Class.Q.Grav. Vol.10(1993) pp.947–960. 8-Dec-1992

THE GAUSS-CODACCI EQUATION ON A REGGE SPACETIME.II.

Leo Brewin
Department of Mathematics
Monash University
Clayton, Vic. 3168
Australia

A geometric expression for the Gauss-Codacci equation on a simplicial (Regge) spacetime will be presented. It will be derived by arguing that the operator associated with the parallel transportation of a vector around a timelike bone may also be de-composed into a product of operators associated with the Cauchy surface and its embedding in the spacetime. It will then be shown that this result is, for a class of weak simplicial spacetimes, term by term equivalent with the usual continuum version of the contracted Gauss-Codacci equation. This leads, for this class of weak simplicial spacetimes, to a simple relationship between the 4-defect, 3-defect and the extrinsic curvature terms.

1. Introduction.

Consider a typical "3+1" formulation of the Regge calculus [1,2,3]. To each spacelike leg in each Cauchy surface there is an associated timelike bone (obtained as the time evolution of that leg). This leg is surrounded by a set of tetrahedra. Likewise the timelike bone is surrounded by a set of 4-dimensional tubes being generated from the time evolution of the tetrahedra. Given the table of leg lengths it is not hard to calculate the defects on both the spacelike leg and the timelike bone. An interesting challenge is to seek an expression by which the defect on the timelike bone can be related to that on the leg and the embedding of the tetrahedra in the spacetime. Since the defects are related to the parallel transport of vectors it is natural to consider the relationships between parallel transport in the full 4-dimensional spacetime and that in the 3-dimensional Cauchy surface. These relationships, for a smooth spacetime, are represented by the Gauss-Codacci equations [4]. Our challenge is to find similar relationships for a simplicial spacetime.

The approach adopted in an earlier paper [5] was to interpret the Gauss-Codacci equation as a differential equation on the simplicial spacetime. This required certain assumptions to be made. It would be nice if an approach could be developed which did not begin with such assumptions. It is for this reason that a purely geometrical approach was sought.

The basic idea here will be to ask what is the relationship between the processes of parallel transportation within the full 4-dimensional spacetime and that of parallel transportation within the 3-dimensional Cauchy surface. Our approach will begin by constructing two different sets of tetrads in each tetrahedron. The first set will be used to define the parallel transport of vectors in the full 4-dimensional spacetime. The second set of tetrads will be tailored for the purpose of parallel transport within each Cauchy surface. It will then be shown that this second set of tetrads is easily constructed from the first set. This fact will then lead us directly to the desired result, namely, a simplicial counterpart of the Gauss-Codacci equations.

The notation used will be similar to that used in references [3,5]. Our attention will be confined to one timelike bone and the set of 4-dimensional tubes attached to that bone. The Cauchy surface will be denoted by Σ, the tetrahedra in Σ by $s_i, i = 1, 2, 3\cdots$ and the 4-dimensional tube generated by s_i will be represented by T_i. The spacelike bone (a leg in

Σ) will be represented by σ' while the timelike bone generated by σ' will be denoted by σ. The defects on the bones, σ and σ', will be written as α and α' respectively. The triangular interface between s_i and s_j will be written as s_{ij} while the interface between the pair of tubes T_i and T_j will be written as T_{ij}. Where confusion may arise between a simplex index and other indices the simplex index will be written in parenthesis. For example $t^{\mu}_{i\,(j)}$ are the components of the i^{th} tetrad vector in the tube T_j.

The restriction to a simplicial space with just one timelike bone may seem un-reasonable. However the metric near any bone, in any generic simplicial spacetime, will be determined by just the set of tetrahedra surrounding that bone. Thus the generic simplicial spacetime can be examined one bone at a time.

2. The Gauss-Codacci equation.

Our first task is to obtain a rule for the parallel transport of a general vector from T_j to T_{j+1}. Our starting point will be one of the basic assumptions of the Regge calculus, namely, that any pair of tubes can be covered by one flat metric. Parallel transport in such a frame is then very easily expressed. Let $x^{\mu}{}_{(j)}$ and $x^{\mu}{}_{(j+1)}$ be a pair of coordinate frames for T_j and T_{j+1} respectively. Let the coordinates of the frame covering T_j and T_{j+1} be $\bar{x}^{\mu}_{jj+1}$. Suppose $\bar{v}^{\mu}_{(j)}$ and $\bar{v}^{\mu}_{(j+1)}$ are the successive components of a vector parallel transported from T_j to T_{j+1}, in the $\bar{x}^{\mu}_{jj+1}$ frame, and that $v^{\mu}_{(j)}$ and $v^{\mu}_{(j+1)}$ are the corresponding components with respect to the local coordinates $x^{\mu}_{(j)}$ and $x^{\mu}_{(j+1)}$. Clearly

$$\bar{v}^{\mu}_{(j+1)} = \bar{v}^{\mu}_{(j)}$$

which when rexepressed in terms of the coordinates $x^{\mu}_{(j)}$ and $x^{\mu}_{(j+1)}$ leads to

$$v^{\mu}{}_{(j+1)} = A^{\mu}{}_{\nu(jj+1)} v^{\nu}{}_{(j)} \tag{2.1}$$

for an appropriate choice of $A^{\mu}{}_{\nu(jj+1)}$. The $A^{\mu}{}_{\nu(jj+1)}$ form a non-singular matrix and can be computed solely from the leg-lengths and the coordinates in T_j and T_{j+1}. Williams and Ellis[6] provide details of the construction of this matrix.

Similarly, the parallel transport of vectors with respect to the metric of the Cauchy surface is represented by

$$v'^{\mu}{}_{(j+1)} = A'^{\mu}{}_{\nu(jj+1)} v'^{\nu}{}_{(j)} \tag{2.2}$$

for an appropriate choice of the $A'^{\mu}{}_{\nu(jj+1)}$. In this expression it is normal to assume that $v'^{\mu}{}_{(j)}$ is tangent to the Cauchy surface in s_j. In this case $v'^{\mu}{}_{(j+1)}$ will also be tangent to the Cauchy surface, this time in s_{j+1}.

