B of one of cycles C_{t+1}, \dots, C_T , say C_j . By construction, B is good. That is, all r -paths that start in B and are initially disjoint from C_j are coloured with r colours in the original colouring. Moreover, by our choice of blocks, we have not recoloured any of these r -paths. Thus C' contains all r edges of B but then, since B has r distinct colours in the colouring Y , the cycle C' is not Y -undercoloured. Hence Y is an r -acyclic edge colouring, as required.

It remains to show how to produce the colouring Y . Firstly note that by construction, blocks in S are vertex-disjoint. Hence we can (properly) recolour each block B in S independently, in any order. We now describe how to recolour a block $B \in S$.

Suppose first that the edges of B do not form a cycle. If the two end edges of B have the same colour then choose one such edge e and recolour e with a colour which does not appear on e or any edge which is adjacent to e in G . This rules out $2d-1$ colours, so there is always an available colour. Then repeatedly, while B still has any internal edge with the same colour as another edge of B , choose one such edge e and recolour e with a colour which does not appear either on an edge adjacent to e or on an edge of B . Then at most $2d-4$ colours appear on edges which are adjacent to e but not lying on B , and at most $r-1$ colours appear on the edges of B because B has a repeated colour. Hence there are at least

$$(r-2)d - (2d-4) - (r-1) = (r-4)(d-1) + 1 \geq 1$$

choices of colour. Therefore the recolouring procedure can continue edge by edge until

B has no repeated colours.

Finally suppose that the edges of B form a cycle. We simply proceed as described for internal edges, above, recolouring one edge of B at a time until B has no repeated colours. \square

For $d \geq 3$, Theorem 1.1 will follow from Theorem 2.1 if we can produce an edge colouring with $(r-2)d$ colours such that conditions (i)–(iii) a.a.s. hold. This we will do in the next section for the case n even and $d \geq 3$ and in Section 4 for the case that n is odd.

3 The probabilistic argument

Let M_1, \dots, M_d be d independent uniformly chosen perfect matchings on the vertex set $[n]$, and let G be the multiset union of these perfect matchings. Fix integers a_1, \dots, a_{r-2} such that $a_1 + \dots + a_{r-2} = n/2$ and $|a_i - a_j| \leq 1$ for $1 \leq i, j \leq r-2$. Independently partition each matching M_i uniformly at random into $r-2$ sets, the i th containing a_i edges. Call each of these parts a *partial matching* and colour each partial matching with a distinct colour, giving $(r-2)d$ colours in all. Let this probability space of $(r-2)d$ -edge-coloured d -regular multigraphs on $[n]$ be denoted by \mathcal{G}_n , or \mathcal{G} for short.

To justify the choice of using $r-2$ colours on each perfect matching, consider the union S of edges of $r-1$ colours R coming from *different* perfect matchings. If paths involving just S proliferate, then there will be too many undercoloured cycles and our method will not work. Given an edge at the end of a path in S , the expected number of extensions of the path is $1 + o(1)$, since there are $r-2$ other matchings to look at, and each has a colour in R with probability $1/(r-2) + o(1)$. Splitting each perfect matching into fewer than $r-2$ colours would give expected number of extensions greater than 1, leading to a great number of undercoloured cycles and a breakdown of our approach.

We would like to establish that assumption (i) of Theorem 2.1 holds for $G \in \mathcal{G}$. To this end, we now estimate the number of k -cycles in the union of $r-1$ random partial matchings. Label the partial matchings $U_1, \dots, U_{(r-2)d}$. The falling factorial is denoted by $[x]_a = x(x-1)\dots(x-a+1)$ for all $a, x \in \mathbb{Z}^+$.

Proposition 3.1 *Suppose that $2 \leq k \leq n$. For $G \in \mathcal{G}$, the expected number of undercoloured k -cycles in G is $O(1/k)$.*

Proof. Fix $\{c_1, \dots, c_{r-1}\} \subseteq [(r-2)d]$ and let $U = U_{c_1} \cup \dots \cup U_{c_{r-1}}$ be the union of the $r-1$ chosen colour classes. Let C_k denote the random variable counting the number of k -cycles of U . Since there are $O(1)$ choices for the set of $r-1$ colours, it suffices to show that $\mathbf{E}C_k = O(1/k)$.

By relabelling if necessary, let M_1, \dots, M_u denote the perfect matchings which have nonempty intersection with U , and suppose that U contains \hat{t}_s partial matchings from M_s . Note for later use that

$$\sum_{1 \leq s \leq u} \hat{t}_s = r-1. \tag{1}$$

Then $|M_s \cap U| = t_s = \hat{t}_s n / (2(r-2)) + \varepsilon_s$, where $\varepsilon_s = O(1)$ for $1 \leq s \leq u$. We count the possibilities for U , with a distinguished k -cycle C . There are $[n]_k$ ways to choose a labelled k -cycle C with a given orientation and start vertex. (Later we will divide by $2k$ to remove the reference to the start vertex and orientation.) Let \mathcal{I} be the set of all proper edge colourings of the edges of C using the set of ‘‘colours’’ $[u] = \{1, \dots, u\}$. That is,

$$\mathcal{I} = \{\mathbf{i} = (i_1, \dots, i_k) \in [u]^k \mid i_s \neq i_{s+1} \text{ for } 1 \leq s \leq k-1, i_k \neq i_1\}.$$

Each \mathbf{i} describes how to assign the edges of C to the matchings M_s , where the ℓ th edge of C (from the given start vertex in the given orientation) is assigned to the matching M_{i_ℓ} . This covers all possible ways to assign the edges of C to the matchings M_1, \dots, M_u . Given $\mathbf{i} \in \mathcal{I}$, define $j_s = j_s(\mathbf{i})$ by

$$j_s = |M_s \cap C| = |\{\ell \in [k] \mid i_\ell = s\}| \quad \text{for } 1 \leq s \leq u$$

and let $\mathbf{j} = \mathbf{j}(\mathbf{i}) = (j_1, \dots, j_u)$.

