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FRIEDMANN COSMOLOGIES VIA THE REGGE CALCULUS

Leo Brewin
Department of Physics
University of British Columbia
British Columbia, V6T2A6
Canada

The detailed construction of six Regge spacetimes, each being an approximation to a time symmetric Friedmann dust filled universe, will be presented. These spacetimes are a generalization of those originally constructed by Collins and Williams [1] . This paper will present new methods for the subdivision of each Cauchy surface into a set of tetrahedra, for the construction of the general 4-dimensional block and for the implementation of the constraints of homogeneity and isotropy. A new action sum for pure dust in a Regge spacetime will also be presented. The evolution of the Regge spaces will be seen to terminate prior to the full collapse of the universe. This will be shown to occur when the particle horizon for an observer at the centre of one tetrahedron has contracted so as to just touch the vertices of that tetrahedron. It is argued that this is a generic feature and will occur in any Regge spacetime whenever the local curvature becomes too large.

1. Introduction

The Regge spacetimes that will be discussed in this paper are based on those originally presented by Collins and Williams [1] . Their spacetimes were chosen as models of the evolution of a dust filled universe with a given a topology of $R \otimes S^3$. This model was chosen for its dynamics could be easily compared with that of the standard Friedmann dust cosmology. This comparison would then provide a measure of the suitability of the Regge Calculus [2] as a tool in the construction of numerical spacetimes.

Collins and Williams chose to foliate the spacetime into a sequence of Cauchy surfaces each with topology S^3 and with each such surface being subdivided into a set of tetrahedra. The regions between pairs of Cauchy surfaces were built from sets of short 4-dimensional tubes. Each Cauchy surface was required to be, in some way, homogeneous and isotropic. As these symmetries could not be imposed directly Collins and Williams chose to mimic these symmetries by imposing certain restrictions on the distribution of tetrahedra in each Cauchy surface and to constrain the leg lengths to being all equal. As a consequence it was only possible to construct models in which there were 5,16 or 600 equilateral tetrahedra in each Cauchy surface. This step also reduced the number of dynamical variables to just two, namely, the typical leg length in one Cauchy surface and the energy density of the dust.

Rather than dealing with the full set of Regge equations Collins and Williams chose to look at the simpler set of equations obtained by reducing the time step to zero. They retained only one equation after this step. The resulting numerical solutions of this differential equation did show a reasonable similarity to the exact Friedmann cosmology. It was not suprising that the most accurate solutions were obtained from the models consisting of 600 tetrahedra in each Cauchy surface.

Collins and Williams enjoyed some good fortune in obtaining this accord bewteen the numerical and analytical soultions. They chose an energy-momentum Lagrangian that did not represent pure dust and their argument that some of the differential equations are trivially satisfied was not correct. One of the purposes of this paper is to correct these inaccuracies. The principal differences between the approach of Collins and Williams and that to be presented here involves the choice of the Lagrangian for pure dust, the subdivision of the spacetime into the set of 4-dimensional blocks and the implementation of the constraints.

The basic structure of their spacetimes (ie. filled with dust and of topology $R \otimes S^3$) will be retained.

In section § 2 the energy-momentum Lagrangian for pure dust will be presented. In the following four sections § 3,4,5 and 6 a detailed analysis of the topological and metrical properties of our model spaces will be presented. The discussion in the two sections § 3 and 4 will be confined to the method by which the individual Cauchy surfaces are constructed. The combinatorial structure of the three primary models used by Collins and Williams will be presented in section § 3. Three related models, the secondary models, will be developed in the following section by subdividing each of the primary models. An analysis of a general construction of a 4-dimensional block for any Regge spacetime will be given in section § 5. This scheme will then be adapted in section § 6 for this particular set of models. In section § 7 the issue of whether the constraints should be imposed before or after the action is varied will be addressed. It will be shown that the two possible sets of equations will generally not be equivalent. The field equations derived by imposing the constraints before the action is varied will be presented in section § 9 and will be the basic equations from which the solutions of section § 12 are obtained. In sections § 10 and 11 the equations by which the numerical spacetimes are compared with the exact Friedmann cosmology are derived. This comparison is made in section § 12 where it will be noted that the evolution of our discrete spaces does look much like the expected behaviour. There is, however, one unexpected feature in that the evolution of the numerical space terminates before the universe collapses to zero volume. This point will be investigated in the final section § 13 by employing the approach of Collins and Williams in developing a related set of equations by reducing the time step to zero. It will be shown that the premature termination of the evolution is not a numerical accident but that it arises from the collapse of a particle horizon onto the vertices of the individual tetrahedra.

2. The Energy-Momentum action integral

The non-vacuum Regge action has been defined [1] as

$$I = 2\sum_i \theta_i A_i \; - \; \frac{16\pi G}{c^4}\sum_j \int_{M_j} \mathcal{L}\sqrt{-g}d^4x \tag{2.1}$$

where θ_i is the defect on the bone with area A_i and M_j is a typical block of the spacetime. Unlike the gravitational part of the action, in which there are contributions from the lower dimensional interfaces, there are no such terms in the matter action since $\mathcal{L}\sqrt{-g}$ depends only on g and ∂g.

For pure dust it is well known [3,4] that

$$\int_{M_j} \mathcal{L}\sqrt{-g}d^4x = \sum_i e_i \int ds_i \tag{2.2}$$

where the summation includes all of the particles inside M_j , e_i is the energy of the i^{th} particle and s_i is the proper distance measured along the path of that particle. It is quite obvious from this matter action that, within each M_j, the particle travels along segments of geodesics and that their energies are conserved.

At this point no assumptions will be made upon the continuity of the particles energy and the direction of its trajectory as the particle crosses each Cauchy surface. Of course on a smooth spacetime these quantities are continuous. This situation arises from the fact that the equations of motion of the matter field may be written as a conservation law. It is not clear whether the same situation applies in the Regge calculus. This point will be discussed further in section § 8.

For simplicity it will be assumed throughout this paper that there is only one particle within each M_j. Substituting the matter action(2.2) back into the general action sum(2.1) leads to

$$I = 2\sum_i \theta_i A_i \; - \; \frac{16\pi G}{c^4}\sum_i e_i \Delta s_i \tag{2.3}$$

where Δs_i is the proper distance of the section of the geodesic within the 4-dimensional block. The associated field equations are

$$0 = 2\sum_i \theta_i \frac{\partial A_i}{\partial L_k} - \frac{16\pi G}{c^4}\sum_i e_i \frac{\partial \Delta s_i}{\partial L_k} \qquad k = 1, 2, 3, \cdots \tag{2.4}$$

where L_k is any one of the dynamical variables.

Collins and Williams chose a different matter action. They argued that if the temporal extent of each block was sufficiently small then the energy density was essentially constant everywhere within each block. Thus they put

$$\int_{M_j} \mathcal{L}\sqrt{-g}d^4x = \rho c^2 \int_{M_j} \sqrt{-g}d^4x = \rho c^2 V(M_j)$$

where ρc^2 is the energy density and $V(M_j)$ is the 4-volume of the block. However for this matter action the associated energy-momentum tensor (when computed on a smooth spacetime) is

$$T^{\mu\nu} = -\rho c^2 g^{\mu\nu} \ .$$

This certainly does not represent pure dust and consequently this form of the matter action will not be considered any further in this paper.

3. Subdividing each Cauchy surface

In this section our only interest is in the subdivision of each Cauchy surface into a set of tetrahedra. As this will reduce to a pure combinatoric problem there will be no need to introduce a metric at this stage. That issue will be dealt with in a later section § 6.

In all six different spacetimes will be considered in this paper. Three of the spacetimes, the primary models, will have the same basic structure as the three Collins and Williams models. The three remaining spacetimes, the secondary models, will be obtained by a subdivision of the three primary models. In each of the models each Cauchy surface is chosen to have the topology of a 3-sphere. For each of the three primary models each Cauchy surface is constructed as a regular subdivision of the 3-sphere into a set of regular tetrahedra. This is a classical problem in combinatorial topology and is fully discussed by Coxeter [5] . The numbers of vertices, legs, etc. in each of the three primary models is listed in Table I.

P_2	N_0	N_1	N_2	N_3
3	5	10	10	5
4	8	24	32	16
5	120	720	1200	600

Table I. The numbers of simplicies in the three primary models.

4. The generalized subdivisions

If M is a simplicial complex with the topology of S^3 then another simplicial complex M' can be generated by subdividing each tetrahedron of M into a set of new tetrahedra. If the process of subdivision has been properly chosen then it will be possible to repeat that process indefinitely thus generating an infinite sequence of simplicial complexes each of which is topologically equivalent to S^3.

The subdivision scheme to be used here is represented by Fig (4.1). This scheme is chosen in preference to others for it ensures that the incidence numbers (eg. the number of legs meeting at each vertex) will remain bounded for any number of subdivisions. This avoids any problems that might arise by overcrowding the vertices with legs.