Since the metric and the connection on the Cauchy surface are inherited from the full 4-dimensional spacetime it is natural to wonder if the $A'^{\mu}{}_{\nu(jj+1)}$ can be obtained from the $A^{\mu}{}_{\nu(jj+1)}$. Such a relation does exist and will be the subject of the remainder of this section. Let $v'^{\mu}{}_{(j)}$ be any vector tangent to s_j. Now consider the vector $A^{\mu}{}_{\nu(jj+1)} v'^{\nu}{}_{(j)}$. This is a vector in T_{j+1} which in general will not be tangent to s_{j+1}. It can be made tangent to s_{j+1} by applying a boost in the plane containing $n^{\mu}_{(j)}$ and $n^{\mu}_{(j+1)}$, the future directed timelike unit-normals to s_j and s_{j+1} respectively. This boost transformation maps the rest frame of T_j into a rest frame of T_{j+1}. A little thought shows that the resulting vector must be $v'^{\mu}{}_{(j+1)}$. That is

$$v'^{\mu}{}_{(j+1)} = B^{\mu}{}_{\rho(jj+1)} A^{\rho}{}_{\nu(jj+1)} v'^{\nu}{}_{(j)} \tag{2.3}$$

where $B^{\mu}{}_{\nu}$ are the components of the boost matrix.

Comparing (2.2) and (2.3) we see that

$$A'^{\mu}{}_{\nu(jj+1)} = B^{\mu}{}_{\rho(jj+1)} A^{\rho}{}_{\nu(jj+1)}$$

This is our desired relation between the $A^{\mu}{}_{\nu(jj+1)}$ and the $A'^{\mu}{}_{\nu(jj+1)}$. It shows explicitly how the operator for parallel transport in the Cauchy surface can be built from the associated operator for the full spacetime and the embedding of the Cauchy surface in the spacetime. The parallels with the continuum are clear, the $A_{(jj+1)}$ can be viewed as the connection on the spacetime, the $A'_{(jj+1)}$ as the connection on the Cauchy surface and the $B_{(jj+1)}$ as the extrinsic curvature of the Cauchy surface.

The preceeding analysis focused on properties local to one interface. Curvature, however, arises from the global relationship of a sequence of interfaces. Thus now consider the parallel transport of a vector, with respect to the metric of the 4-dimensional spacetime, around a simple loop starting from T_1, passing once through each of the other T_j and returning to T_1. The nett effect is to rotate the vector, in a plane normal to the bone, through an angle equal to the defect on σ. This rotation operator must equal the ordered product of each of the $A_{(jj+1)}$. Thus

$$\exp(\alpha\, U)^\mu{}_\nu = \left(A_{(m1)}A_{(m-1m)}\cdots A_{(jj+1)}\cdots A_{(23)}A_{(12)}\right)^\mu{}_\nu \tag{2.4}$$

where $U^\mu{}_\nu$ is the normalized bi-vector normal to σ (ie. $U^\mu{}_\nu W^\nu = 0$ for any vector W^μ parallel to σ) and α is the defect on σ.

In a similar fashion, the nett effect of parallel transporting a vector, with respect to the metric of the Cauchy surface, along a non-trivial loop starting and finishing in s_1 and enclosing the leg σ' once, will be given by

$$\exp(\alpha'\, U')^\mu{}_\nu = \left(B_{(m1)}A_{(m1)}B_{(m-1m)}A_{(m-1m)}\cdots B_{(jj+1)}A_{(jj+1)}\cdots B_{(23)}A_{(23)}B_{(12)}A_{(12)}\right)^\mu{}_\nu \tag{2.5}$$

where α' is the defect on the leg σ' and $U'^\mu{}_\nu$ is the normalized bi-vector normal to that leg (ie. $U'^\mu{}_\nu W^\nu = 0$ for any vector W^μ tangent to the Cauchy surface and parallel to σ').

It is important to notice that there is some considerable freedom in choosing the $A_{(jj+1)}$. Indeed the coordinates in each of the m tubes $T_j, j = 1, 2, \cdots m$ may be freely chosen. This in turn will impose a choice on each of the $A_{(jj+1)}$. Conversely, one could choose any coordinates in T_1 and extend these coordinates throughout the remaining $m-1$ tubes by an appropriate choice of the $A_{(jj+1)}, j = 1, 2, \cdots m-1$. The components of $A_{(m1)}$ could not then be freely chosen but would be fully determined by the coordinate transformation from T_m back into T_1. Clearly, only $m-1$ of the $A_{(jj+1)}, j = 1, 2, \cdots m$ can be freely chosen. For our analysis it is simplest to choose

$$I = A_{(m1)} = A_{(m-1m)} = \cdots A_{(jj+1)} \cdots = A_{(34)} = A_{(23)}$$

where I is the identity operator. This corresponds to the familar picture of cutting the set of tubes along the interface T_{12} and mapping that set of tubes into a Minkowski space.

Substituting this into the above relations, (2.4,2.5), will lead to

$$\exp(\alpha' U')^\mu{}_\nu = \left(B_{(m1)}B_{(m-1m)}\cdots B_{(jj+1)}\cdots B_{(23)}B_{(12)}\right)^\mu{}_\rho \exp(\alpha U)^\rho{}_\nu \tag{2.6}$$

Now let β_j be the boost angle and $V_j^\mu{}_\nu$ be the normalized bi-vector normal to the interface s_{jj+1} (ie. $V_j^\mu{}_\nu W^\nu = 0$ for any vector W^ν tangent to s_{jj+1}). The Lorentz transformation associated with the interface s_{jj+1} is then $\exp(\beta_j\, V_j)^\mu{}_\nu$. Consequently

$$\exp\left(\alpha' \, U'\right)^\mu{}_\nu = \left\{\prod_{j=1}^{m} \exp\left(\beta_j\, V_j\right)\right\}^\mu{}_\rho \exp\left(\alpha\, U\right)^\rho{}_\nu \tag{2.7}$$

where m equals the number of tubes surrounding the bone. This equation can also be re-written as

$$\exp\left(\alpha\, U\right)^\mu{}_\nu = \left\{\prod_{j=1}^{m} \exp\left(-\beta_j\, V_j\right)\right\}^\mu{}_\rho \exp\left(\alpha' \, U'\right)^\rho{}_\nu \tag{2.8}$$

This is the main result of this paper. It displays an explicit decomposition of the curvature of the spacetime in terms of the intrinsic and extrinsic curvatures of the Cauchy surface.