Let $F(s, t)$ be the number of ways to select t independent edges on s vertices, where $s \geq 2t$. Then

$$F(s, t) = \binom{s}{2t} \frac{(2t)!}{t! 2^t} = \frac{s!}{(s-2t)! t! 2^t}. \quad (2)$$

Given \mathbf{i} , the number of ways to complete each edge set $M_s \cap C$ to the matching $M_s \cap U$ is

$$\prod_{s=1}^u F(n - 2j_s, t_s - j_s).$$

(This completes all the colour classes in M_s which are involved in U . There are other factors counting the number of ways to specify the remaining colour classes in each matching. However, this factor will cancel with the corresponding factor in the total number of graphs G . Therefore we can leave this factor out. Similarly we omit the factor counting the number of ways to partition each perfect matching into its constituent partial matchings.) Let $b \geq 0$ be any constant such that $2\sqrt{bn \log n} \leq k$. (We will set the value of b later, depending on k . In particular we will sometimes take $b = 0$.) It is easy to check that $[n]_k \leq n^{-b} \prod_{i=1}^u [n]_{j_i}$, by expanding the logarithm of the falling factorials. We have

$$\begin{aligned} \mathbf{E}C_k &= \frac{[n]_k}{2k} \sum_{\mathbf{i} \in \mathcal{I}} \prod_{s=1}^u \frac{F(n - 2j_s, t_s - j_s)}{F(n, t_s)} \\ &\leq \frac{n^{-b}}{2k} \sum_{\mathbf{i} \in \mathcal{I}} \prod_{s=1}^u \frac{[n]_{j_s} F(n - 2j_s, t_s - j_s)}{F(n, t_s)}. \end{aligned} \quad (3)$$

We now estimate the latter ratio, dropping the subscripts on t and j . From (2) and using Stirling’s formula, since $t_s \leq n/2$ and $t_s \rightarrow \infty$, we have

$$\frac{[n]_j F(n - 2j, t - j)}{F(n, t)} = \frac{(n - 2j)! t! 2^j}{(n - j)! (t - j)!} = O(\phi(t, j)) (2\rho f(x, \rho))^j \quad (4)$$

where $x = j/t$, $\rho = t/n$,

$$f(x, \rho) = (1 - 2x\rho)^{(1-2x\rho)/x\rho} (1 - x\rho)^{-(1-x\rho)/x\rho} (1 - x)^{-(1-x)/x}$$

and $\phi(t, j) = \sqrt{t/(t-j+1)}$. (Note that $\phi(t, j) = O(1)$ provided that j/t is bounded away from 1.) Hence, reinstating subscripts, (3) says

$$\mathbf{E}C_k = O\left(\frac{n^{-b}}{k}\right) \sum_{\mathbf{i} \in \mathcal{I}} \left(\prod_{s=1}^u \phi(t_s, j_s)\right) \left(\prod_{\ell=1}^k 2\rho_{i_\ell} f(x_{i_\ell}, \rho_{i_\ell})\right) \quad (5)$$

where $x_s = j_s/t_s$ and

$$\rho_s = t_s/n = \frac{\hat{t}_s}{2(r-2)} + O(n^{-1}). \quad (6)$$

Now for a given sequence \mathbf{i} we have $\prod_{s=1}^u \phi(t_s, j_s) = O(n^{\psi(\mathbf{i})})$ where

$$\psi(\mathbf{i}) = \begin{cases} 0 & \text{if } j_s \leq t_s/2 \text{ for } 1 \leq s \leq u, \\ u/2 & \text{otherwise.} \end{cases}$$

(Note that if $j_s > t_s/2$ for some s then $k \geq j_s > t_s/2 \geq n/4(r-2) + O(1) = \Theta(n)$.)

Define the $u \times u$ matrix A_j with (s', s) entry given by

$$a_{s's} = \begin{cases} a_s := 2\rho_s f(x_s, \rho_s) & \text{if } s' \neq s, \\ 0 & \text{if } s' = s. \end{cases}$$

Then (5) can be written as

$$\mathbf{E}C_k = O\left(\frac{n^{-b}}{k}\right) \sum_{\mathbf{i} \in [u]^k} n^{\psi(\mathbf{i})} \prod_{\ell=1}^k a_{i_{\ell-1}i_\ell}. \quad (7)$$

(The subtraction in the indices should be read cyclically, so $i_0 = i_k$.) If $\psi(\mathbf{i}) > 0$ for some $\mathbf{i} \in [u]^k$ then $k = \Theta(n)$ and we can set $b = u/2$, and otherwise set $b = 0$ giving

$$\mathbf{E}C_k = O(1/k) \sum_{\mathbf{i} \in [u]^k} \prod_{\ell=1}^k a_{i_{\ell-1}i_\ell}$$

in both cases.

The proof is completed in Lemmas 3.2, 3.3, 3.4 where we prove that

$$\sum_{\mathbf{i} \in [u]^k} \prod_{\ell=1}^k a_{i_{\ell-1}i_\ell} = O(1). \quad (8)$$

Lemma 3.2 *If $k = O(\sqrt{n})$ then $\sum_{\mathbf{i} \in [u]^k} \prod_{\ell=1}^k a_{i_{\ell-1}i_\ell} = O(1)$.*

Proof. For $k = o(n)$ we have $x_s = o(1)$ for all s , and expanding the logarithm of f about x_s gives

$$\log f(x_s, \rho_s) = \left(\frac{3\rho_s}{2} - \frac{1}{2} \right) x_s + O(x_s^2) \quad (9)$$

for all s . In particular, suppose that $k = O(\sqrt{n})$. Then $x_s = O(n^{-1/2})$ and $x_s k = O(1)$, which implies that $j_s \log f(x_s, \rho_s) = O(1)$ for all s . Let Δ be the $u \times u$ matrix with (s', s) th entry given by

$$\alpha_{s's} = \begin{cases} 2\rho_s & \text{if } s' \neq s, \\ 0 & \text{if } s' = s. \end{cases}$$

Then since Δ is nonnegative,

$$\begin{aligned} \sum_{\mathbf{i} \in [u]^k} \prod_{\ell=1}^k a_{i_{\ell-1}i_\ell} &= \sum_{\mathbf{i} \in [u]^k} \left(\prod_{s=1}^u f(x_s, \rho_s)^{j_s} \right) \left(\prod_{\ell=1}^k \alpha_{i_{\ell-1}i_\ell} \right) \\ &= O(1) \sum_{\mathbf{i} \in [u]^k} \prod_{\ell=1}^k \alpha_{i_{\ell-1}i_\ell} \\ &= O(1) \text{trace}(\Delta^k). \end{aligned}$$

But the trace of Δ^k is $O(|\lambda_1(\Delta)|^k)$, where λ_1 denotes the eigenvalue with maximum absolute value. Since Δ is nonnegative, $|\lambda_1(\Delta)|$ is bounded above by the maximum row sum of Δ . The s th row sum of Δ is

$$2 \sum_{\ell \neq s} \rho_\ell = \sum_{\ell \neq s} \frac{\hat{t}_\ell}{r-2} + O(n^{-1}) \leq 1 + O(n^{-1})$$

since $\hat{t}_s \geq 1$. Hence

$$\text{trace}(\Delta^k) = O(1)(1 + O(n^{-1}))^k = O(1),$$

as required. \square

So now we can assume that $k \geq \sqrt{n}$. Let

$$\mathcal{S} = \{\mathbf{i} \in [u]^k \mid f(x_s, \rho_s) \leq 1 \text{ for } 1 \leq s \leq u\},$$

and for $1 \leq s \leq u$ let

$$\mathcal{L}_s = \{\mathbf{i} \in [u]^k \mid f(x_s, \rho_s) > 1\}.$$

Lemma 3.3 *We have $\sum_{\mathbf{i} \in \mathcal{S}} \prod_{\ell=1}^k a_{i_{\ell-1}i_\ell} = O(1)$.*