By inspection of Fig (4.1) one can easily deduce the numbers of the various simplicies appearing in the three secondary models. This information is listed in Table II.

		Number of simplicies		
Index	Simplex	$P_2 = 3$	$P_2 = 4$	$P_2 = 5$
1	$(12, 13)$	30	96	3600
2	$(1, 12)$	20	48	1440
3	$(12, 1234)$	30	96	3600
1	$(12, 13, 14)$	20	64	2400
2	$(12, 13, 23)$	10	32	1200
3	$(1, 12, 13)$	30	96	3600
4	$(12, 13, 1234)$	60	192	7200
1	$(1, 12, 13, 14)$	20	64	2400
2	$(12, 13, 14, 1234)$	40	128	4800

Table II. The indicies and numbers of all simplicies in the three secondary models.

5. The general 4-tube

There can be no sense of dynamics until the region between each pair of Cauchy surfaces has been subdivided into a set of 4-dimensional blocks. These blocks will be the fundamental blocks of the spacetime. The various tetrahedra in each Cauchy surface will be viewed as the "past" and "future" spacelike faces of the basic block.

The trajectory of any one tetrahedron, its worldtube, will be a 4-dimensional timelike tube in the spacetime. The section of this tube that lies between a pair of Cauchy surfaces will be used as the typical 4-dimensional block.

Denote the sequence of Cauchy surfaces by S_i, $i = 0, \pm 1, \pm 2, \cdots$ and the tubes by T_i, $i = 1, 2, 3, \cdots$. The tetrahedron that lies at the intersection of T_i with S_j will be represented by s_{ij}. The section of T_i that lies between S_j and S_{j+1}, including s_{ij} and s_{ij+1}, will be denoted by T_{ij}.

Consider now one leg (ab) of s_{ij} and follow its path as it evolves forward into the leg $(a'b')$ of s_{ij+1}. This leg will generate a 2-dimensional surface which will not, in general, be planar (ie. the leg may rotate as it evolves) and therefore would not be suitable as a face of T_{ij}. This difficulty can be overcome by introducing one diagonal leg for each face and then re-defining the face so that it is composed of just two triangles. The diagonal can be chosen to join either a with b' or b with a'. The two possible faces generated by these choices will not, unless the original face was planar, be coincident. There would appear to be no obvious reason for choosing one diagonal instead of the other. The chosen diagonal will be referred to as the principal diagonal and the rejected diagonal as the auxiliary diagonal. Once all of the principal diagonals have been chosen the connection matrix of the generic block will have been fully specified.

It is important to note that the defects on each of these two triangles must be equal. Consider one principal diagonal with vertices (a) and (b') . There will be just two bones on this diagonal, namely, (abb') and $(ba'b')$. That the defects on these two bones must be equal should be obvious once one recalls that each bone is surrounded by the same sequence of 4-tubes and that any vector that is parallel transported around any one of these bones will be subjected to a rotation that depends only on that sequence of 4-tubes. As the angle of rotation of the vector is equal to the defect it follows that the defects on the two adjacent bones must be equal.

It is not difficult to prove that the geometry of the generic block is fully determined by specifying the twelve leg lengths in both s_{ij} and s_{ij+1}, the lengths of the four legs that separate the Cauchy surfaces and the lengths of just six of the diagonals. This provides 22 pieces of data from which one may deduce the ten $g_{\mu\nu}$'s and the coordinates of three of the eight vertices. The coordinates of the remaining five vertices may be chosen freely.

In the subsequent sections of this paper the timelike legs will be referred to as struts while the spacelike and diagonal legs will be referred to as legs and diagonals respectively.

6. A simple 4-tube

The constraints that are to be imposed on our Regge spacetimes are

i) that each and every Cauchy surface is a model of the geometry of a 3-sphere and

ii) that the sequence of Cauchy surfaces be parallel and

iii) that there is no twist or shear along each 4-tube and

iv) that there is no spatial variation in the energy densities.

The first and the last constraints are intended to convey the notions of homogeneity and isotropy. The usual definitions of these symmetries (that the space admits certain continuous symmetry groups) can not be applied here for the symmetry groups of a simplicial complex are discrete. The second and third constraints may be viewed as a choice of lapse and shift functions.

The scenario envisaged here is that of each and every tetrahedron being subjected to a homogeneous and isotropic collapse (or expansion). It is quite a simple step to translate these requirements into restrictions on the leg, strut and diagonal lengths.

Choose any one tube segment T_{ij}. Denote the leg lengths of s_{ij}, in the base tetrahedron of T_{ij}, by L_{ab}, the lengths of s_{ij+1} by $L_{a'b'}$, the strut lengths by t_a and the diagonal lengths by $d_{ab'}$. In some later parts of the discussion it will be necessary to distinguish between the future, present and past quantities associated with the current Cauchy surface. Thus $L^{\uparrow}_{ab}$ will represent the value of L_{ab} in s_{ij+1} while $t^{\uparrow}_{a}$ will be the length of the strut (aa') and $d^{\uparrow}_{ab}$ will be the length of the diagonal (ab'). Similar definitions will apply to the related quantities $L^{\downarrow}_{ab}, t^{\downarrow}_{a}$ and $d^{\downarrow}_{ab}$ with the one exception that the vertices a' and b' are now vertices of s_{ij-1} rather than s_{ij+1}. The constraints may now be expressed (for all valid combinations of i and j) as

i) $L_{ij} = \alpha_{ij} L$,

ii) $t_i = t$,

iii) $d_{ij'} = d_{ji'}$,

iv) $e_i/V_i = e_j/V_j$,

where the α_{ij}'s are absolute constants, L and t are typical leg and strut lengths and V_i is the 3-volume of the i^{th} tetrahedron. Both L and t are constant across each Cauchy surface but may vary from slice to slice. It should be clear that not all of the $L's, t's$ and $d's$ can be independent for this class of tubes.

If the coordinates (x^μ) of the vertices a, b, c, d of s_{ij} are chosen as

$$
\begin{aligned}
a &: (0,0,0,0)\ , \\
b &: (1,0,0,0)\ , \\
c &: (0,1,0,0)\ , \\
d &: (0,0,1,0)\ ,
\end{aligned}
\tag{6.1}
$$

then the coordinates of the vertices a', b', c', d' of s_{ij+1} may be chosen as

$$
\begin{aligned}
a' &: (0,0,0,1)\ , \\
b' &: (\mu,0,0,1)\ , \\
c' &: (0,\mu,0,1)\ , \\
d' &: (0,0,\mu,1)\ .
\end{aligned}
\tag{6.2}
$$

This choice of coordinates ensures that the first condition is automatically satisfied. The quantity μ is being used as a conformal factor and may be computed as

$$
\mu = \frac{L_{a'b'}}{L_{ab}} = \frac{L_{a'c'}}{L_{ac}} = \cdots = \frac{L_{c'd'}}{L_{cd}}\ .
$$

Assume for the moment that all ten of the g_{ij} are known. The $L's$ and $t's$ could then be calculated from the equations

$$
\begin{aligned}
-t_a^2 &= g_{44} \,, \\
-t_b^2 &= (1-\mu)^2 g_{11} - 2(1-\mu) g_{14} + g_{44} \,, \\
-t_c^2 &= (1-\mu)^2 g_{22} - 2(1-\mu) g_{24} + g_{44} \,, \\
-t_d^2 &= (1-\mu)^2 g_{33} - 2(1-\mu) g_{34} + g_{44} \,,
\end{aligned} \tag{6.3}
$$

$$
\begin{aligned}
L_{ab}^2 &= g_{11} \,, \\
L_{ac}^2 &= g_{22} \,, \\
L_{ad}^2 &= g_{33} \,, \\
L_{bc}^2 &= g_{11} + g_{22} - 2g_{12} \,, \\
L_{bd}^2 &= g_{11} + g_{33} - 2g_{13} \,, \\
L_{cd}^2 &= g_{22} + g_{33} - 2g_{23} \,.
\end{aligned} \tag{6.4}
$$

Since all of the $t's$ must be equal (from condition (ii)) the last three equations in (6.3) may be reduced to

$$
\begin{aligned}
g_{14} &= \frac{1}{2}(1-\mu) g_{11} \,, \\
g_{24} &= \frac{1}{2}(1-\mu) g_{22} \,, \\
g_{34} &= \frac{1}{2}(1-\mu) g_{33} \,.
\end{aligned} \tag{6.5}
$$

These equations together with (6.4) and the first equation of (6.3) may be used to fully determine the ten $g'_{ij}s$ as functions of only the $L's, t's$ and μ. Consequently all of the $d's$ must be expressible as functions of those quantities. Contrast this situation with that for the generic tube in which six of the $d's$ are truely independent.