It seems reasonable to ask in what way can the equations (2.8) be compared with the usual Gauss-Codacci equations [4]

$$\begin{gathered} {}^{(3)}R^\mu{}_{\alpha\beta\gamma} = \perp R^\mu{}_{\alpha\beta\gamma} - \epsilon\left\{K^\mu{}_\gamma K_{\alpha\beta} - K^\mu{}_\beta K_{\alpha\gamma}\right\} \\ K^\mu{}_{\mu|\alpha} - K^\mu{}_{\alpha|\mu} = \perp R_{\mu\alpha} n^\mu \end{gathered} \tag{2.9}$$

where $\epsilon = \pm 1$ according to the choice of a Euclidian ($\epsilon = +1$) or Lorentzian ($\epsilon = -1$) signature.

There are number of difficulties here, for example (2.8) applies to discrete metrics whereas (2.9) is applicable only to smooth metrics. This problem can be dealt with in two ways. The first approach would be to consider a continuous one parameter family of simplicial spacetimes whose limit was a given smooth spacetime. The second approach takes the opposite point of view, to employ a continuous one parameter family of smooth spacetimes converging to a given simplicial geometry. The first approach is highly non-trivial. It is not hard to accept that the metrics of the smooth and simplicial spacetimes will converge but it is much harder to see how the Riemann tensor for the simplicial spacetime (distributed over many bones pointing in many different directions) can settle down to a smooth tensor on the smooth spacetime. However, Cheeger, Muller and Schrader [7] have shown that this does in fact occur (under certain conditions, eg. that the simplices do not become long and skinny or short and squat). Their work also shows that all of the defects must vanish in the limit of the sequence. That is, if a given smooth spacetime can be well approximated by a simplicial spacetime, then the defects will all be very small. So for the purposes of numerical relativity, it is sufficient to assume that all of the defects are very small.

The second approach mentioned above is much easier to pursue. Thus our aim will be to show that when (an appropriate form of) the Gauss-Codacci equations is evaluated on the sequence of smooth spacetimes one obtains a linearized form of (2.8). Our assumptions will be

- that the simplicial spacetime consists of one timelike bone and its neighbouring simplices,
- that the simplicial spacetime and Cauchy surfaces are almost flat,
- that the metric of the simplicial spacetime can be arbitrarily approximated by a continuous one parameter family of smooth metrics and
- that the timelike bone stands normal to the Cauchy surface in the flat space limit.

The last assumption is equivalent to requiring the shift vector to vanish. This is done only to simplify the calculations.

Our approach will be to integrate

$$R = {}^{(3)}R - 2\epsilon\,(Kn^{\mu})_{;\mu} - \epsilon(K^{\mu}{}_{\nu}K^{\nu}{}_{\mu} - K^2) \qquad (2.10)$$

throughout a region containing the bone and to then compare the result with a linearised version of a contracted form of (2.8). The analysis will lead to simplicial counterparts for each of the terms in (2.10).

In the rest of this paper it will be assumed that the metric signature is Euclidian (ie. $\epsilon = +1$). This will simplify the parts of the discussions involving angles between various pairs of vectors. Of course the final results can be easily re-cast in a form suitable for spacetimes with a Lorentzian signature. However, the terms timelike and spacelike will continue to be used for they aid in identifying which bones are being discussed – a spacelike bone lies within the Cauchy surface whereas the timelike bone σ is the bone generated by lifting the (spacelike) leg σ' off the Cauchy surface.

3. Weak simplicial spaces.

Consider a continuous one parameter family of simplicial spaces. The parameter will be denoted by λ and will be chosen so that when $\lambda \to 0$ the defects and boosts vanish. It will be assumed that all defects, bi-vectors etc. are smooth differentiable functions of λ near $\lambda = 0$.

Our starting point will be to expand the defects, bi-vectors etc. in powers of λ. Thus assume that

$$\begin{aligned}
\alpha &= \overset{1}{\alpha}\,\lambda + \overset{2}{\alpha}\,\frac{\lambda^2}{2} + O(\lambda^3)\,,\\
\alpha' &= \overset{1}{\alpha}{}'\,\lambda + \overset{2}{\alpha}{}'\,\frac{\lambda^2}{2} + O(\lambda^3)\,,\\
\beta_i &= \overset{1}{\beta}_i\,\lambda + \overset{2}{\beta}_i\,\frac{\lambda^2}{2} + O(\lambda^3)\,,\\
U^\mu{}_\nu &= \overset{0}{U}{}^\mu{}_\nu + \overset{1}{U}{}^\mu{}_\nu\,\lambda + O(\lambda^2)\,,\\
U'^\mu{}_\nu &= \overset{0}{U}{}'^\mu{}_\nu + \overset{1}{U}{}'^\mu{}_\nu\,\lambda + O(\lambda^2)\,,\\
V_i^\mu{}_\nu &= \overset{0}{V}{}_i^\mu{}_\nu + \overset{1}{V}{}_i^\mu{}_\nu\,\lambda + O(\lambda^2)
\end{aligned} \tag{3.1}$$

where each coefficient of the form $\overset{\mathrm{n}}{X}$ is independent of λ. The expansion of (2.8) to order λ^3

can be written as

$$\alpha U + \frac{\alpha^2}{2} U \cdot U = \alpha' U' + \frac{\alpha'^2}{2} U' \cdot U' - \sum_{i=1}^{n} \beta_i V_i \cdot (I + \alpha' U') \\ + \frac{1}{2}\Big(\sum_{i=1}^{n} \beta_i V_i\Big) \cdot \Big(\sum_{j=1}^{n} \beta_j V_j\Big) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} \beta_i \beta_j [V_i, V_j] + O(\lambda^3) \tag{3.2}$$

where terms of the form $X \cdot Y$ and $[X, Y]$ are defined as

$$(X \cdot Y)^{\mu}{}_{\nu} = X^{\mu}{}_{\rho} Y^{\rho}{}_{\nu}$$
$$[X, Y]^{\mu}{}_{\nu} = X^{\mu}{}_{\rho} Y^{\rho}{}_{\nu} - Y^{\mu}{}_{\rho} X^{\rho}{}_{\nu} \, .$$

Substituting (3.1) in (3.2), collecting common terms and setting the coefficients of λ, λ^2 to zero leads to the pair of equations

$$\overset{1}{\alpha} \overset{0}{U} = \overset{1}{\alpha'} \overset{0}{U'} - \sum_{i=1}^{n} \overset{1}{\beta}_i \overset{0}{V}_i \tag{3.3a}$$

$$\overset{2}{\alpha} \overset{0}{U} + 2\overset{1}{\alpha} \overset{1}{U} + (\overset{1}{\alpha})^2 \overset{0}{U} \cdot \overset{0}{U} = \overset{2}{\alpha'} \overset{0}{U'} + 2\overset{1}{\alpha'} \overset{1}{U'} + (\overset{1}{\alpha'})^2 \overset{0}{U'} \cdot \overset{0}{U'} \\ - \sum_{i=1}^{n} (2\overset{1}{\beta}_i \overset{1}{V}_i + \overset{2}{\beta}_i \overset{0}{V}_i) - \sum_{i=1}^{n}\sum_{j=1}^{i-1} \overset{1}{\beta}_i \overset{1}{\beta}_j [\overset{0}{V}_i, \overset{0}{V}_j] \tag{3.3b}$$