Proof. If $\mathbf{i} \in \mathcal{S}$ then $f(x_s, \rho_s) \leq 1$ for all s . Therefore, elementwise $A_{\mathbf{j}} \leq \Delta$ where Δ is the $u \times u$ matrix defined in Lemma 3.2. Hence since Δ is nonnegative,

$$\sum_{\mathbf{i} \in \mathcal{S}} \prod_{\ell=1}^k a_{i_{\ell-1}i_\ell} \leq \sum_{\mathbf{i} \in \mathcal{S}} \prod_{\ell=1}^k \alpha_{i_{\ell-1}i_\ell} \leq \sum_{\mathbf{i} \in [u]^k} \prod_{\ell=1}^k \alpha_{i_{\ell-1}i_\ell} = \text{trace}(\Delta^k) = O(1)$$

as shown in the proof of Lemma 3.2. \square

Now it suffices to prove the following result, since combining Lemmas 3.3 and 3.4 establishes (8) for $k > \sqrt{n}$, as $[u]^k = \mathcal{S} \cup \mathcal{L}_1 \cup \dots \cup \mathcal{L}_u$.

Lemma 3.4 *Suppose that $k > \sqrt{n}$. For $1 \leq s \leq u$ we have $\sum_{\mathbf{i} \in \mathcal{L}_s} \prod_{\ell=1}^k a_{i_{\ell-1}i_\ell} = O(1)$.*

Proof. Fix $s \in [u]$. For any sequence $\mathbf{j} = (j_1, \dots, j_u)$ of nonnegative integers with $j_1 + \dots + j_u = k$, let $\mathcal{L}_{s,\mathbf{j}}$ be defined by

$$\mathcal{L}_{s,\mathbf{j}} = \{\mathbf{i} \in \mathcal{L}_s \mid \mathbf{j}(\mathbf{i}) = \mathbf{j}\}.$$

We will prove that there exists some $\varepsilon > 0$ such that

$$|\lambda_1(A_{\mathbf{j}})| \leq 1 - \varepsilon \tag{10}$$

for all sequences \mathbf{j} with $\mathcal{L}_{s,\mathbf{j}} \neq \emptyset$. There are at most $O(n^u)$ such sequences \mathbf{j} . Hence, since $A_{\mathbf{j}}$ is nonnegative,

$$\sum_{\mathbf{i} \in \mathcal{L}_s} \prod_{\ell=1}^k a_{i_{\ell-1}i_\ell} = \sum_{\mathbf{j}} \sum_{\mathbf{i} \in \mathcal{L}_{s,\mathbf{j}}} \prod_{\ell=1}^k a_{i_{\ell-1}i_\ell} \leq \sum_{\mathbf{j}} \text{trace}(A_{\mathbf{j}}^k) = O(n^u)(1 - \varepsilon)^k = O(1).$$

It remains to establish (10).

Fix a sequence \mathbf{j} with $\mathcal{L}_{s,\mathbf{j}} \neq \emptyset$. Note that

$$|\lambda_1(A_{\mathbf{j}})| = \sqrt{|\lambda_1(A_{\mathbf{j}}^2)|}. \tag{11}$$

(This can be proved by considering the Jordan forms.) We will bound $|\lambda_1(A_{\mathbf{j}}^2)|$ by the maximum row sum of $A_{\mathbf{j}}^2$. Now $A_{\mathbf{j}} = (J - I)D$ where J is the $u \times u$ matrix of 1's, I is the identity matrix and D is the diagonal matrix with main diagonal a_1, \dots, a_u . Thus

$$A_{\mathbf{j}}^2 = (J - I)D(J - I)D = JDJD - JD^2 - DJD + D^2.$$

Let

$$E_w = \sum_{i=1}^u a_i^w$$

for $w \geq 1$, and let σ_ℓ be the ℓ th row sum. Since $JDJ = E_1J$ we have

$$\begin{aligned} \sigma_\ell(A_{\mathbf{j}}^2) &= \sigma_\ell(JDJD) - \sigma_\ell(JD^2) - \sigma_\ell(DJD) + \sigma_\ell(D^2) \\ &= E_1^2 - E_2 - a_\ell E_1 + a_\ell^2. \end{aligned} \tag{12}$$

We find the following technical result useful, postponing its proof.

Claim 1 *There exists $\delta > 1/3$ such that $f(x, \rho) \leq 1$ for all $\rho \leq \delta$.*

By assumption $a_s > 2\rho_s$, so by Claim 1 we have $\rho_s > \delta$. Suppose that also $\rho_\ell > \delta$ for some $\ell \neq s$. Then $\sum_\ell \rho_\ell > 2\delta$, whereas $\sum_\ell \rho_\ell = (r-1)/2(r-2) + O(n^{-1})$ by (1) and (6). This gives a contradiction unless $r = 4$. In the case $r = 4$, the contradiction comes anyway because then each t_i is within $O(1)$ of a multiple of $n/4$, and so $\rho_s > \delta$ implies $\rho_s = 1/2 + O(1/n)$, which again is too large. Hence $a_\ell \leq 2\rho_\ell \leq 2\delta$ for all $\ell \neq s$. (Note that this also says that the sets $\mathcal{S}, \mathcal{L}_1, \dots, \mathcal{L}_u$ are pairwise disjoint, but we do not need this for our proof.) In particular, $a_\ell \leq a_s$ for all $\ell \neq s$. Since the derivative of (12) with respect to a_ℓ is $E_1 - a_\ell > 0$ we may now assume that $a_\ell = 2\rho_\ell = \hat{t}_\ell/(r-2) + O(n^{-1})$ for all $\ell \neq s$. This also shows that the s th row sum is not maximal.

Fixing the value of \hat{t}_s and a_s , it can be seen that E_2 is minimised by maximising u , with $\hat{t}_\ell = 1$ for all $\ell \neq s$. Fixing E_1 , the terms $-a_\ell E_1 + a_\ell^2$ in (12) are also maximised up to $O(n^{-1})$ terms by taking $\hat{t}_\ell = 1$, since $1/(r-2) + O(n^{-1})$ is now the minimum possible value of a_ℓ or $E_1 - a_\ell$. Thus, (12) (with $\ell \neq s$) is maximised by taking $\hat{t}_\ell = 1$ for all $\ell \neq s$, which means that $u = r - \hat{t}_s$. We now have $E_w = a_s^w + (u-1)(r-2)^{-w} + O(n^{-1})$. Setting $a_0 = a_s - \hat{t}_s/(r-2)$, (1) gives $E_1 = a_0 + (r-1)/(r-2) + O(n^{-1})$, and applying (12) with $\ell \neq s$ gives

$$\sigma_\ell(A_j^2) \leq 1 + O(n^{-1}) + \frac{a_0(2r-3-2\hat{t}_s)}{r-2} - \frac{\hat{t}_s(\hat{t}_s-1)}{(r-2)^2}.$$