Consider a typical pair of diagonals ab' and ba'. Their lengths may be calculated from

$$
\begin{aligned}
d_{ab'}^2 &= \mu^2 g_{11} + 2\mu g_{14} + g_{44} \,, \\
d_{ba'}^2 &= g_{11} - 2g_{14} + g_{44} \,.
\end{aligned}
$$

Both of these equations are, in fact, equivalent and may be reduced to

$$d_{ab'}^2 = d_{ba'}^2 = -t_a^2 + L_{ab} L_{a'b'} \; . \tag{6.6}$$

This equation applies to each pair of diagonals in the tube. Notice that the requirement that $d_{ab'} = d_{ba'}$ for each and every ab' and ba' leads directly to the equations (6.5) and thus does not provide any new information.

It is interesting to note that this class of 4-tubes can also be constructed by truncating a full 4-simplex. To show that this is true first recall that the various parallel (spacelike) cross-sections of the 4-tube are dilated images of the base tetrahedron. Thus the worldlines of the vertices when extended beyond one of the two end tetrahedra will meet at one common point. This construction yields a 4-simplex which in turn can be truncated to duplicate the original 4-tube segment.

6.1 The geometrical quantities

A variety of geometrical data, the heights, areas, volumes and defects associated with this class of 4-tubes will be required in the subsequent discussion. The derivation of the formulae for these quantities is a relatively trivial but tedious task.

The typical tetrahedron of the preceeding section possessed no special symmetries other than that the ratios of the legs should remain constant throughout the evolution of that tetrahedron and that the faces of the tube should be planar. In this section a much more specific class of tetrahedra will be considered in which the six leg lengths are chosen so that

$$\begin{aligned} L^2_{ab} &= L^2_{ac} = L^2_{ad} \,, \\ L^2_{bc} &= L^2_{bd} = L^2_{cd} \,. \end{aligned} \tag{6.7}$$

These conditions will be derived, in a later section § 10, by requiring the geometry of each Cauchy surface to look like that of a 3-sphere.

If the following quantities are introduced

$$\begin{aligned} e &= L^2_{ab} \,, \\ f &= L^2_{bc} \,, \\ g &= e - f/2 \,, \\ h &= \frac{1}{2}(1-\mu)e \,, \\ j &= -t^2 \,, \end{aligned} \tag{6.8}$$

then the components of the metric, in the coordinate frame of the previous section, are

$$(g_{ij}) = \begin{pmatrix} e & g & g & h \\ g & e & g & h \\ g & g & e & h \\ h & h & h & j \end{pmatrix} \,. \tag{6.9}$$

The contravariant components are

$$(g^{ij}) = \begin{pmatrix} E & G & G & H \\ G & E & G & H \\ G & G & E & H \\ H & H & H & J \end{pmatrix} , \tag{6.10}$$

with

$$\begin{aligned} E &= \frac{2h^2 - j(e+g)}{(e-g)(3h^2 - j(e+2g))} , \\ G &= \frac{gj - h^2}{(e-g)(3h^2 - j(e+2g))} , \\ H &= \frac{h}{3h^2 - j(e+2g)} , \\ J &= \frac{-(e+2g)}{3h^2 - j(e+2g)} . \end{aligned} \tag{6.11}$$

6.2 Heights

The height of the 4-tube segment may be calculated as the length of the vector that is normal to and joins both of the end-tetrahedra of the tube. The covariant components of the normal to the base, $n_i(abcd)$, can be easily shown to be

$$n_i(abcd) = (0, 0, 0, 1)_i = \delta_i^4 .$$

Consider now the path, parallel to $n^i(abcd)$, which passes through the vertex (a). The points on this path are characterized by

$$x^i(\tau) = \tau n^i(abcd)$$

where τ is some parameter. The point in the upper tetrahedron through which this path passes must have $x^4 = 1$. Thus, at this point, $\tau = 1/n^4$ and the (real) length of this path is

$$\Delta s = \left| \frac{g_{ij} n^i n^j}{(n^4)^2} \right|^{1/2}$$

which may be simplified to

$$\Delta s = \left| \frac{1}{g^{44}} \right|^{1/2} = \left(t^2 + \frac{3(1-\mu)^2}{12 - 4f/e} L_{ab}^2 \right)^{1/2} . \qquad (6.12)$$

Upon eliminating μ this equation becomes

$$\Delta s = \left(t^2 + \frac{3e}{12e - 4f} (L_{ab} - L_{a'b'})^2 \right)^{1/2} . \qquad (6.13)$$

6.3 Areas

The set of two-dimensional faces of the 4-tubes, the bones, consists of spacelike triangles and timelike trapeziums. Calculating the areas of these faces is a simple task.
For any spacelike triangle with leg lengths L_{ab}, L_{ac} and L_{bc} the area, B, may be calculated from

$$B = \frac{1}{2} \left((L_{ab} L_{ac})^2 - \frac{1}{4} (L_{ab}^2 + L_{ac}^2 - L_{bc}^2)^2 \right)^{1/2} . \qquad (6.14)$$

Suppose that the timelike trapezium has struts of square length $-t^2$ and legs of square length L_{ab}^2 and $L_{a'b'}^2$. The (real) area is then

$$A = \left(\frac{L_{ab} + L_{a'b'}}{2} \right) \left(t^2 + \frac{1}{4} (L_{ab} - L_{a'b'})^2 \right)^{1/2} . \qquad (6.15)$$

6.4 Volumes

The volume of a typical tetrahedron in any one of our models is given by

$$V = \frac{L_{ab}^3}{12} \left(3\frac{f}{e} - 1 \right)^{1/2} . \qquad (6.16)$$

6.5 Defects

In the previous section the areas of all of the bones were defined to be real. Regge however chose to define the area of a timelike bone as being a real number and that for a spacelike bone as purely imaginary. The defects associated with these bones were then defined so that the contribution to the action would be a real number. If the same Regge action is to be used with our modified definition of areas then a change in the definition of the defects will be required.

Any reasonable definition of the angle between any pair of vectors must satisfy the addition property, namely,

$$\theta(\mathbf{u},\mathbf{v}) = \theta(\mathbf{u},\mathbf{w}) - \theta(\mathbf{v},\mathbf{w}) \tag{6.17}$$

for any tripple of colinear vectors $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$. For vectors that lie in a spacelike plane define

$$\theta(\mathbf{v},\frac{\partial}{\partial x}) = \begin{cases} \mathrm{Cos}^{-1}(\mathbf{v}\cdot\frac{\partial}{\partial x}/\lambda) & \text{in the upper half plane,} \\ \pi + \mathrm{Cos}^{-1}(\mathbf{v}\cdot\frac{\partial}{\partial x}/\lambda) & \text{in the lower half plane,} \end{cases} \tag{6.18a}$$

while for the vectors that lie in a timelike plane define

$$\theta(\mathbf{v},\frac{\partial}{\partial x}) = \begin{cases} \mathrm{Cosh}^{-1}(\mathbf{v}\cdot\frac{\partial}{\partial x}/\lambda) & \text{in I,} \\ \mathrm{Sinh}^{-1}(\mathbf{v}\cdot\frac{\partial}{\partial x}/\lambda) & \text{in II,} \\ -\,\mathrm{Cosh}^{-1}(\mathbf{v}\cdot\frac{\partial}{\partial x}/\lambda) & \text{in III,} \\ -\,\mathrm{Sinh}^{-1}(\mathbf{v}\cdot\frac{\partial}{\partial x}/\lambda) & \text{in IV,} \end{cases} \tag{6.18b}$$

where the four regions I,II,III and IV are as indicated in Fig (6.1) and $\lambda = |\mathbf{v}||\partial/\partial x|$. These definitions are chosen so that θ is an increasing function in the counter-clockwise direction and that the right hand side of (6.17) is independent of $\mathbf{w}$. In particular, if $\mathbf{u}$ is in region I and $\mathbf{v}$ is in region II then

$$\theta(\mathbf{v},\mathbf{u}) = \mathrm{Sinh}^{-1}(v_\mu u^\mu) \tag{6.19}$$

provided $|v_\mu v^\mu| = |u_\mu u^\mu| = 1$.

The total increment in the Lorentzian angle for one complete loop of the origin is zero whereas for the Euclidean angle the total increment in 2π. Thus the defect for a timelike bone will be computed as

$$\alpha = 2\pi - \sum \alpha_i \ ,$$

while for a spacelike bone the defect will be

$$\beta = \sum \beta_i \ .$$

The α_i and β_i are just the angles between adjacent 3-dimensional faces of the 4-tubes. The typical arrangement of 4-tubes that surround either a timelike or spacelike bone are indicated in Fig (6.2). The circle in the centre of each diagram is a shorthand representation of the bone.