Since $\lambda = 0$ corresponds to flat space it follows that

$$\overset{0}{U} = \overset{0}{U'}$$

and

$$0 = \overset{0}{U}{}^{\mu}{}_{\nu} \overset{0}{V}{}^{\nu}{}_{\mu}$$

It can also be shown that (see the Appendix)

$$\overset{1}{V}_i = \overset{1}{\rho}_i \overset{0}{U} \tag{3.4a}$$

$$[\overset{0}{V}_i, \overset{0}{V}_j] = \sin \overset{0}{\theta}_{ij} \overset{0}{U} \tag{3.4b}$$

where $\rho_i = O(\lambda^1)$ is the angle, minus $\pi/2$, between the bone σ and the interface s_{ii+1}. The angle $\theta_{ij} = O(\lambda^0)$ is the *signed* angle from s_{ii+1} to s_{jj+1}. Consequently

$$\overset{1}{\alpha} = \overset{1}{\alpha}'$$

$$\overset{2}{\alpha} = \overset{2}{\alpha}' - \sum_{i=1}^{n} 2\overset{1}{\beta}_i \overset{1}{\rho}_i - \sum_{i=1}^{n}\sum_{j=1}^{i-1} \overset{1}{\beta}_i \overset{1}{\beta}_j \, \sin\theta_{ij}$$

which when substituted into

$$\alpha = \overset{1}{\alpha}\,\lambda + \overset{2}{\alpha}\,\frac{\lambda^2}{2} + O(\lambda^3)$$

leads to

$$\alpha = \alpha' - \sum_{i=1}^{n} \beta_i \sin\rho_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} \beta_i\beta_j \sin\theta_{ij} + O(\lambda^3)\,. \tag{3.5}$$

It will now be shown that this important result can be compared, *term by term*, with an integral version of a contracted form of (2.9), namely

$$^{(4)}R = {}^{(3)}R - 2\epsilon\,(Kn^\mu)_{;\mu} - \epsilon\left\{K^\mu{}_\nu K^\nu{}_\mu - K^2\right\}$$

The following arguments are intended to provide plausible arguments, rather than a proof, that the simplicial and continuum equations are equivalent (for weak simplicial spacetimes). The region of integration M will be chosen so as to contain the timelike bone. This will be done as follows. Choose an orthonormal set of coordinates (p, q, u, v) in a neighbourhood of σ such that the points on the bone σ are described by $u = v = 0$. Now construct the 2-dimensional disk D by setting p, q=constant. The 2-metric on D is a conical metric whose curvature gives rise to the defect angle α. The symmetry of the 4-dimensional metric implies that there are two killing vectors, each parallel to σ. This means that a 4-dimensional region can be generated by dragging the disk D along the trajectories of this pair of Killing vectors. The region M is then defined to be the smallest such region, for a given D, which contains the bone σ. This generates a $2+2$ foliation of M. Each leaf of M is a copy of D and is characterized by the coordinates p, q of the intersection of this leaf with the bone σ.

A similar construction applies within each Cauchy surface. The orthonormal coordinates can be chosen as (p', u', v') with the points of the bone σ' being described by $u' = v' = 0$. The disk D' is defined by p' =constant and can be used to generate M' by dragging D' along the spacelike bone σ'. This construction is displayed in Figure (1). Notice that M' also arises from the intersection of M with the Cauchy surface.

As the metric in M is not differentiable it is necessary to approximate the metric by smooth functions in M (such approximations can always be made arbitrarily accurate, see eg. [8]). Let g be a continuous one parameter sequence of smooth metrics on M and let $R(g)$ be the associated curvature. Denote the parameter by γ and choose it so that $\gamma \to 0$ corresponds to the discrete metric on M. Similar definitions apply for h and $R(h)$ in M'. Actually these sequences also depend implicitly on the parameter λ. However, in the following, λ will be fixed while the limit $\gamma \to 0$ is taken. It will be further assumed that the smooth metric, lapse and curvatures are differentiable functions of γ near $\gamma = 0$. Thus assume that

$$\begin{aligned} R(h) &= \overset{0}{R}(h) + \overset{1}{R}(h)\,\gamma + O(\gamma^2)\,, \\ R(g) &= \overset{0}{R}(g) + \overset{1}{R}(g)\,\gamma + O(\gamma^2)\,. \end{aligned} \tag{3.6}$$

The functions $\overset{0}{R}(h)$ and $\overset{0}{R}(g)$ should properly be viewed as distributions on M. Both will look like delta functions on the bone. These distributions will also depend smoothly on the parameter λ and since $\lambda \to 0$ corresponds to flat space it is reasonable to expect that

$$\lim_{\lambda \to 0} \overset{0}{R}(h) = 0\,,$$

$$\lim_{\lambda \to 0} \overset{0}{R}(g) = 0\,.$$

The integral to be evaluated may now be written as

$$\int_M R(g)\sqrt{-g}\,d^4x = \int_M R(h)\sqrt{-g}\,d^4x + 2\int_M (Kn^\mu)_{;\mu}\sqrt{-g}\,d^4x + \int_M \left\{K^\mu{}_\nu K^\nu{}_\mu - K^2\right\}\sqrt{-g}\,d^4x \tag{3.7}$$

The easiest term to tackle is the single term on the left hand side. It can be expressed as a double integral over a disk such as D and a double integral over the bone σ. However, in the later integral, the defect angle is constant. These facts are well known (see [8,9]) and lead to

$$\begin{aligned}\lim_{\gamma\to 0}\int_M R(g)\sqrt{-g}\,d^4x &= \lim_{\gamma\to 0}\int_\sigma\int_D R(g)\sqrt{-g}\,d^4x\\ &= 2\alpha A\end{aligned} \tag{3.8}$$

where A is the area of σ and α the defect on σ.