This will be at most $1 - \varepsilon$, implying (10) by (11), provided

$$a_0 < \frac{\hat{t}_s(\hat{t}_s-1)}{(r-2)(2r-3-2\hat{t}_s)} - \varepsilon_1$$

for some $\varepsilon_1 > 0$. Thus we are done if

$$a_s < \frac{\hat{t}_s}{r-2} \left(1 + \frac{\hat{t}_s-1}{2r-3-2\hat{t}_s} \right) - \varepsilon_1$$

or, for some $\varepsilon_2 > 0$,

$$f(x_s, \rho_s) < 1 + \frac{2(r-2)\rho_s - 1}{2r-3-4(r-2)\rho_s} - \varepsilon_2. \quad (13)$$

We now assume $r \geq 5$, treating $r = 4$ separately later. The bound in (13) is increasing in ρ_s , which is greater than $\delta > 1/3$, so we are immediately done if

$$f(x_s, \rho_s) < 1 + \frac{2(r-2)/3 - 1}{2r-3-4(r-2)/3}.$$

But this expression is increasing in r , so it suffices to prove that

$$f(x_s, \rho_s) < 4/3$$

since $r \geq 5$. Later we prove the following.

Claim 2 For all $\rho \leq 2/5$ and $0 \leq x \leq 1$, $f(x, \rho) < 4/3$.

Thus we may assume $\rho_s > 2/5$, and we are done by (13) if $f(x_s, \rho_s) < 18/11$. Note that $M_{i_s} \cap E(C)$ is a matching, so

$$j_s \leq |E(C) \setminus M_{i_s}| \leq \frac{n(r-1-\hat{t}_s)}{2(r-2)} + O(1)$$

and hence

$$x_s \leq \frac{r-1}{2\rho_s(r-2)} - 1 + O(1/n). \quad (14)$$

Since $\rho_s > 2/5$ and $r \geq 5$, this implies that $x_s \leq 2/3 + O(1/n)$, since $\rho_s > 2/5$ and $r \geq 5$. The following claim (proved below) completes the proof of Case 1 when $r \geq 5$.

Claim 3 For all $\rho \leq 1/2$ and $0 \leq x \leq 3/4$, $f(x, \rho) < 3/2$.

Now suppose that $r = 4$. Here the only possibility is $u = 2$, $\rho_s = 1/2$ (since one full matching must be used) and $\rho_{3-s} = 1/4 + O(n^{-1})$. The largest row sum of A_j is $a_s = f(x_s, 1/2) < 3/2$, by (14) and Claim 3, while $a_{3-s} \leq 2\rho_{3-s} = 1/2$. The largest eigenvalue of A_j is $\sqrt{a_1 a_2} = \sqrt{a_s/2} + O(n^{-1})$, and (10) follows.

To complete the proof of Lemma 3.4 we must prove our claims. First note that $\partial \ln f(x, \rho) / \partial \rho = (\ln(1-z) - \ln(1-2z) - z) / x\rho^2$ where $z = x\rho$. By differentiation, it follows that

$$\frac{\partial \ln f(x, \rho)}{\partial \rho} \geq 0 \quad (15)$$

for $x\rho < 1/2$, which applies in the whole region of interest.

Proof of Claim 1. We first consider $\rho = 1/3$. Note that

$$\frac{d \ln f(x, 1/3)}{dx} = \frac{\ln((3-x)^3(1-x)/(3-2x)^3)}{x^2}$$

and the numerator, having strictly negative derivative for $0 \leq x \leq 1$, is strictly negative for $0 < x \leq 1$. Since $f(0, 1/3) = 1$ (by continuity), the claim follows with $\rho = 1/3$. Careful examination of the argument (using continuity of the relevant derivatives, as functions of ρ) shows that it applies for some $\rho = \delta > 1/3$ as well. It then follows for $\rho < \delta$ by (15).

Proof of Claim 2. By (15) we only need to consider $\rho = 2/5$. Let $f_0(x) = \frac{d}{dx} \ln f(x, 2/5)$ and $f_1(x) = \exp(2x^2 f_0(x)) = (5-2x)^5(1-x)^2/(5-4x)^5$. Then $f_1(x)$ has derivative $(5-8x)g(x)$ where $g(x) > 0$ on $(0, 1)$, $f_1(0) = 1$ and $f_1(1) = 0$. Hence, for some $x_0 \in (5/8, 1)$, $f_1 > 1$ on $(0, x_0)$ and $f_1 < 1$ on $(x_0, 1)$. So f_0 is positive on $(0, x_0)$ and negative on $(x_0, 1)$. We compute $f_0(3/4) > 0$ and $f_0(4/5) < 0$, so the maximum value of $\ln f(x, 2/5)$ occurs for x in $((3/4), (4/5))$. Differentiation shows that $h(z) = (1/z - 1) \ln(1-z)$ is increasing on $(0, 1)$ and thus the maximum of $f(x, 2/5) = \exp(2h(4x/5)) - h(2x/5) - h(x)$ is at most $\exp(2h(16/25)) - h(3/10) - h(3/4) \approx 1.156 < 4/3$ as required.

Proof of Claim 3. We have

$$\frac{d \ln f(x, 1/2)}{dx} = -\frac{1}{x^2} \ln \left(\frac{1-x}{(1-x/2)^2} \right) > 0$$

and so (recalling (15)) we only need to check that $f(3/4, 1/2) \approx 1.378 < 3/2$.

This completes the proof of Lemma 3.4. \square

Having proved Proposition 3.1, we could try to obtain an r -acyclic edge colouring using just one extra colour (as in [11]). The approach would be as follows: randomly choose an edge in each of the undercoloured cycles and recolour it with the new colour. Then use the Lovász Local Lemma and properties of random regular graphs to deduce that the resulting colouring is proper. But cycles which had fewer than $r-1$ colours originally would still be undercoloured even after adding the new colour. Another problem occurs if the edge chosen for recolouring in a particular undercoloured cycle is coloured with a colour that does not appear anywhere else on the cycle. We do not pursue this approach. Instead we will show how to obtain an r -acyclic colouring without introducing any new colours, by investigating the structure of the undercoloured cycles and paths more closely and then applying Theorem 2.1 for the recolouring.

Corollary 3.5 *A.a.s. there are at most $\log^2 n$ undercoloured cycles in $G \in \mathcal{G}$.*

Proof. Proposition 3.1 proves that the expected number of short undercoloured cycles is $O\left(\sum_{k=2}^n \frac{1}{k}\right) = O(\log n)$. The result follows by Markov's inequality. \square

Thus assumption (i) of Theorem 2.1 holds a.a.s. for $G \in \mathcal{G}$. Now we turn our attention to showing that assumption (ii) of Theorem 2.1 holds a.a.s. for $G \in \mathcal{G}$. We say that an undercoloured path is *extensive* if it has more than $\sqrt{n} \log n$ edges. Otherwise it is *unextensive*.