The explicit formulae for the defects in the primary models are not too hard to derive. Let $n_\mu(ijkl)$ be a normalized (ie. $n_\mu n^\mu = \pm 1$) normal to the 3-dimensional face $(ijkl)$. Consider first the timelike bones. Each such bone is surrounded by 3,4 or 5 *identical* 4-tubes. The defect will therefore be

$$\alpha = 2\pi - n\alpha_1 \ , \tag{6.20}$$

with $n = 3, 4$ or 5 and α_1 is the interior angle within one 4-tube. In the coordinate frame of section § 6 the covariant components of the normalized normals to the two 3-dimensional faces adjacent to the bone $(aba^\uparrow)$ are

$$n_\mu(aba^\uparrow c) = \frac{+\delta^2_\mu}{(g^{22})^{1/2}} \ ,$$

$$n_\mu(aba^\uparrow d) = \frac{-\delta^3_\mu}{(g^{33})^{1/2}} \ .$$

The $\pm$ signs have been chosen to ensure that the n_μ point in the counter-clockwise direction around the bone. Consequently

$$\cos\alpha_1 = \frac{-g^{\mu\nu}\delta^2_\mu \delta^3_\nu}{(g^{22}g^{33})^{1/2}} \ ,$$

which may be reduced to

$$\cos\alpha_1 = \frac{(\Delta L^{\uparrow})^2 + 2(\Delta t^{\uparrow})^2}{2(\Delta L^{\uparrow})^2 + 6(\Delta t^{\uparrow})^2}\ , \tag{6.21}$$

in which

$$\begin{aligned} \Delta L^{\uparrow} &= L^{\uparrow}_{ab} - L_{ab}\ , \\ \Delta t^{\uparrow} &= t^{\uparrow}\ . \end{aligned}$$

The spacelike bones are surrounded by four 4-tubes : two identical 4-tubes that lie in the future of the bone and two identical 4-tubes that lie in the past of the bone. Denote the interior angles in the future and past 4-tubes by $\beta^{\uparrow}$ and $\beta^{\downarrow}$ respectively. The defect is then expressed as

$$\beta = 2(\beta^{\uparrow} + \beta^{\downarrow}) \tag{6.22}$$

The two normalized normals associated with the future 4-tubes for the bone (abc) have covariant components

$$\begin{aligned} n_\mu(abcd) &= \frac{+\delta^4_\mu}{(-g^{44})^{1/2}}\ , \\ n_\mu(abca^{\uparrow}) &= \frac{-\delta^3_\mu}{(+g^{33})^{1/2}}\ . \end{aligned}$$

The future interior angle, $\beta^{\uparrow}$, is the principal solution of

$$\begin{aligned} \sinh\beta^{\uparrow} &= \frac{-g^{\mu\nu}\delta^3_\mu\delta^4_\nu}{(-g^{33}g^{44})^{1/2}}\ , \\ &= \frac{\Delta L^{\uparrow}}{(8(\Delta L^{\uparrow})^2 + 24(\Delta t^{\uparrow})^2)^{1/2}}\ . \end{aligned} \tag{6.23}$$

Similarily, for $\beta^{\downarrow}$ the appropriate formula is

$$\sinh\beta^{\downarrow} = \frac{\Delta L^{\downarrow}}{(8(\Delta L^{\downarrow})^2 + 24(\Delta t^{\downarrow})^2)^{1/2}}\ , \tag{6.24}$$

with

$$\Delta L^{\downarrow} = L^{\downarrow}_{ab} - L_{ab} \; ,$$
$$\Delta t^{\downarrow} = t^{\downarrow} \; .$$

7. The field equations : I

The point to be considered in this section is whether the constraints can be imposed before or after the variations of the action are performed. Local variations can only be performed on the unconstrained action and will lead to field equations which are functions of only the local geometry of the spacetime. However, if the action is constrained prior to the process of extremization the resulting field equations will be functions of the global geometry. These field equations need not be identical with the previous set since small scale variations have been excluded. It is also possible that the boundary terms in the global variations may introduce extra terms which would not arise when performing local variations.

The complete action sum for the Regge Friedmann dust cosmology is

$$I = 2\sum_i A_i\alpha_i + 2\sum_i B_i\beta_i - \frac{16\pi G}{c^4}\sum_i e_i\Delta s_i \tag{7.1}$$

This sum includes the contributions to I from all of the objects in the spacetime. It is easier however to focus attention on only those terms in the above expression which arise from the objects *on* or *between* a pair of Cauchy surfaces. Let $J_i^{\uparrow}$ denote the contribution to I from the terms associated with those objects on and between S_i and S_{i+1}. Thus

$$J_i^{\uparrow} = \left(2\sum_{j=1}^{N_1} A_j\alpha_j + \sum_{j=1}^{N_2} B_j\beta_j - \frac{16\pi G}{c^4}\sum_{j=1}^{N_3} e_j\Delta s_j \right)_i^{\uparrow} \tag{7.2}$$

where the final pair of indices on the right hand side indicates that that expression should be evaluated on the slice between S_i and S_{i+1}. The numbers N_1, N_2 and N_3 represent the number of legs, triangles and tetrahedra in each Cauchy surface, (these have already been listed in section § 3 and 4). In a similar fashion $J_i^{\downarrow}$ may be defined as

$$J_i^{\downarrow} = J_{i-1}^{\uparrow} \tag{7.3}$$

and represents the terms that arise from the slice between S_i and S_{i-1}.

Let $L_j, j = 1, 2, 3, \cdots N_1$, represent the leg lengths in S_i, $L_{j'}$ the leg lengths in S_{i+1}, t_j the strut and d_j the diagonal lengths between S_i and S_{i+1}.

The first set of field equations are obtained by the unconstrained extremization of $J^{\uparrow} + J^{\downarrow}$ (the redundant subscripts on the J's have been dropped since the form of the field equations will be independent of the chosen pair of Cauchy surfaces). These equations will be referred to as the *local field equations* and may be written (symbolicaly) as

$$0 = \frac{\partial(J^{\uparrow} + J^{\downarrow})}{\partial L_i}\,, \tag{7.4}$$

$$0 = \frac{\partial J^{\uparrow}}{\partial t_i}\,, \tag{7.5}$$

$$0 = \frac{\partial J^{\uparrow}}{\partial d_i}\,. \tag{7.6}$$

The constraints would now be imposed upon these equations.

Let $(J^{\uparrow})^c$ and $(J^{\downarrow})^c$ be the constrained partial actions. The notation $(\cdots)^c$ will denote that the quantity in the brackets has been constrained. The second set of field equations are

$$0 = \frac{\partial(J^{\uparrow} + J^{\downarrow})^c}{\partial L}\,, \tag{7.7}$$

$$0 = \frac{\partial(J^{\uparrow})^c}{\partial t}\,, \tag{7.8}$$

and will be referred to as the *global field equations.* There is quite a simple relation between the global and constrained local field equations which can be obtained by applying the chain rule to the global field equations. The result is

$$\frac{\partial (J^\uparrow + J^\downarrow)^c}{\partial L} = \sum_i \left(\frac{\partial J^\uparrow + J^\downarrow}{\partial L_i}\right)^c \left(\frac{\partial L_i}{\partial L}\right)^c + \left(\frac{\partial J^\uparrow + J^\downarrow}{\partial d_i}\right)^c \left(\frac{\partial d_i}{\partial L}\right)^c , \tag{7.9}$$

$$\frac{\partial (J^\uparrow)^c}{\partial t} = \sum_i \left(\frac{\partial J^\uparrow}{\partial t_i}\right)^c \left(\frac{\partial t_i}{\partial t}\right)^c + \left(\frac{\partial J^\uparrow}{\partial d_i}\right)^c \left(\frac{\partial d_i}{\partial t}\right)^c . \tag{7.10}$$

This shows clearly that the global field equations are linear combinations of the constrained local field equations.

One can quite easily see that if the constrained local field equations does have a solution then that solution will also be a solution of the global field equations. It should also be clear that the converse need not always be true. Thus one can not assert, except in some special cases, that the two sets of field equations and their associated solutions are equivalent. Which of the two sets of field equations are to be considered more fundamental is a question of philosophy. In choosing the global field equations one is guaranteed of obtaining equations that are compatible with the constraints. The alternative approach, to use the constrained local field equations, might lead to an inconsistent set of equations. If this situation did arise then one would be forced to remove the constraints and allow the Regge space to evolve in accord with the (unconstrained) local field equations. These problems will be avoided in this paper by considering only the global field equations.

However, for the primary models of section § 3, the global and constrained local field equations can be shown to have identical solutions. As each vertex, leg, triangle, tetrahedron and 4-tube in any Cauchy slice is identical to any other vertex, leg etc. in that Cauchy slice it should be obvious that the equations (7.9) and (7.10) may be reduced to

$$\frac{\partial (J^\uparrow + J^\downarrow)^c}{\partial L} = N_1 \left(\frac{\partial J^\uparrow + J^\downarrow}{\partial L_0}\right)^c \frac{\partial L_0}{\partial L} + N_1 \left(\frac{\partial J^\uparrow + J^\downarrow}{\partial d_0}\right)^c \frac{\partial d_0}{\partial L} , \tag{7.11}$$

$$\frac{\partial (J^\uparrow)^c}{\partial t} = N_0 \left(\frac{\partial J^\uparrow}{\partial t_0}\right)^c \frac{\partial t_0}{\partial t} + N_0 \left(\frac{\partial J^\uparrow}{\partial d_0}\right)^c \frac{\partial d_0}{\partial t} , \tag{7.12}$$

where L_0, t_0 and d_0 are the lengths of one typical leg, strut and diagonal. Suppose that it is possible to choose the diagonals so that

$$0 = \left(\frac{\partial J^{\uparrow}}{\partial d_0}\right)^c = \left(\frac{\partial J^{\downarrow}}{\partial d_0}\right)^c . \tag{7.13}$$

Then a solution to the global field equations will also be a solution of the constrained local field equations. In pursuing this line of investigation it will be necessary to temporarily ignore the third constraint and instead ask what is the appropriate relation between d, L and t so that the equation (7.13) is identically satisfied.