Now consider the integral $\int_M R(h)\sqrt{-g}\,d^4x$. It too can be factored into a pair of double integrals, thus leading to

$$\begin{aligned}\lim_{\gamma\to 0}\int_M R(h)\sqrt{-g}\,d^4x &= \lim_{\gamma\to 0} A\int_D R(h)\sqrt{d}\,d^2x\\ &= \lim_{\gamma\to 0} A\int_{D'} R(h)\sqrt{d'}\,\frac{\partial(u,v)}{\partial(u',v')}\,d^2x\end{aligned}$$

where d and d' are the induced metrics on D and D' respectively. The map of the integral from D to D' was obtained by a projection along the trajectories of the vectors parallel to σ and normal to σ'. If it were not for the Jacobian this integral would have the value α', the defect on the disk D'. It is not hard to see that the Jacobian equals $U^\mu{}_\nu U'^\nu{}_\mu/2$ which from (3.1) is seen to equal $1 + O(\lambda^2)$. Thus, to second order in λ,

$$\lim_{\gamma\to 0}\int_M R(h)\sqrt{-g}\,d^4x = 2\alpha' A \tag{3.9}$$

The third integral in (3.7) is $\int_M (Kn^\mu)_{;\mu}\,d^4x$ where n^μ is the unit timelike normal to each leaf and K is the extrinsic curvature. This can be rewritten as a surface integral over the three parts of the boundary of M, namely, the top and bottom leaves and the cylinder that joins the two leaves. The contributions from the top and bottom leaves will cancel (by construction of M) thus leaving

$$\int_M (Kn^\mu)_{;\mu}\sqrt{-g}\, d^4x = \int_C Kn^\mu r_\mu\, d^3x$$

where C is the cylindrical part of ∂M and r^μ is the outward pointing normal to C. It is not hard to see that in the process of generating M from successive copies of D' the cylinder C is generated from successive copies of the boundary of D'. Consequently the integral over C can be written as a double integral over σ and a single integral around $\partial D'$,

$$\begin{aligned}\int_M (Kn^\mu)_{;\mu}\sqrt{-g}\, d^4x &= \int_C Kn^\mu r_\mu\, d^3x \\ &= A\int_{\partial D'} Kn^\mu r_\mu\, ds \qquad (3.10)\end{aligned}$$

Consider now the path $\partial D'$ in a neighbourhood of one of the interfaces, s_{12} say. Denote this segment of $\partial D'$ by $\partial D'_{12}$. Along this path the normal n^μ changes from its value in s_1 to its value in s_2 all of which takes place in passing through s_{12}. Thus the integral on the right hand side of (3.10) can only be evaluated by making a suitable choice for n^μ, r^μ and K along $\partial D'_{12}$. The basic idea will be to construct a set of basis vectors throughout $s_1 \cup s_2$ from which an interpolation of n^μ along the path can be made.

Let $t^\mu, u^\mu, v^\mu, w^\mu$ be an orthonormal set of unit vectors chosen so that t^μ and u^μ are tangent to s_{12} and v^μ is normal to T_{12}. This prescription does not uniquely determine t^μ and u^μ (other valid choices can be obtained by a rotation in the plane spanned by t^μ and u^μ).

A related set of orthonormal vectors can be constructed from the normal n^μ to M', the tangent vector m^μ to $\partial D'_{12}$ and the pair k^μ, l^μ to complete the set (a $2+1$ example appears in Figure (2)). It is important to notice that all of these vectors can be defined throughout $s_1 \cup s_2$ and that they are functions of only one coordinate, namely, the distance measured from T_{12} (ie. along the integral curves of v^μ). Thus the integral in (3.10) is path independent and so the path $\partial D'_{12}$ may be deformed into a straight line segment in $s_1 \cup s_2$ normal to

s_{12}. Alternatively, the disk D' could have been chosen to have a boundary consisting of such straight line segments. It follows then that

$$0 = r^{\mu}_{;\nu}$$

on the section of C generated by $\partial D'_{12}$.

The interpolation can now be written as

$$\begin{aligned} k^{\mu}(v) &= t^{\mu} \\ l^{\mu}(v) &= u^{\mu} \\ m^{\mu}(v) &= v^{\mu}\cos\beta(v) - w^{\mu}\sin\beta(v) \\ n^{\mu}(v) &= w^{\mu}\cos\beta(v) + v^{\mu}\sin\beta(v) \end{aligned} \tag{3.11}$$

where v is the distance measured along the integral curves of v^{μ} with $v = \beta(v) = 0$ on T_{12} and with $v > 0$ in T_2. Notice that since the metric inside $s_1 \cup s_2$ is flat

$$0 = t^{\mu}_{;\nu} = u^{\mu}_{;\nu} = v^{\mu}_{;\nu} = w^{\mu}_{;\nu} \tag{3.12}$$

in $s_1 \cup s_2$. From this and (3.11) it follows that

$$\begin{aligned} K^{\mu}{}_{\nu} &= \perp (n^{\mu}{}_{;\nu}) \\ &= m^{\mu} m_{\nu} \frac{d\beta}{dm} \end{aligned} \tag{3.13}$$

where m is the distance measured along $\partial D'_{12}$, $m = 0$ on s_{12} and $m > 0$ in s_2. This result can also be found in [9] . Consider now the term $n^{\mu} r_{\mu}$. If $\rho(v)$ is defined as $\pi/2$ minus the angle from r^{μ} to n^{μ} then $n^{\mu} r_{\mu} = \sin\rho(v)$. However, from (3.11) it is not hard to see that $\rho(v)$ is constant along $\partial D'_{12}$ and therefore

$$\begin{aligned} \int_{\partial D'_{12}} K n^{\mu} r_{\mu}\, ds &= \int \sin\rho(v) \frac{d\beta}{dm}\, dm \\ &= \beta \sin\rho \end{aligned}$$

where $\rho = \rho(0)$ is ($\pi/2$ minus) the angle between the timelike and spacelike bones and where $\beta = \beta_{s_2} - \beta_{s_1}$ is the angle between the unit normals of s_1 and s_2 (ie. the boost from a rest frame of T_j to a rest frame of T_{j+1}). There is one such result for each of the interfaces, such as T_{12}, attached to the timelike bone. Thus one obtains

$$\lim_{\gamma\to 0} \int_M (Kn^\mu)_{;\mu} \sqrt{-g}\, d^4x = A \sum_{i=1}^{n} \beta_i \sin\rho_i \tag{3.14}$$