Lemma 3.6 *The expected number of undercoloured k -paths in $G \in \mathcal{G}$ is $O(n)$ if $1 \leq k \leq \sqrt{n} \log n$, and moreover it is $O(n^{-1})$ if $\sqrt{8n \log n} \leq k \leq \sqrt{n} \log n$.*

Proof. Let $\{c_1, \dots, c_{r-1}\} \subseteq [(r-2)d]$. Form the union $U = U_{c_1} \cup \dots \cup U_{c_{r-1}}$ of the $r-1$ chosen colour classes. Let P_k be the number of k -paths in U . Since there are $O(1)$ ways to choose the $r-1$ colours, it suffices to show that $\mathbf{E}P_k$ is bounded by $O(n)$, and that this bound can be reduced to $O(n^{-1})$ if $k \geq \sqrt{8n \log n}$.

The calculations are almost identical to the calculations for $\mathbf{E}C_k$ from Proposition 3.1, except in two respects. Firstly, there are $O([n]_{k+1})$ ways to choose the vertices of the k -path, a factor $O(nk)$ larger than for a k -cycle. Secondly, the set \mathcal{I} of colourings for the edges of the k -path using the colours $[u]$ contains some k -tuples where the first and last entry are the same (and thus cannot be used to properly colour a k -cycle). However, each k -tuple in \mathcal{I} can be extended in $O(1)$ ways to give elements of $[u]^{k+1}$ which can be used to properly colour a $(k+1)$ -cycle (unless $u = 2$ and k is odd in which case we can extend each k -tuple in $O(1)$ ways to give elements of $[u]^{k+2}$ which properly

colour a $(k+2)$ -cycle). Since $2\rho f(x, \rho) = e^{o(1)}$ is bounded, (by (9), say), the calculations of Proposition 3.1 show that

$$\sum_{\mathbf{i} \in [u]^k} \prod_{\ell=1}^k a_{i_{\ell-1}i_{\ell}} = O(1).$$

Hence $\mathbf{E}P_k = O(nk) \mathbf{E}C_k$. Substitute $b = 0$, $b = 2$ into (7) to complete the proof. \square

Using Markov's inequality we obtain the following results.

Corollary 3.7 *A.a.s. there are no extensive undercoloured paths in $G \in \mathcal{G}$.*

Proof. If $k > \sqrt{n} \log n$ then any undercoloured k -path must contain an undercoloured subpath of length $\lfloor \sqrt{n} \log n \rfloor$. By Lemma 3.6 the expected number of undercoloured paths of this length is $O(n^{-1})$. Hence there are a.a.s. no undercoloured paths of length $\lfloor \sqrt{n} \log n \rfloor$, which implies that there are a.a.s. no extensive undercoloured paths. \square

Corollary 3.8 *A.a.s. there are no undercoloured cycles of more than $\sqrt{n} \log n$ edges.*

Lemma 3.9 *A.a.s. no two distinct vertices of a short undercoloured cycle C are joined by an unextensive undercoloured path which is not contained in C .*

Proof. Let k, t be integers such that $2 \leq k \leq \log^5 n$ and $1 \leq t \leq \sqrt{n} \log n$. For $G \in \mathcal{G}$ let $X_{k,t}$ be the number of ordered pairs (C, P) where C is an undercoloured k -cycle in G and P is an undercoloured t -path in G whose endvertices are distinct vertices of C , with the internal vertices of P disjoint from C . Compare $X_{k,t}$ with the number of such pairs (C, P) in which the endvertices of P are not required to lie on C (that is, they may be arbitrary). The calculations are almost identical, except that in the first case (for $X_{k,t}$) there are $O(k^2)$ ways to choose the two endvertices of the path, but in the second case there are $O(n^2)$. Therefore, from Proposition 3.1 and Lemma 3.6,

$$\mathbf{E}X_{k,t} = \mathbf{E}C_k \cdot \mathbf{E}P_t \cdot O(k^2/n^2) = O(k/n).$$

So the expected number of short undercoloured cycles with two vertices joined by an unextensive undercoloured path is

$$O(1) \sqrt{n} \log n \sum_{k=2}^{\log^5 n} \frac{k}{n} = O\left(\frac{\log^{11} n}{\sqrt{n}}\right) = o(1),$$

Hence there are a.a.s. none of these structures. \square

Lemma 3.10 *A.a.s. there are no unextensive undercoloured paths between two vertices in disjoint short undercoloured cycles.*

Proof. Fix integers k, k', t such that $2 \leq k, k' \leq \log^5 n$ and $1 \leq t \leq \sqrt{n} \log n$. Let $X(k, k', t)$ be the number of triples (C, C', P) where C is an undercoloured k -cycle in G , C' is a disjoint undercoloured k' -cycle in G , and P is an undercoloured t -path which has one endvertex in C , the other endvertex in C' and no other vertex in $C \cup C'$. Arguing as in Lemma 3.9 gives

$$\mathbf{E}X(k, k', t) = \mathbf{E}C_k \cdot \mathbf{E}C_{k'} \cdot \mathbf{E}P_t \cdot O(kk'/n^2) = O(1/n).$$

It follows that the expected number of pairs of disjoint short undercoloured cycles joined by unextensive undercoloured paths is

$$O\left(\frac{\log^{11} n}{\sqrt{n}}\right) = o(1).$$

Therefore there are a.a.s. no such structures. \square

Lemma 3.11 *A.a.s. no two short undercoloured cycles intersect in a single vertex.*

Proof. Fix integers k, k' such that $2 \leq k, k' \leq \log^5 n$. Let $X(k, k')$ be the number of ordered pairs (C, C') where C is an undercoloured k -cycle in G and C' is an undercoloured k' -cycle in G which intersects C in exactly one vertex. Arguing as in Lemma 3.9 gives

$$\mathbf{E}X(k, k') = \mathbf{E}C_k \cdot \mathbf{E}C_{k'} \cdot O(k/n) = O(1/n).$$

Therefore the expected number of ordered pairs of undercoloured cycles which intersect in a single vertex is

$$O\left(\frac{\log^{10} n}{n}\right) = o(1).$$

Hence there are a.a.s. no such structures. \square

Corollary 3.12 *A.a.s. $G \in \mathcal{G}$ satisfies assumption (ii) of Theorem 2.1.*

Proof. This follows by combining Lemmas 3.7, 3.9, 3.10 and 3.11. \square

Lemma 3.13 *A.a.s. $G \in \mathcal{G}$ satisfies assumption (iii) of Theorem 2.1.*

Proof. Let C be a long undercoloured k -cycle. By definition and using Corollary 3.8 we may assume that $\log^5 n < k \leq \sqrt{n} \log n$. Fix the vertices, edges and edge colours of C . Also fix a start vertex and a direction on C , and let the vertices of C be labelled (v_1, v_2, \dots, v_k) from the chosen start vertex v_1 , in the chosen direction. While the cycle is equipped with direction, we will write blocks as ordered $(r-1)$ -tuples consisting of the $r-1$ vertices of the block in the transversal order. Given a vertex v of C , let F_v be the subgraph induced by the vertices in $(V(G) \setminus V(C)) \cup \{v\}$ which can be reached by

a path P of length at most r from v , where $V(P) \cap V(C) = \{v\}$. We will call F_v the r -neighbourhood of v . Let $m = (d-1)^r - 1$. Observe that F_v has at most

$$1 + (d-2) + (d-2)(d-1) + (d-2)(d-1)^2 + \cdots + (d-2)(d-1)^{r-1} = m + 1$$

vertices. Suppose that F_v has exactly $m + 1$ vertices. Then F_v is a tree and every path $P = (v, w_1, \dots, w_r)$ in G which is initially disjoint from C has $\{w_1, \dots, w_r\} \cap C = \emptyset$.