A substitution of (7.2) into (7.6) leads to

$$0 = \left(2\alpha\frac{\partial A}{\partial d} - \frac{16\pi G}{c^4}P_2 e\frac{\partial \Delta s}{\partial d}\right)^c \tag{7.14}$$

in which $P_2 = 3, 4$ or 5 and the redundant subscripts have been discarded. The quantity A represents the area of the 2-dimensional timelike face of a typical 4-tube and must be viewed as function of L_1, L_2, t_1, t_2 and d (ie. the two legs, the two struts and the one diagonal). This face is composed of two adjacent triangles that share the diagonal as their common edge. It is then just a simple matter to show that

$$0 = \left(\frac{\partial A(L_1, L_2, t_1, t_2, d)}{\partial d}\right)^c$$

for any L_1, L_2 and t (the common value of t_1 and t_2) if and only if

$$d^2 = -t^2 + L_1 L_2 . \tag{7.15}$$

The equation (7.14) now reduces to

$$0 = \left(\frac{\partial \Delta s}{\partial d}\right)^c . \tag{7.16}$$

One can attempt to investigate this equation in the same way as for the previous equation by asking what is the appropriate dependence of d on L_i and t_j such that this equation is identically satisfied. It is possible that the relationship so derived might differ from that just previously derived. In order to avoid this conflict a different approach will be used. The idea is to choose the dependence of Δs on L_i, t_i and d so that (7.16) is identically satisfied. Recall that Δs is the length of the geodesic segment of the dust particle's worldline that is contained within one 4-tube. The points at which the worldline enters (somewhere in S_i) and leaves (somewhere in S_{i+1}) have not been explicitly chosen. Our aim is to see if it is possible to choose these points so that the above equation is satisfied. Notice the subtle change in the way the Regge equation is being employed – rather than being used to evolve the geometry it is now being used to evolve the energy momentum field variables.

In equation (7.16) all of the constraints have been applied after the differentiation of Δs. As the first two constraints do not involve any of the $d's$ it is possible to introduce these constraints prior to the differentiation of Δs. The purpose of this step is that it now allows one to invoke symmetry arguments in the analysis of (7.16). Suppose that the geodesic segment (along which the particle travels) is not normal to the lower tetrahedron. The dependence of Δs on the six diagonals is therefore not symmetric and consequently Δs can not be simultaneously extremized with respect to variations in all six diagonals. The conclusion then is that the geodesic segment must be normal to the lower tetrahedron. However the upper and lower tetrahedra are parallel and thus when the particle crosses the upper tetrahedron there will be no change in the direction of the particles trajectory. Thus not only does the particle travel along geodesic segments within each segment of the 4-tube but also its trajectory is a geodesic throughout the whole 4-tube.

This argument has shown that for the three primary models it is possible to ensure that the global and constrained local field equations are equivalent by requiring, in addition to the previously mentioned constraints, that the particle's trajectory be a geodesic of the 4-tube and not just of its segments. For the three secondary models it is certainly true that the global and constrained local field equations will *not* be equivalent. However it may still be possible to ensure that the equations (7.6) vanish identically. Certainly the $\partial A/\partial d$ terms will vanish when the first three constraints are imposed. Whether one can or can not ensure that the remaining terms in this set of equations can be made to vanish by an appropriate choice of trajectories is an open question that will not be pursued here.

8. The field equations : II

Another important question that must be answered before the field equations can be solved is this : Exactly what are the dynamical variables that are to be evolved? As there are two non-trivial field equations (7.7 and 7.8) there can be no more than two dynamical variables, one of which must be L. When the same problem is posed in the ADM [6] formalism one finds that the dynamical variables are the radius of the 3-sphere and the particle's energy; the lapse and shift functions being freely chosen. This would suggest that the dynamical variables for our problem should be L and e. However this may lead to a conflict for there is no guarantee that the two field equations can be solved under the original assumption that e is always constant. A possible resolution of this conflict is presented in the following speculative comparison of the ADM and Regge approaches.

A typical Regge action sum is

$$I = 2\sum_i \theta_i A_i - \frac{16\pi G}{c^4}\sum_i I_M(T_i)$$

in which θ_i is the defect on the i^{th} bone, A_i is the area of that bone and $I_M(T_i)$ is the contribution to the matter action sum from those sources within the tube T_i. This action is a function of the $L's, d's$ and $t's$ and its extremization with respect to these variables may be formally expressed as

$$0 = \frac{\partial I}{\partial L_i} \quad , \quad i = 1, 2, \ldots$$

$$0 = \frac{\partial I}{\partial d_i} \quad , \quad i = 1, 2, \ldots$$

$$0 = \frac{\partial I}{\partial t_i} \quad , \quad i = 1, 2, \ldots$$

As the $L's$ are the carriers of the 3-geometry their associated Regge equations should be related to the ADM equations associated with the ${}^{(3)}g_{ij}$. This will be expressed as

$$0 = \frac{\partial I}{\partial L_i} \qquad \leftrightarrow \qquad 0 = \frac{\partial I'}{\partial {}^{(3)}g_{ij}}$$

where I' is the ADM action. Similarily the t_i specify the separations between successive slices and should, therefore, lead to the relation

$$0 = \frac{\partial I}{\partial t_i} \qquad \leftrightarrow \qquad 0 = \frac{\partial I'}{\partial N}$$

where N is the lapse function. The remaining Regge equations, those associated with the $d's$, are not so easily identified with their ADM counterparts. The best analogy that can be offered is that the diagonal legs determine the amount of sideways displacement of the vertices from slice to slice. The shift functions, N_i, serve a similar role in the ADM theory. Thus the expected relation would be

$$0 = \frac{\partial I}{\partial d_i} \qquad \leftrightarrow \qquad 0 = \frac{\partial I'}{\partial N_i}\ .$$

It is well known that in the ADM formalism the constraint equations are first integrals of the evolution equations. Thus the constraints need only be satisfied once for they will continue to be satisfied provided the evolution equations are satisfied. The same situation can not be expected to occur in the Regge calculus. The reason for this is quite simply that there does not appear to be, in the Regge calculus, a direct analogue of the coordinate freedoms that are available in the ADM formalism. The lapse and shift functions in the ADM formalism may be chosen arbitrarily by making an appropriate choice of coordinates. The coordinate transformations do not effect the geometry of the spacetime and their only effect is to shift and distort the coordinate curves. If such transformations exist in the Regge calculus then they might involve any combination of

i) a re-labelling of the vertices,

ii) a redistribution of the vertices and a subsequent adjustment of the leg lengths,

iii) an introduction of extra vertices, legs etc.

The first transformation, which can always be applied, will certainly not lead to a reduction in the number of active field equations. The second class of transformations may lead to a symmetry of the field equations but since the bones are shifted and their defects altered that transformation can not also be a symmetry of the geometry. The opposite situation may arise when using the third class of transformations – it is easy to preserve the geometry but there is no guarantee that the subdivided spacetime will be a new solution of the Regge field equations.

If it is true that a *continuous* class of transformations of the one Regge spacetime into itself does not exist then none of the Regge equations can be considered trivial. Thus *all* of the Regge equations will need to be used to evolve *all* of the leg, strut and diagonal lengths. This argument reveals an important point : that even though the lapse and shift functions in the ADM formalism may be chosen arbitrarily their Regge counterparts, the struts and diagonals, cannot and must be evolved along with the 3-geometry.

This approach is alien to the usual philosophy of numerical relativity. However, for the models to be described here, it is possible to overcome this difficulty. The idea is to increase the number of unkowns by allowing some of the parameters of the energy-momentum field to vary. There will then be more unknowns than field equations. This will allow a free choice to be made for the lengths of some of the struts and diagonals. In section § 2 it was argued that the rest energy of the particle was always constant and that the trajectory of the particle was a global geodesic. These conditions were then relaxed by allowing the rest energy to vary from slice to slice and the trajectory to bend as it passed through each Cauchy surface (the trajectory would then fail to be a global geodesic but it would remain a geodesic within each segment of the 4-tube). The reason for introducing these relaxed conditions should now be apparent – it allows one to freely choose the strut and diagonal lengths.