There remains one pair of terms to evaluate, the integral of the squares of the extrinsic curvature. This is not an easy task. The difficulty is that the $K^\mu{}_\nu$ behave like delta-functions. Thus quadratic terms in the $K^\mu{}_\nu$ are not likely to make much sense. However it is easy to see, from (3.13), that formally, $K^\mu{}_\nu K^\nu{}_\mu - K^2 = 0$. This shows that if any sense is to be made from these terms then the result should depend on contributions from *distinct pairs* of interfaces. This could also be gleaned from the very result that we are endeavouring to establish, namely the last term in (3.5). Thus our approach will be to split the integral as follows

$$\begin{aligned}\int_M \left\{K^\mu{}_\nu K^\nu{}_\mu - K^2\right\} \sqrt{-g}\, d^4x = &\sum_{i=1}^{n}\sum_{j=1}^{i-1} \int_M \left\{(K^\mu{}_\nu)_i\,(K^\nu{}_\mu)_j - (K)_i(K)_j\right\} \sqrt{-g}\, d^4x\\ = A&\sum_{i=1}^{n}\sum_{j=1}^{i-1} \int_{M^*} \left\{(K^\mu{}_\nu)_i\,(K^\nu{}_\mu)_j - (K)_i(K)_j\right\} \sqrt{d'}\, d^2x\end{aligned}$$

The second equality follows from arguments similar to that used in (3.9), namely, that each integral may be factored into a pair of double integrals, one over the bone σ and one over the disk D'. From (3.13) it follows that

$$(K^\mu{}_\nu)_i\,(K^\nu{}_\mu)_j - (K)_i(K)_j = -\sin^2\theta_{ij} \left(\frac{d\beta}{dm}\right)_i \left(\frac{d\beta}{dm}\right)_j$$

The integral can be evaluated by adopting $(m)_i$ and $(m)_j$ as coordinates for which the

Jacobian is $1/\sin\theta_{ij}$ and therefore

$$\begin{aligned}
\int_M \left\{K^\mu{}_\nu K^\nu{}_\mu - K^2\right\} \sqrt{-g}\, d^4x &= -A\sum_{i=1}^{n}\sum_{j=1}^{i-1} \int_{D'} \sin\theta_{ij} \left(\frac{d\beta}{dm}\right)_i \left(\frac{d\beta}{dm}\right)_j (dm)_i (dm)_j \\
&= -A\sum_{i=1}^{n}\sum_{j=1}^{i-1} \sin\theta_{ij} \left(\int \frac{d\beta}{dm} dm\right)_i \left(\int \frac{d\beta}{dm} dm\right)_j \\
&= -A\sum_{i=1}^{n}\sum_{j=1}^{i-1} \beta_i\beta_j \sin\theta_{ij} \qquad (3.15)
\end{aligned}$$

It is now easy to see that a substitution of (3.8,3.9,3.14,3.15) into (3.7) will lead directly to (3.5) as claimed.

It is possible to further simplify (3.5) in the instance where the bone σ is normal to the Cauchy surface as $\lambda \to 0$. Consider the relation

$$2R(g)_{\mu\nu} n^\mu n^\nu = -\left(Kn^\mu\right)_{;\mu} - \left\{K^\mu{}_\nu K^\nu{}_\mu - K^2\right\} .$$

For this type of weak simplicial spacetime, $n^\mu = q^\mu + O(\lambda)$ where q^μ is parallel to the timelike bone, $R(g)_{\mu\nu} q^\mu = 0$ and $R(g) = O(\lambda)$. Thus $R(g)_{\mu\nu} n^\mu n^\nu = O(\lambda^3)$ and consequently

$$\left(Kn^\mu\right)_{;\mu} = \left\{K^\mu{}_\nu K^\nu{}_\mu - K^2\right\} + O(\lambda^3) .$$

The integral version should clearly be

$$A\sum_{i=1}^{n} \beta_i \sin\rho_i = -A\sum_{i=1}^{n}\sum_{j=1}^{i-1} \beta_i\beta_j \sin\theta_{ij} + O(\lambda^3) \qquad (3.16)$$

which leads to a simplification of (3.5) to

$$\alpha = \alpha' - \frac{1}{2}\sum_{i=1}^{n} \beta_i \sin\rho_i + O(\lambda^3) . \qquad (3.17)$$

In an earlier paper [5] it was argued that

$$\alpha = \alpha' - \frac{1}{2 < q_\mu n^\mu >} \sum_i \beta_i \sin \rho_i \tag{3.18}$$

(the notation has been changed slightly to agree with that in this paper). For this class of weak simplicial spacetimes, where $n^\mu = q^\mu + O(\lambda)$, $< q_\mu n^\mu >= O(1)$. Thus this result agrees exactly with our present result (3.17).

Discussion.

It is natural to ponder whether or not results similar to (3.5) and (3.17) can be obtained *without* the two main assumptions, namely, that the defects are very small and that the timelike bone stands normal to the Cauchy surface in the flat space limit. It is probably not too hard to relax the later condition. This would correspond to having a non-zero shift vector. The analysis would be complicated by the appearance of extra terms in the perturbation expansions given in (3.4). However, the above procedure could possibly be carried through without too much difficulty. In contrast, relaxing the small defect condition will lead to some considerable difficulties. No longer would it be possible to expand the exponentials in (2.8). Consequently it would not be possible to obtain a linear dependence on the two defects α and α'. It is hard to see how, for a non-weak spacetime, that (2.8) and (3.18) can be made to agree. This is not a serious problem, for, as has been stated before, accurate numerical simulations of smooth spacetimes by way of the Regge calculus can only be obtained when all of the defects are very small. This puts us back into the weak spacetime regime.

Appendix.

The following analysis applies to a three dimensional space. However when the results are generalized to four dimensional spaces (a straightforward task) the results remain unchanged.

Consider three legs of a tetrahedron, as depicted in Figure (3), and suppose that the vectors u^μ_1, u^μ_2 are spacelike and that c^μ is timelike. To each pair of these vectors one can construct a bi-vector. The angles between the faces of this tetrahedron can then be computed by contracting pairs of these bi-vectors. The purpose of this appendix is to derive some simple relationships amongst the various angles.