We make the following inductive definition of *untouched* blocks (with respect to the given start vertex and direction). The block $B_1 = (v_1, \dots, v_{r-1})$ is untouched. Now suppose that the first ℓ untouched blocks B_1, B_2, \dots, B_ℓ have been identified, for $1 \leq \ell$. Suppose that $B_\ell = (v_j, v_{j+1}, \dots, v_{j+r-2})$. Then the next untouched block is $B_{\ell+1} = (v_t, v_{t+1}, \dots, v_{t+r-2})$, where t is the *minimum* over all $s \geq j + r - 1$ such that there is no path of length $2, \dots, r$ which is internally disjoint with C and which connects any vertex of $B_1 \cup \dots \cup B_\ell$ to any of $v_t, v_{t+2}, \dots, v_{t+r-2}$.

Once the first ℓ untouched blocks have been identified, they eliminate at most $r(r-1)m\ell$ choices for the starting position of the next untouched block. To see this, note that there are at most $(r-1)m\ell$ vertices in $C \setminus (B_1 \cup \dots \cup B_\ell)$ which belong to F_w for some $w \in B_1 \cup \dots \cup B_\ell$. Each of these vertices rules out $r-1$ possible starting positions, giving $(r-1)^2 m\ell$ positions, as well as the $(r-1)\ell$ positions occupied by the vertices of $B_1 \cup \dots \cup B_\ell$. So there are at least ℓ untouched blocks in any subgraph of C which consists of a path of $r(r-1)m\ell$ edges. In particular, we can certainly define the first $\log^2 n$ untouched blocks in any long undercoloured cycle, for a given start vertex and direction.

Let $p_\ell(C)$ be the probability that the first ℓ untouched blocks around C (from some given start vertex in a given direction) are bad, for $\ell \geq 0$. Note that $p_0(C) = 1$. We will prove that

$$p_{\ell+1}(C) \leq (1-q)p_\ell(C) \tag{16}$$

for $0 \leq \ell < \log^2 n$, where $q = (2(r-1))^{-(r-1)m} = O(1)$. For fixed ℓ with $0 \leq \ell < \log^2 n$, suppose that the first ℓ untouched blocks are all bad. Let $F_B = \cup_{v \in V(B)} F_v$ for all blocks B . Condition on the vertices, edges and edge colourings in F_{B_i} for $1 \leq i \leq \ell$. The total number of vertices fixed (in these r -neighbourhoods and in C itself) is y , where $y \leq k + (r-1)m\ell = O(\sqrt{n} \log n)$. Let $B_{\ell+1}$ be the next untouched block in C . We prove that the probability that $B_{\ell+1}$ is good, conditioned on the vertices, edges and edge colourings in $C \cup F_{B_1} \cup \dots \cup F_{B_\ell}$, is at least q .

Instead of working with partial matchings, we first argue about perfect matchings and then $(r-2)$ -colour each perfect matching later. For ease of notation, write B for $B_{\ell+1}$ and F for $F_{B_{\ell+1}}$. First we show that with probability at least $2^{-(r-1)m}$, the subgraph F is the union of $r-1$ disjoint trees, each with $m+1$ vertices, which are disjoint from $C \setminus B$ and from the r -neighbourhoods of the previous ℓ untouched blocks. Call this event \mathcal{A} . Suppose that there are i_j fixed edges from the j th perfect matching, with $i_1 + \dots + i_d = y$. There are $\binom{n-y}{(r-1)m}$ ways to choose vertices to make F the union of $r-1$ disjoint trees of $m+1$ vertices each, such that the vertices in $F \setminus B$ are disjoint from the y fixed vertices. Fill in the edges of F from the d perfect matchings in a canonical way. Namely, start with the first vertex v in B . Look at the $d-2$ vertices which have

been selected to be the neighbours of v in F_v . Assign an edge to each in ascending order, starting from the lowest-labelled vertex and the lowest-labelled allowable perfect matching (that is, one which is not touching v). Once you have done that, proceed to the $d-1$ vertices which have been chosen to neighbour the lowest-labelled new neighbour of v . Assign edges from the perfect matchings in ascending order. Continue until each edge in F_v has been assigned. Then do the same for the remaining vertices in B .

Suppose that after this process, a_j edges of F have been assigned from the j th perfect matching, for $1 \leq j \leq d$. Then $a_1 + \dots + a_d = (r-1)m$. Then the probability that a randomly chosen perfect matching on $n - 2i_j$ vertices contains these a_j specified edges is

$$\frac{1}{(n - 2i_j - 1)(n - 2i_j - 3) \cdots (n - 2i_j - 2a_j - 1)} \geq \frac{1}{n^{a_j}}.$$

It follows that

$$\mathbf{P}(\mathcal{A}) \geq \frac{[n - y]_{(r-1)m}}{n^{(r-1)m}} \geq 2^{-(r-1)m}.$$

(This uses the fact that $y = o(1)$, so $n - y \geq n/2$.)

For block B to be good, each perfect matching must be $(r-2)$ -coloured such that the colouring of $F \cup C$ is proper and such that there is no undercoloured path of length at least r starting from a vertex of B and using edges of F . For each perfect matching, order the colour classes of that matching arbitrarily. We stress that every edge of F has already been assigned to a perfect matching, and we must merely decide to which colour class of that perfect matching it belongs. Use the first colour (of the appropriate perfect matching) on any edges of F which are at distance 1 or 2 from C , use the second colour (of the appropriate perfect matching) on any edges of F which are at distance 3 or 4 from C , and so on, finally using the $r/2$ 'th colour (of the appropriate perfect matching) on all edges of F at distance $r-1$ or r from C , if r is even, or using the $(r-1)/2$ 'th colour (of the appropriate perfect matching) on all edges at distance $r-2$ or $r-1$ from C and the $(r+1)/2$ 'th colour (of the appropriate perfect matching) on all edges at distance r from C , if r is odd. Since $r/2 \leq r-2$ if $r \geq 4$ is even, and $(r+1)/2 \leq r-2$ if $r \geq 5$ is odd, this is always possible. Clearly any path of length r starting from $v \in C$ and using edges of F will involve r distinct colours.