The answer to the question posed at the beginning of this section is that there are two dynamical variables, namely, the particle's energy and the length of the typical spacelike leg. It is quite clear that this approach is much closer in spirit to the usual ADM approach.

9. The field equations : III

For the three primary models of section § 3 it is a simple matter to show that

$$(J^\uparrow)^c = D_j \left(\frac{12}{E_j}(\alpha A)^\uparrow + 2(\beta B) - \frac{16\pi G}{c^4}(e\Delta s)^\uparrow \right)^c \tag{9.1}$$

where the subscript j is an index to the particular model. The coefficients D_j and E_j have the values

j	D_j	E_j
1	5	3
2	16	4
3	600	5

For the first subdivsion of the primary models, section § 4, the partial action is

$$\begin{aligned}(J^\uparrow)^c = D_j \Bigg(& \frac{12}{E_j}(\alpha A)^\uparrow_1 + 6(\alpha A)^\uparrow_2 + 6(\alpha A)^\uparrow_3 \\ & + 3(\beta B)^\uparrow_1 + 2(\beta B)^\uparrow_2 + (\beta B)^\uparrow_3 + 6(\beta B)^\uparrow_4 \\ & - \frac{32\pi G}{c^4}(e\Delta s)^\uparrow_1 - \frac{64\pi G}{c^4}(e\Delta s)^\uparrow_2 \Bigg)^c .\end{aligned} \tag{9.2}$$

The subscripts in this expression correspond exactly to the indices listed in Table II of section § 4. The coefficients D_j and E_j for these models have the values

j	D_j	E_j
4	10	3
5	32	4
6	1200	5

For each of the six models there are only two field equations, namely

$$0 = \frac{\partial (J^{\uparrow})^c}{\partial t} \tag{9.3}$$

and

$$0 = \frac{\partial (J^{\uparrow} + J^{\downarrow})^c}{\partial L} \ . \tag{9.4}$$

These equations, when fully expanded, can be greatly simplified by using Regge's identity [2] which states that

$$0 = \sum_i A_i \delta\alpha_i + \sum_i B_i \delta\beta_i$$

for any small variations in the defects on the interior of the simplicial geometry. Thus the derivatives of the defects would appear in the field equations only if the Cauchy surfaces have a boundary. The Cauchy surfaces in our models do not have boundaries. Using the formulae of section § 6 one can show that, when $t^{\uparrow} = t^{\downarrow}$, the above field equations for the three primary models may be written as

$$0 = \frac{12}{E_j}\left(\alpha\bar{L}\right)^{\uparrow}\left(\frac{\Delta t}{\Delta h}\right)^{\uparrow} - \frac{16\pi G}{c^4} e^{\uparrow}\left(\frac{\Delta t}{\Delta s}\right)^{\uparrow} \ , \tag{9.5}$$

$$\begin{aligned} 0 = &\frac{3}{E_j}\left(\alpha^{\uparrow}\left(2\Delta h + \bar{L}\frac{\Delta L}{\Delta h}\right)^{\uparrow} + \alpha^{\downarrow}\left(2\Delta h + \bar{L}\frac{\Delta L}{\Delta h}\right)^{\downarrow}\right)\\ &+ 4\sqrt{3}(\beta^{\uparrow} + \beta^{\downarrow})L \\ &- \frac{16\pi G}{c^4}\left(e^{\uparrow}\left(\frac{\Delta L}{\Delta s}\right)^{\uparrow} + e^{\downarrow}\left(\frac{\Delta L}{\Delta s}\right)^{\downarrow}\right) \ , \end{aligned} \tag{9.6}$$

where

$$\Delta h^{\uparrow} = \left((\Delta t)^2 + \frac{1}{4} (\Delta L^{\uparrow})^2 \right)^{1/2} ,$$

$$\Delta s^{\uparrow} = \left((\Delta t)^2 + \frac{3e}{12f - 4e} (\Delta L^{\uparrow})^2 \right)^{1/2} ,$$

$$\alpha^{\uparrow} = 2\pi - E_j \mathrm{Cos}^{-1} \left(\frac{2(\Delta t)^2 + (\Delta L^{\uparrow})^2}{6(\Delta t)^2 + 2(\Delta L^{\uparrow})^2} \right) ,$$

$$\beta^{\uparrow} = \mathrm{Sinh}^{-1} \left(\frac{\Delta L^{\uparrow}}{(24(\Delta t)^2 + 8(\Delta L^{\uparrow})^2)^{1/2}} \right) ,$$

$$\bar{L}^{\uparrow} = \left(L^{\uparrow}_{ab} + L_{ab} \right) /2 ,$$

$$\Delta L^{\uparrow} = L^{\uparrow}_{ab} - L_{ab} ,$$

$$\Delta t = t^{\uparrow} = t^{\downarrow} ,$$

with similar formulae applying for the $(\cdots)^{\downarrow}$ quantities. A slightly less complicated version of the second field equation (9.6) can be obtained by using (9.5) to eliminate the terms involving $e^{\uparrow}$ and $e^{\downarrow}$. The result is

$$\begin{aligned} 0 =& \frac{3}{4E_j} \left\{ \alpha^{\uparrow} \left(4\Delta h^{\uparrow} - (\bar{L}\frac{\Delta L}{\Delta h})^{\uparrow} \right) + \alpha^{\downarrow} \left(4\Delta h^{\downarrow} - (\bar{L}\frac{\Delta L}{\Delta h})^{\downarrow} \right) \right\} \\ & + 4\sqrt{3}(\beta^{\uparrow} + \beta^{\downarrow}) L . \end{aligned} \tag{9.7}$$

This equation can also be obtained directly from the action sum be taking independent variations in L and Δs instead of L and Δt.

10. The tangent 3-sphere

If the discrete Regge spacetimes are to be viewed as an approximation to one smooth Einstein spacetime then it will be necessary to establish a link between the characteristic parameters of those spacetimes. Such a relationship will be derived in this section by making explicit the requirement that the geometries of the Cauchy surfaces of the Regge and Einstein spacetimes shall be similar. The basis of the approach is to choose the leg lengths and the radius of the 3-sphere so that the vertices of the discrete model lie on the surface of the 3-sphere.

It is well known that a 3-sphere may be viewed as a surface embedded in E^4. A useful parametric representation of this surface is

$$\begin{aligned} x^1 &= R\cos\theta \\ x^2 &= R\sin\theta\cos\phi \\ x^3 &= R\sin\theta\sin\phi\cos\alpha \\ x^4 &= R\sin\theta\sin\phi\sin\alpha \end{aligned}$$

where R is the radius of the 3-sphere and x^1, x^2, x^3, x^4 are the four Cartesian coordinates of E^4. The metric in this space is simply

$$ds^2 = \sum_i (dx^i)^2 \ .$$

Consider now one leg of the $N_3 = 5$ primary model. There will be a total of five vertices in the set of three tetrahedra that share this leg as a common edge. Choose the labels of the vertices of the leg as (1) and (2) and those for the three vertices that surround the leg as $(3), (4)$ and (5). Since all of the leg lengths are equal it must be possible to align the coordinates so that the parameters (polar coordinates) of the vertices are of the form

vertex	θ	ϕ	α
1	0	0	0
2	θ_2	0	0
3	θ_3	ϕ_3	0
4	θ_3	ϕ_3	$2\pi/3$
5	θ_3	ϕ_3	$4\pi/3$

The values of θ_2, θ_3 and ϕ_3 are determined by the requirement that

$$L_{12}^2 = L_{ij}^2$$

for all valid combinations of i and j. This leads to the equations

$$\begin{aligned} L_{12}^2 &= 2R^2(1 - \cos\theta_2) , \\ L_{13}^2 &= 2R^2(1 - \cos\theta_3) , \\ L_{23}^2 &= 2R^2(1 - \cos\phi_3)\sin^2\theta_2 , \\ L_{34}^2 &= 3R^2(\sin\theta_3 \sin\phi_3)^2 , \end{aligned}$$

with the solution

$$\begin{aligned} \theta_2 &= \theta_3 , \\ \cos\theta_2 &= -\frac{1}{4} , \\ \cos\phi_3 &= -\frac{1}{3} . \end{aligned}$$

In the process one also obtains

$$R = \sqrt{0.4}\, L_{12} . \tag{10.1}$$

This is the desired relationship that links the geometries of the discrete and continuous models. This construction ensures, since all of the vertices, legs, etc., of the primary models are equivalent, that the 3-sphere touches *each* vertex of the discrete model and not just the subset used here. The same procedure may be applied to the remaining two primary models. The results are as follows.

i) $N_3 = 16$

The leg $(1, 2)$ is now surrounded by four tetrahedra. The polar coordinates are chosen as

vertex	θ	ϕ	α
1	0	0	0
2	θ_2	0	0
3	θ_3	ϕ_3	0
4	θ_3	ϕ_3	$2\pi/4$
5	θ_3	ϕ_3	$4\pi/4$
6	θ_3	ϕ_3	$6\pi/4$