Begin by constructing orthonormal sets of vectors for each of the three legs. Thus choose a^μ, b^μ and c^μ as an orthonormal set of vectors derived from c^μ. Similarly for $u^\mu_1, v^\mu_1, w^\mu_1$ and $u^\mu_2, v^\mu_2, w^\mu_2$. Orient a^μ and b^μ so that a^μ lies in the face spanned by u^μ_1 and c^μ. Likewise, orient v^μ_1, w^μ_1 so that v^μ_1 lies in the same face. It will be convenient to have a similar arrangement for the second spacelike leg. Thus construct a second set of vectors a'^μ, b'^μ and c'^μ by a rotation of a^μ, b^μ and c^μ around c^μ. Orient these vectors so that both a'^μ and v^μ_2 lie in the face spanned by u^μ_2 and c^μ. These sets of vectors are displayed in Figure (3).

It is rather easy to establish the following relationships.

$$\begin{aligned} a'^\mu &= a^\mu \cos\theta + b^\mu \sin\theta \\ b'^\mu &= b^\mu \cos\theta - a^\mu \sin\theta \\ c'^\mu &= c^\mu \end{aligned} \tag{A.1}$$

$$\begin{aligned} u^\mu_1 &= a^\mu \cos\rho_1 - c^\mu \sin\rho_1 \\ v^\mu_1 &= c^\mu \cos\rho_1 + a^\mu \sin\rho_1 \\ w^\mu_1 &= b^\mu \end{aligned} \tag{A.2}$$

$$\begin{aligned} u^\mu_2 &= a'^\mu \cos\rho_2 - c'^\mu \sin\rho_2 \\ v^\mu_2 &= c'^\mu \cos\rho_2 + a'^\mu \sin\rho_2 \\ w^\mu_2 &= b^\mu \end{aligned} \tag{A.3}$$

The angles ρ_1, ρ_2 etc. are defined in Figure (3).

Now consider the bi-vectors associated with the legs (parallel to) u_1^μ, u_2^μ and c^μ defined by

$$\begin{aligned} V_1^{\mu\nu} &= v_1^\mu w_1^\nu - v_1^\nu w_1^\mu \\ V_2^{\mu\nu} &= v_2^\mu w_2^\nu - v_2^\nu w_2^\mu \\ U^{\mu\nu} &= a^\mu b^\nu - a^\nu b^\mu \end{aligned} \tag{A.4}$$

Two important results will now be derived. One will be a proof of the claim in section § 3 that, for weak simplicial spacetimes,

$$\overset{1}{V}_i = \overset{1}{\rho}_i \overset{0}{U} \tag{3.4a}$$

$$[\overset{0}{V}_1, \overset{0}{V}_2] = \sin \overset{0}{\theta} \overset{0}{U} \tag{3.4b}$$

and the other result will be an *elementary* derivation of the result

$$\alpha = \alpha' - \frac{1}{2} \sum_{i=1}^{n} \beta_i \sin \rho_i + O(\lambda^3) \,. \tag{3.17}$$

To prove (3.4a) first notice that from (A.2-4) it follows that

$$V_1^{\mu\nu} = (c^\mu b^\nu - c^\nu b^\mu) \cos \rho_1 + (a^\mu b^\nu - a^\nu b^\mu) \sin \rho_1 \,.$$

For a weak spacetime $\rho_1 = O(\lambda)$ and $V_1^{\mu\nu} = O(1)$ thus,

$$V_1^{\mu\nu} = (\overset{0}{c}{}^\mu \overset{0}{b}{}^\nu - \overset{0}{c}{}^\nu \overset{0}{b}{}^\mu) + (\overset{0}{a}{}^\mu \overset{0}{b}{}^\nu - \overset{0}{a}{}^\nu \overset{0}{b}{}^\mu) \overset{1}{\rho}_1 \,.$$

Comparing this with the perturbation expansion for $V_1^{\mu\nu}$,

$$V_1^{\mu\nu} = \overset{0}{V}{}_1^{\mu\nu} + \overset{1}{V}{}_1^{\mu\nu} \lambda + O(\lambda^2) \,,$$

leads directly to (3.4a). One also obtains

$$\overset{0}{V}{}_1^{\mu\nu} = (\overset{0}{c}{}^\mu \overset{0}{b}{}^\nu - \overset{0}{c}{}^\nu \overset{0}{b}{}^\mu) \,.$$

The corresponding result for $\overset{0}{V}{}_2^{\mu\nu}$ is

$$\overset{0}{V}{}_2^{\mu\nu} = (\overset{0}{c}{}^{\mu}\,\overset{0}{b}{}'^{\nu} - \overset{0}{c}{}^{\nu}\,\overset{0}{b}{}'^{\mu})\ .$$

Combining these last two results with (A.1) will lead to (3.4b).

The elementary proof of (3.5) proceeds as follows. For the two sets of basis vectors generated from u_1^μ and u_2^μ the unit normal to the spacelike face spanned by u_1^μ and u_2^μ can be written as

$$\begin{aligned} n^\mu &= v_1^\mu \cos\beta_1^+ + w_1^\mu \sin\beta_1^+ \\ &= v_2^\mu \cos\beta_2^- - w_2^\mu \sin\beta_2^-\ . \end{aligned}$$

Using (A.1-3) each of the right hand sides can be expressed in terms of the basis a^μ, b^μ and c^μ. It follows, upon comparing coefficients, that

$$\begin{aligned} \cos\beta_1^+\ \cos\rho_1 &= \cos\beta_2^-\ \cos\rho_2 \\ \cos\beta_1^+\ \sin\rho_1 &= \cos\beta_2^-\ \sin\rho_2\ \cos\theta + \sin\beta_2^-\ \sin\theta \\ \sin\beta_1^+ &= \cos\beta_2^-\ \sin\rho_2\ \sin\theta - \sin\beta_2^-\ \cos\theta\ . \end{aligned} \tag{A.5}$$

A simple relation for $\cos\theta'$ can be obtained by forming the scalar product $u_1^\mu u_{2\mu}$, leading to

$$\cos\theta' = \cos\rho_1\ \cos\rho_2\ \cos\theta + \sin\rho_1\ \sin\rho_2\ . \tag{A.6}$$

Now suppose that the spacetime is weak. This means that $\beta_1^+, \beta_2^-, \rho_1$ and ρ_2 are all $O(\lambda)$. Expanding the above relations leads to

$$\rho_1 = (\beta_2^- + \beta_1^+ \cos\theta)/\sin\theta + O(\lambda^2) \tag{A.7}$$

$$\rho_2 = (\beta_1^+ + \beta_2^- \cos\theta)/\sin\theta + O(\lambda^2) \tag{A.8}$$

$$\cos\theta' = \cos\theta - \frac{1}{2}(\rho_1^2 + \rho_2^2)\cos\theta + \rho_1\rho_2 + O(\lambda^3) \tag{A.9}$$