Given that event \mathcal{A} holds, the probability that this configuration occurs after each perfect matching is $(r-2)$ -coloured is at least

$$\left(\frac{1}{r-2} - O(n^{-1}) \right)^{(r-1)m} > (r-1)^{-(r-1)m},$$

since each edge in a particular perfect matching is coloured with a given colour of that matching with probability $1/(r-2) + O(n^{-1})$.

Putting this together (and reintroducing the subscript for $B_{\ell+1}$), the probability that $B_{\ell+1}$ is good, given that B_1, \dots, B_ℓ are bad, is at least $q = (2(r-1))^{-(r-1)m}$. This shows that (16) holds, for $0 \leq \ell < \log^2 n$. Since $p_0(C) = 1$, the probability that the first $\log^2 n$ untouched blocks in a given k -cycle C are bad (from a given start vertex and in a given direction), conditioned upon C being a subgraph of the edge-coloured graph,

is bounded above by $(1 - q)^{\log^2 n} = O(n^{-2})$. This is uniform over all k -cycles, with $\log^5 n < k \leq \sqrt{n} \log n$, and all choices of start vertex and direction.

Let C_k be the number of undercoloured k -cycles in G . Then the expected number of bad long undercoloured cycles is bounded above by

$$\sum_{k=\log^5 n+1}^{\sqrt{n} \log n} \mathbf{E} C_k \cdot 2k \cdot O(n^{-2}) = O(n^{-1})$$

using Proposition 3.1. Therefore Markov's inequality guarantees there are a.a.s. no long bad undercoloured cycles of length at most $\sqrt{n} \log n$. This completes the proof. \square

Proof of Theorem 1.1 for n even. First note that for $d = 2$, Theorem 1.1 is clearly true since a 2-regular graph is the union of disjoint cycles. There are $2(r-2) \geq r$ colours available, so we can easily form a proper r -acyclic edge colouring of any 2-regular graph.

Now assume that $d \geq 3$. Combining Corollary 3.5, Corollary 3.12 and Lemma 3.13, we find that a.a.s. $G \in \mathcal{G}$ satisfies conditions (i), (ii), (iii) of Theorem 2.1. The event that $G \in \mathcal{G}$ is simple holds with probability which tends to a non-zero constant (see [4] or [13]). Applying Theorem 2.1 to simple elements of \mathcal{G} , we conclude that a.a.s. $G \in d\mathcal{G}_{n,1}$ satisfies $A'_r(G) \leq (r-2)d$. Hence by contiguity, a.a.s. $G \in \mathcal{G}_{n,d}$ satisfies $A'_r(G) \leq (r-2)d$, as required. \square

4 Odd number of vertices

Let n be even and d an even constant, $d \geq 4$. Consider the probability space which we denote by $\mathcal{G}_{n,d}^{(\text{choose})}$, obtained by taking a random $G \in \mathcal{G}_{n,d}$ and selecting a set of $d/2$ edges of G (which we call distinguished edges) uniformly at random. Given such a graph G with $d/2$ distinguished edges, we can form a graph G' by adding a new vertex $n+1$ together with d new edges joining vertex $n+1$ to each endvertex of the distinguished edges, then deleting the distinguished edges. Note that G' is a d -regular multigraph which is simple if and only if the $d/2$ chosen edges are non-adjacent (otherwise repeated edges are formed). We call this operation *pegging* the edges. Let $\mathcal{G}_{n,d}^{(\text{peg})}$ denote the probability space of all graphs G' obtained by the above procedure, restricting the resulting probability space to simple graphs. We say that two sequences of probability spaces $\mathcal{A}_n, \mathcal{B}_n$ on the same underlying set Ω_n are *asymptotically equivalent* if the total variation distance between $\mathbf{P}_{\mathcal{A}_n}$ and $\mathbf{P}_{\mathcal{B}_n}$ is $o(1)$. Robinson and Wormald [12, Section 3] proved that

$$\mathcal{G}_{n,d}^{(\text{peg})} \text{ and } \mathcal{G}_{n+1,d} \text{ are asymptotically equivalent.} \quad (17)$$

The calculations of the previous section involve d independent, uniformly chosen perfect matchings. Let $(d\mathcal{G}_{n,1})^{(\text{choose})}$ be the probability space obtained by taking a random $G \in d\mathcal{G}_{n,1}$ and then choosing a set of $d/2$ distinguished edges uniformly at random from G . Then $(d\mathcal{G}_{n,1})^{(\text{peg})}$ denotes the probability space obtained from $(d\mathcal{G}_{n,1})^{(\text{choose})}$ by

pegging the $d/2$ distinguished edges, restricting the resulting probability space to simple graphs.

Lemma 4.1 *The probability spaces $\mathcal{G}_{n,d}^{(\text{peg})}$ and $(d\mathcal{G}_{n,1})^{(\text{peg})}$ are contiguous.*

Proof. For ease of notation, write $\mathcal{A} = \mathcal{G}_{n,d}$ and $\hat{\mathcal{A}} = d\mathcal{G}_{n,1}$, and write $\mathcal{B} = \mathcal{G}_{n,d}^{(\text{peg})}$ and $\hat{\mathcal{B}} = (d\mathcal{G}_{n,1})^{(\text{peg})}$.

Consider an event $X(= X_n)$ in \mathcal{B} or $\hat{\mathcal{B}}$. For a d -regular graph G on $[n]$, let $R_X(G)$ be the probability that choosing a set of $d/2$ distinguished edges of G uniformly at random, and pegging these edges, gives an element of X . Fix $\varepsilon > 0$. Then

$$\mathbf{P}_{\mathcal{B}}(X) = \sum_{R_X(G) \leq \varepsilon} \mathbf{P}_{\mathcal{A}}(G)R_X(G) + \sum_{R_X(G) > \varepsilon} \mathbf{P}_{\mathcal{A}}(G)R_X(G).$$

The first summand is at most ε , which implies that

$$\varepsilon \sum_{R_X(G) > \varepsilon} \mathbf{P}_{\mathcal{A}}(G) < \mathbf{P}_{\mathcal{B}}(X) < \varepsilon + \sum_{R_X(G) > \varepsilon} \mathbf{P}_{\mathcal{A}}(G). \quad (18)$$

Similarly

$$\varepsilon \sum_{R_X(G) > \varepsilon} \mathbf{P}_{\hat{\mathcal{A}}}(G) < \mathbf{P}_{\hat{\mathcal{B}}}(X) < \varepsilon + \sum_{R_X(G) > \varepsilon} \mathbf{P}_{\hat{\mathcal{A}}}(G). \quad (19)$$