Equality of the leg lengths leads to

$$\theta_2 = \theta_3$$

$$\theta_2 = \pi/2$$

$$\phi_3 = \pi/2$$

and

$$R = \sqrt{0.5}\, L_{12}\,. \tag{10.2}$$

ii) $N_3 = 600$

There are five tetrahedra around the leg $(1,2)$. The polar coordinates are chosen as

vertex	θ	ϕ	α
1	0	0	0
2	θ_2	0	0
3	θ_3	ϕ_3	0
4	θ_3	ϕ_3	$2\pi/5$
5	θ_3	ϕ_3	$4\pi/5$
6	θ_3	ϕ_3	$6\pi/5$
7	θ_3	ϕ_3	$8\pi/5$

with

$$\theta_2 = \theta_3$$

$$\cos\theta_2 = \frac{\cos(2\pi/5)}{1-2\cos(2\pi/5)}$$

$$\cos\phi_3 = \frac{\cos(2\pi/5)}{1-\cos(2\pi/5)}$$

and

$$R = \sqrt{\frac{1-2\cos(2\pi/5)}{2-6\cos(2\pi/5)}}\, L_{12}\,. \tag{10.3}$$

A slightly different approach will be used for the three secondary models. Each secondary model was constructed by subdividing the associated primary model. The coordinates of the vertices in the secondary model are readily deduced from those of the primary model. Some of these coordinates are already known (ie. from the vertices common to both the primary and secondary models) while the others will be deduced by a linear combination of the known coordinates. Consider, for example, the first primary model with $N_3 = 5$. The Cartesian coordinates of the vertices of one typical tetrahedron may be computed from the known values of the polar coordinates. The results are

$$
\begin{aligned}
(x^\mu)_1 &= (\ \ 1\,, & 0\,, & & 0\,, & & 0\,)&, \\
(x^\mu)_2 &= (\ -\tfrac{1}{4}\,, & \tfrac{\sqrt{15}}{4}\,, & & 0\,, & & 0\,)&, \\
(x^\mu)_3 &= (\ -\tfrac{1}{4}\,, & -\tfrac{\sqrt{15}}{12}\,, & & \tfrac{\sqrt{30}}{6}\,, & & 0\,)&, \\
(x^\mu)_4 &= (\ -\tfrac{1}{4}\,, & -\tfrac{\sqrt{15}}{12}\,, & & -\tfrac{\sqrt{30}}{12}\,, & & \tfrac{\sqrt{90}}{12}\,)&, \\
(x^\mu)_5 &= (\ -\tfrac{1}{4}\,, & -\tfrac{\sqrt{15}}{12}\,, & & -\tfrac{\sqrt{30}}{12}\,, & & -\tfrac{\sqrt{90}}{12}\,)&.
\end{aligned}
$$

There are only two types of introduced vertex, (12) on the leg $(1,2)$ and (1234) on the interior of the tetrahedron $(1,2,3,4)$. Their coordinates are defined as

$$
\begin{aligned}
(x^\mu)_{12} &= \frac{\lambda_{12}}{2}\sum_{i=1}^{2}(x^\mu)_i\,, \\
(x^\mu)_{1234} &= \frac{\lambda_{1234}}{4}\sum_{i=1}^{4}(x^\mu)_i\,.
\end{aligned}
$$

The scale factors λ_{12} and λ_{1234} are chosen so that the introduced vertices lie on the 3-sphere. This is expressed as

$$
R^2 = (x_\mu x^\mu)_{12} = (x_\mu x^\mu)_{1234}
$$

and leads to

$$
\begin{aligned}
\lambda_{12} &= \frac{4}{\sqrt{6}}\,, \\
\lambda_{1234} &= 4\,.
\end{aligned}
$$

One may now directly calculate the leg lengths between pairs of adjacent vertices with the result that

$$L_{1,12}^2 = \left(2 - \frac{3}{\sqrt{6}}\right) R^2 \,,$$
$$L_{12,13}^2 = \frac{5}{3} R^2 \,, \tag{10.4}$$
$$L_{12,1234}^2 = 2\left(1 - \frac{1}{\sqrt{6}}\right) R^2 \,.$$

Once again this same procedure may be applied to the remaining two secondary models for which the results are

i) $N_3 = 192$

$$L_{1,12}^2 = \left(2 - \sqrt{2}\right) R^2 \,,$$
$$L_{12,13}^2 = R^2 \,, \tag{10.5}$$
$$L_{12,1234}^2 = \left(2 - \sqrt{2}\right) R^2 \,,$$

ii) $N_3 = 7200$

$$L_{1,12}^2 \approx 0.09788696740970 R^2 \,,$$
$$L_{12,13}^2 \approx 0.10557280900008 R^2 \,, \tag{10.6}$$
$$L_{12,1234}^2 \approx 0.05350202106454 R^2 \,.$$

11. The Friedmann dust cosmology

The relationships between L and R that have just been derived provides a connection between the discrete and continuous models at one instant in the evolution of the models. It is fortunate that for this simple problem a complete solution of the Einstein equations for the continuous spacetime is known. With this solution it is possible to compare the *dynamics* of the discrete and continuous models.

It is well know that for a Friedmann dust cosmology the total energy E is [7]

$$E = \frac{3\pi c^4}{4G} R_0 \tag{11.1}$$

where R_0 is the radius of the 3-sphere at the moment of time-symmetry. The corresponding formula for the discrete spacetimes would be

$$E = \sum_i e_i$$

where e_i is the energy of the i^{th} particle. For the three primary models this may be reduced to

$$E = N_3 e \tag{11.2}$$

where $N_3 = 5, 16$ or 600 is the number of identical tetrahedra in one Cauchy surface and e is the typical energy in one tetrahedron. For the three secondary models the appropriate expression is

$$E = N_3^{(1)} e_1 + N_3^{(2)} e_2 = \left(N_3^{(1)} + N_3^{(2)} \frac{V_2}{V_1} \right) e_1 \tag{11.3}$$

where $N_3^{(i)}$ is the number of tetrahedra in the i^{th} class, V_i is the volume and e_i the energy of one tetrahedron in that class. A simple parametric representation of the Friedmann dust cosmology is

$$\begin{aligned} R &= \frac{R_0}{2}(1 + \cos\eta) \ , \\ t &= \frac{R_0}{2c}(\eta + \sin\eta) \end{aligned} \tag{11.4}$$

for $-\pi < \eta < +\pi$.

The above equations (10.1,10.2,10.3,11.2 and 11.3) are used to convert the successive values of L_{ab} and e, obtained by solving the field equations (9.5 and 9.7), into the associated values of R and E. This pair of quantities can then be compared with the analytic values (11.1 and 11.4) for the associated smooth Einstein spacetime.

12. Solving the field equations

There are two distinct phases in solving the field equations. In the first phase one must obtain a consistent set of initial data. The second phase involves the evolution of this data onto the successive Cauchy surfaces. There are a variety of ways to approach each of these phases. Two of these approaches will presented here.

The single boundary condition to be imposed on our solutions is that they should be time symmetric. On such a solution it must be possible to choose the label of each Cauchy surface so that the geometries of the pair of Cauchy surfaces S_n and S_{-n} are identical for every allowed value of n. In this method of labelling there are two distinct classes of solutions which may be distinguished by noting whether $n = 0$ is or is not an allowed label. This distinction is of importance only in the solution of the initial data problem.

The simpler of the two cases is that in which $n = 0$ is *not* allowed. In this case the initial data would consist of $L_1 = L_{-1}$ the typical leg length in $S_1 \equiv S_{-1}$, Δt the typical strut length and e_1 the typical particle energy. However only one of the two field equations (9.5) can be applied to just *two* Cauchy surfaces (the other equation uses data from *three* Cauchy surfaces). Since Δt can be freely chosen, at least on the initial slice, the single equation (9.5) should be viewed as a constraint on L_1 and e_1. One could choose to choose L_1 freely and then compute e_1 or to choose e_1 and then compute L_1.

In the alternative case when $n = 0$ *is* allowed the initial data will involve the typical leg lengths L_0 and $L_1 = L_{-1}$ of both S_0 and $S_1 \equiv S_{-1}$ respectively and the particle's energy $e_1 = e_{-1}$. In this situation both field equations (9.5 and 9.7) must be used. One strategy for solving these equations is to choose L_0 and Δt and then to use an iterative scheme to compute both L_1 and e_1.

In the evolution phase one must solve both field equations at each time step. This is necessary because there is no guarantee that the constraint equation (9.5) is a first solution (ie. first integral) of the evolution equation (9.7). This point was raised in sections § 7 and 8 where it was also noted that one could either evolve both the $L's$ and $\Delta t's$ while holding the $e's$ constant or evolve both the $L's$ and $e's$ while freely choosing the $\Delta t's$.