Now write $\rho_1^2 + \rho_2^2$ as $\rho_1\rho_1 + \rho_2\rho_2$ and use (A.7-8) to eliminate *one* factor each of ρ_1 and ρ_2. The result is

$$\cos\theta' = \cos\theta + \frac{1}{2}(\rho_1\beta_1^+ + \rho_2\beta_2^-)\sin\theta + O(\lambda^3)\ .$$

Now since $\theta' \approx \theta$, it follows that

$$\theta' = \theta - \frac{1}{2}(\rho_1 \beta_1^+ + \rho_2 \beta_2^-) + O(\lambda^3)\ .$$

Finally, the defects are defined by $\alpha = 2\pi - \sum_i \theta$ and $\alpha' = 2\pi - \sum_i \theta'$, thus

$$\alpha' = \alpha + \frac{1}{2}\sum_i (\rho_i \beta_i^+) + (\rho_{i+1}\beta_{i+1}^-) + O(\lambda^3)\ .$$

The summation includes all tetrahedra that meet on the timelike leg. A little bit of thought reveals that this sum can be re-arranged as a sum over the triangles meeting on the leg and therefore

$$\alpha = \alpha' - \frac{1}{2}\sum_i \rho_i \beta_i + O(\lambda^3)$$

where $\beta_i = \beta_i^+ + \beta_i^-$ is the boost associated with this triangle. Finally, observe that since $\rho_i = O(\lambda)$ and $\beta_i = O(\lambda)$ then $\rho_i \beta_i = \beta_i \sin \rho_i + O(\lambda^3)$. Thus we may also write the previous expression as

$$\alpha = \alpha' - \frac{1}{2}\sum_i \beta_i \sin \rho_i + O(\lambda^3)$$

which agrees exactly with the earlier result (3.17).

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Fig.1 This figure displays, for a "2+1" spacetime, the regions M, M' and C as defined in section § 3. The cylinder C is the set of points generated by the boundary $\partial D'$. The region M' is the set of points enclosed by D' while M is the set of points inside the cylinder C. The triangles s_1 and s_2 form part of the Cauchy surface. The dotted line is the segment $\partial D'_{12}$. This segment is later deformed into a geodesic segment, see Fig.2

Fig.2 This figure displays the choice of the unit orthonormal basis vectors used in the calculation of the integral $(Kn^{\mu})_{;\mu}$. The dashed path is the segment $\partial D'_{12}$ along which the vectors n^{μ} and m^{μ} are interpolated. The vector w^{μ} is parallel to T_{12} and n^{μ} is normal to the Cauchy surface. The figure has been drawn for a "2+1" spacetime, hence the absence of the vectors t^{μ} and k^{μ}.

Fig.3 This figure defines the various vectors and angles introduced in equations (A.1-3).

$$\boldsymbol{C} \qquad \boldsymbol{C}'$$

$$\boldsymbol{S}_1 \qquad \boldsymbol{S}_2 \qquad \boldsymbol{S}_{12} \qquad \boldsymbol{T}_{12}$$

$$\boldsymbol{S}_1 \qquad \boldsymbol{S}_2 \qquad \boldsymbol{S}_{12} \qquad \boldsymbol{T}_{12}$$

$$\boldsymbol{M} \qquad \boldsymbol{M}' \qquad \boldsymbol{C} \qquad \boldsymbol{\sigma} \qquad \boldsymbol{\sigma}'$$

$$\boldsymbol{u} \qquad \boldsymbol{v} \qquad \boldsymbol{w}$$

$$\boldsymbol{l} \qquad \boldsymbol{m} \qquad \boldsymbol{n}$$

$$\boldsymbol{a} \qquad \boldsymbol{a}' \qquad \boldsymbol{c}$$

$$\boldsymbol{u}_1 \qquad \boldsymbol{v}_1 \qquad \boldsymbol{w}_1$$

$$\boldsymbol{u}_2 \qquad \boldsymbol{v}_2 \qquad \boldsymbol{w}_2$$

$$\boldsymbol{\theta}_{12} \qquad \boldsymbol{\theta}'_{12}$$

$$\frac{\boldsymbol{\pi}}{2} + \boldsymbol{\rho}_1 \qquad \frac{\boldsymbol{\pi}}{2} + \boldsymbol{\rho}_2$$

$$\frac{\boldsymbol{\pi}}{2} + \boldsymbol{\beta}_1^+ \qquad \frac{\boldsymbol{\pi}}{2} + \boldsymbol{\beta}_2^-$$

$$\boldsymbol{\partial D}'_{12} \qquad \boldsymbol{D}'$$

$$\boldsymbol{C} \qquad \boldsymbol{C}'$$

$$\boldsymbol{S}_1 \qquad \boldsymbol{S}_2 \qquad \boldsymbol{S}_{12} \qquad \boldsymbol{T}_{12}$$

$$\boldsymbol{S}_1 \qquad \boldsymbol{S}_2 \qquad \boldsymbol{S}_{12} \qquad \boldsymbol{T}_{12}$$

$$\boldsymbol{M} \qquad \boldsymbol{M}' \qquad \boldsymbol{C} \qquad \boldsymbol{\sigma} \qquad \boldsymbol{\sigma}'$$

$$\boldsymbol{u} \qquad \boldsymbol{v} \qquad \boldsymbol{w}$$

$$\boldsymbol{l} \qquad \boldsymbol{m} \qquad \boldsymbol{n}$$

$$\boldsymbol{a} \qquad \boldsymbol{a}' \qquad \boldsymbol{c}$$

$$\boldsymbol{u}_1 \qquad \boldsymbol{v}_1 \qquad \boldsymbol{w}_1$$

$$\boldsymbol{u}_2 \qquad \boldsymbol{v}_2 \qquad \boldsymbol{w}_2$$

$$\boldsymbol{\theta}_{12} \qquad \boldsymbol{\theta}'_{12}$$

$$\frac{\boldsymbol{\pi}}{2} + \boldsymbol{\rho}_1 \qquad \frac{\boldsymbol{\pi}}{2} + \boldsymbol{\rho}_2$$

$$\frac{\boldsymbol{\pi}}{2} + \boldsymbol{\beta}_1^+ \qquad \frac{\boldsymbol{\pi}}{2} + \boldsymbol{\beta}_2^-$$

$$\boldsymbol{\partial D}'_{12} \qquad \boldsymbol{D}'$$