Assume that $\mathbf{P}_{\mathcal{B}}(X) = o(1)$. The lower bound in (18) shows that $\sum_{R_X(G) > \varepsilon} \mathbf{P}_{\mathcal{A}}(G) \rightarrow 0$. That is, $\mathbf{P}_{\mathcal{A}}(R_X(G) > \varepsilon) \rightarrow 0$, and contiguity of \mathcal{A} and $\hat{\mathcal{A}}$ implies that $\mathbf{P}_{\hat{\mathcal{A}}}(R_X(G) > \varepsilon) \rightarrow 0$. Then the right hand side of (19) says that $\mathbf{P}_{\hat{\mathcal{B}}}(X) = O(\varepsilon)$. Since this is true for all $\varepsilon > 0$ (or if you like, taking $\varepsilon \rightarrow 0$) it follows that $\mathbf{P}_{\hat{\mathcal{B}}}(X) = o(1)$. Thus $\mathbf{P}_{\mathcal{B}}(X) = o(1)$ implies $\mathbf{P}_{\hat{\mathcal{B}}}(X) = o(1)$, and the converse follows by the symmetric argument. Hence \mathcal{B} and $\hat{\mathcal{B}}$ are contiguous, as required. \square

Now let \mathcal{G}_{n+1}^* be the following probability space of $(r-2)d$ -edge-coloured d -regular multigraphs on $[n+1]$. Fix an ordering on the set of $(r-2)d$ colours. With \mathcal{G} as in the previous section, take a random $G \in \mathcal{G}$. This is a $(r-2)d$ -edge-coloured d -regular multigraph on $[n]$. Uniformly choose a set of $d/2$ edges of G and peg these edges to give a graph G' on the vertex set $[n+1]$, with the edges incident to vertex $n+1$ not yet coloured. Now take each neighbour v of $n+1$ in ascending order (according to its label) and recolour the edge $\{v, n+1\}$ with the least colour (with respect to the fixed ordering) which may be used to properly colour this edge. There are at least

$$(r-2)d - 2(d-1) \geq 2$$

colours available at each step, so this is always possible. This gives a proper $(r-2)d$ -edge colouring of G' , and this properly edge-coloured graph G' is an element of \mathcal{G}_{n+1}^* .

Proposition 4.2 *A.a.s. $G' \in \mathcal{G}_{n+1}^*$ satisfies conditions (i), (ii), (iii) of Theorem 2.1.*

Proof. Take a random $G \in \mathcal{G}$ with $d/2$ distinguished edges, as in the construction of $G' \in \mathcal{G}_{n+1}^*$. Form a set D consisting of the endvertices of the distinguished edges.

Consider first the expected number of undercoloured k -cycles in G' , for $2 \leq k \leq n$. If C is an undercoloured k -cycle in G' which does not involve the vertex $n+1$ then C is also an undercoloured k -cycle in G . On the other hand, if C is an undercoloured k -cycle in G' which involves the vertex $n+1$ then there exists an undercoloured $(k-2)$ -path in G with both endvertices in the set D . Considering the calculations of Lemma 3.6, there are $O(n)$ undercoloured k -paths in G , but forcing both endvertices to be in D gives an extra factor $O(n^{-2})$. This implies several statements. Firstly, the expected number of undercoloured k -cycles in G' is bounded above by

$$O(1/k) + O(1/n) = O(1/k)$$

using Proposition 3.1. Summing over k shows that condition (i) of Theorem 2.1 holds. Secondly, there are a.a.s. no short undercoloured cycles in G' involving vertex $n+1$. Thirdly, there are a.a.s. no undercoloured paths in G with both endvertices in the set D , since there are a.a.s. no extensive undercoloured paths in G and the expected number of unextensive undercoloured paths in G with both endvertices in D is at most $O(\sqrt{n} \log n/n) = o(1)$.

Similarly, an undercoloured k -path in G' which does not involve vertex $n+1$ is also present in G . An undercoloured k -path which involves the vertex $n+1$ corresponds to an undercoloured k_1 -path and an undercoloured k_2 -path in G , where $k_1 + k_2 = k - 2$ or $k_1 = k - 1$ and $k_2 = 0$. Therefore the expected number of undercoloured k -paths in G' is bounded above by

$$O(n) + \sum_{k_1} O(1) = O(n),$$

analogous to the calculations of Lemma 3.6. If $k \geq \sqrt{n} \log n$ then we may reduce this bound to

$$O(n^{-1}) + \sum_{k_1} O(n^{-2}) = O(n^{-1})$$

using the same arguments, noting that at least one of k_1, k_2 is greater than $\sqrt{8n \log n}$. Hence there are a.a.s. no extensive undercoloured paths in $G' \in \mathcal{G}_{n+1}^*$.

Now consider the expected number of short undercoloured cycles C with two distinct distinguished vertices which are the ends of an unextensive undercoloured path. We have seen that a.a.s. there are no short undercoloured cycles in G' which contain the vertex $n+1$. By Lemma 3.9, any such structure in G' must have vertex $n+1$ on the unextensive undercoloured path. But then G contains a short undercoloured cycle with a distinguished vertex which is one endvertex of an unextensive undercoloured path, with the other endvertex in D . The expected number of these structures is

$$O(1/k) \cdot O(n) \cdot O(k/n^2) = O(n^{-1})$$

by arguments similar to Lemma 3.9. This follows since for one end of the path there are k choices and for the other there are $O(1)$ choices, instead of a factor of $O(n^2)$ for the free

choice of the two ends. Hence there are a.a.s. no such structures. This also shows that a.a.s. there are no unextensive undercoloured paths between two vertices in disjoint short undercoloured cycles. Next, suppose that G' contains two short undercoloured cycles which intersect in a single vertex. By Lemma 3.11, vertex $n + 1$ must lie on one (or both) of these short undercoloured cycles. But we have shown above that a.a.s. G' has no short undercoloured cycles containing vertex $n + 1$. Putting all this together we see that a.a.s. $G' \in \mathcal{G}_{n+1}^*$ satisfies condition (ii) of Theorem 2.1.

Finally for condition (iii), let C be a long undercoloured cycle in G' . We know that G a.a.s. has no long bad undercoloured cycle, so assume that C contains the vertex $n + 1$. Then there is an undercoloured path in G with both endvertices in the set D , but, as we have shown, a.a.s. there are none of these. \square

Proof of Theorem 1.1 for n odd. Again, the result is trivial when $d = 2$. Now assume that $d \geq 4$ is an even constant. By Proposition 4.2, a.a.s. $G \in \mathcal{G}_n^*$ (for n odd) satisfies conditions (i), (ii), (iii) of Theorem 2.1. The event that $G \in \mathcal{G}_n^*$ is simple holds with probability which tends to a non-zero constant (see [4] or [13], noting that a.a.s. the $d/2$ distinguished edges are pairwise non-adjacent). Therefore a.a.s. $G \in (d\mathcal{G}_{n-1,1})^{(\text{peg})}$ satisfies $A'_r(G) \leq (r - 2)d$. By Lemma 4.1 and (17), it follows that $G \in \mathcal{G}_{n,d}$ a.a.s. satisfies $A'_r(G) \leq (r - 2)d$, as required. \square

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