All of the above strategies have been tried and they all seem to yield indistinguishable values for L and e as a function of t, the accumalated time, provided Δt is very small compared to L_0 (or L_1 if L_0 does not exist). The behaviour of R and E (obtained from L and e by equations (10.1,10.2,10.3,10.4,10.5,10.6,11.2,11.3)) as functions of t are illustrated in Figs (12.1a) and (12.1b). These figures were obtained by solving the initial data problem with the $n = 0$ Cauchy surface included and by evolving L and e while keeping Δt constant.

When Δt is not small compared to L different solutions do arise when different solution strategies are employed. The most noticeable change arises when one solves the evolution equation with a constant but non-small value of Δt. The solutions for this situation, which are represented by Figs (12.2a,12.2b), clearly show (since E is not constant) that the initial data equation is not a first solution of the evolution equation. There appeared to be no significant change in the constant e solutions when the initial value of Δt was increased ten fold.

Each solution, one for each of the six models, exhibits a behaviour similar to that of the smooth Friedmann solution. As the spatial and temporal resolutions are improved (ie. the number of tetrahedra is increased and Δt is reduced) the discrete solution exhibits a greater concordance with the smooth solution. There is, however, one obvious difference between the discrete and smooth solutions : the discrete model can not be evolved to the point where the 3-volume vanishes (ie. $L = 0$). This is not a problem that arises from any numerical difficulty in solving the field equations but is, as will be shown in the following section, a real feature of this class of spacetimes.

The coupled non-linear equations (9.5,9.7) were solved by employing a generalized secant algorithm [8,9] It was found that for the initial data problem the first guess of the particles energy was very crucial in determining whether or not this algorithm succeeded. A successful first guess for e was obtained by temporarily setting $L_0 = L_1$ and then using the initial data equation (9.5) to compute an approximation for e. In the evolution phase the previous value for e was used as a first guess for its new value. For L the first guess was obtained by a quadratic extrapolation of the three previous values for L. It was noticed that at each time step in the evolution phase there was more than one possible solution for L and e. The particular branch that was chosen was that which minimized the change in e. This was the only branch that appeared to have the property that as $\Delta t \to 0$ then $\Delta e \to 0$. All of the other branches would yield valid solutions of the Regge equations but it would seem unlikely that they could be associated with a smooth Einstein spacetime.

13. The continuous time model

There are two features of the solutions that are interesting. The first is that real solutions seem to be possible only for a limited range of L, the typical leg length of one tetrahedron. The largest range of L occurs in the $N_3 = 7200$ model; the final value of L being 4.6% of its initial value. The second feature is that the total energy remains essentially constant, the variation being at most 0.13% through out the evolution of the model. Although these features have been brought to our notice by a numerical solution of the field equations one should be able to deduce these features, assuming that they are real and not just numerical accidents, by analysing the appropriate field equations (9.5,9.7). Unfortunately though these equations appear to be just too involved to allow a simple investigation.

One approach in overcoming this difficulty is to reduce the field equations to a set of differential equations. This will be achieved by developing a new model in which the time variable is continuous. The successive values of L on each Cauchy surface will be approximated be the values of a smooth function $\hat{L}(t)$ evaluated at discrete values of t. The variable t will be used as a time coordinate in the continuous time model and as a label for each Cauchy surface of the discrete time model. The field equations for the continuous model will be derived from those of the discrete model be developing a Taylor series expansion and retaining only the leading terms as the time step is reduced to zero. This approach differs from that of Collins and Williams in that they chose only to retain the zero$^{\text{th}}$ order terms. This procedure will be applied only to the three primary models.

For any choice of time step the exact field equations are (9.5,9.7). Let Δt be the constant value of the separation between pairs of Cauchy surfaces. The (continuous) time coordinate,

t, is then defined as $n\Delta t$ for $n = 1, 2, 3, \cdots$. In section § 9 the $\uparrow\downarrow$ superscripts were used to indicate whether the associated quantity was to be evaluated in the future or past regions of the present Cauchy surface. This notation will be modified for this section by replacing the superscript with a integer subscript. Thus α_n will be the defect on the timelike bone between the n^{th} and $(n+1)^{th}$ Cauchy surfaces. Likewise $\beta^{\uparrow}$ will be rewritten as β_n. Now from the definitions of $\beta^{\uparrow}$ and $\beta^{\downarrow}$ (6.23,6.24) it should be clear that $\beta^{\downarrow} = -\beta_{n-1}$. Thus the defect on the spacelike bone in the n^{th} Cauchy surface is $2(\beta_n - \beta_{n-1})$. Now suppose that the smooth differentiable functions $\hat{L}(t), \hat{\alpha}(t), \hat{\beta}(t)$ and $\hat{e}(t)$ are chosen so that

$$\begin{aligned}
\hat{L}(t_n) &= L_n \\
\hat{\alpha}(t_{n+\frac{1}{2}}) &= \alpha_n \\
\hat{\beta}(t_{n+\frac{1}{2}}) &= \beta_n \\
\hat{e}(t_{n+\frac{1}{2}}) &= e_n
\end{aligned}$$

The functions $\hat{\alpha}, \hat{\beta}$ and $\hat{e}$ are evaluated at $t = t_{n+\frac{1}{2}}$ because the quantities α_n, β_n and e_n are associated with the *pair* of slices $t = t_n$ and $t = t_{n+1}$.
If Δt is small compared to a typical L_n then the Taylor series

$$\begin{aligned}
\beta_n &= \left(\hat{\beta} + \frac{d\hat{\beta}}{dt}\frac{\Delta t}{2} + O\left(\Delta t\right)^2\right)_{t=t_n} \\
\alpha_n &= \left(\hat{\alpha} + \frac{d\hat{\alpha}}{dt}\frac{\Delta t}{2} + O\left(\Delta t\right)^2\right)_{t=t_n} \\
L_{n+1} &= \left(\hat{L} + \frac{d\hat{L}}{dt}\Delta t + O\left(\Delta t\right)^2\right)_{t=t_n} \\
\left(\frac{\Delta s}{\Delta h}\right)_n &= \left(\frac{ds}{dh} + \frac{d}{dt}\left(\frac{ds}{dh}\right)\frac{\Delta t}{2} + O\left(\Delta t\right)^2\right)_{t=t_n} \\
\left(\alpha\bar{L}\frac{\Delta L}{\Delta t}\right)_n &= \left(\hat{\alpha}\hat{L}\frac{d\hat{L}}{dt} + \frac{d}{dt}\left(\hat{\alpha}\hat{L}\frac{d\hat{L}}{dt}\right)\frac{\Delta t}{2} + O\left(\Delta t\right)^2\right)_{t=t_n}
\end{aligned}$$

are valid. A similar set of formulae, for the $\alpha_{n-1}, \beta_{n-1}, L_{n-1}, \cdots$, may be obtained by substituting $-\Delta t$ for Δt. A substitution of these Taylor series into (9.5 and 9.7) and a subsequent application of the usual limiting procedures as $\Delta t \to 0$ will lead to

$$0 = \hat{\alpha}\hat{L}\frac{ds}{dh} - \frac{4\pi G}{3c^4}\hat{e}P_2\,, \tag{13.1}$$

$$0 = \hat{\alpha}\frac{dh}{dt} + \frac{1}{8}\frac{d}{dt}\left(\hat{\alpha}\hat{L}\frac{d\hat{L}}{dh}\right) + \frac{P_2}{\sqrt{3}}\hat{L}\frac{d\hat{\beta}}{dt}\,, \tag{13.2}$$

with

$$\begin{aligned}
\hat{\alpha} &= 2\pi - P_2\mathrm{Cos}^{-1}\left(\frac{2+\lambda^2}{6+2\lambda^2}\right)\,,\\
\hat{\beta} &= \mathrm{Sinh}^{-1}\left(\frac{\lambda}{(24+8\lambda^2)^{\frac{1}{2}}}\right)\,,\\
\frac{dh}{dt} &= (1+\frac{1}{4}\lambda^2)^{\frac{1}{2}}\,,\\
\frac{ds}{dt} &= (1+\frac{3}{8}\lambda^2)^{\frac{1}{2}}\,,\\
\lambda &= \frac{d\hat{L}}{dt}
\end{aligned} \tag{13.3}$$

and $P_2 = 3, 4$ or 5. These are the field equations for the continuous time model. If the features of the discrete solutions are not just numerical accidents then those same features should be apparent in the solutions of this pair of differential equations.

It is not hard to verify that (13.1) is a first integral of (13.2) provided that

$$\frac{d\hat{e}}{dt} = 0\,.$$

This clearly establishes that the energy, $\hat{e}$, is conserved. However this result does not require that e should be constant in the discrete time models (and indeed the numerical solutions do show a tendency for e to vary, particularily in the latter stages of the evolution).

Now consider the first order equation (13.1). As it is now known that $\hat{e}$ is non-zero and constant it follows that the first term in (13.1) must also be non-zero and constant. However from (13.3) it is clear that the product $\hat{\alpha} ds/dh$ is bounded below when $\lambda \to 0$ and is bounded above when $\lambda \to \infty$. An evaluation of (13.1) at each of these limits and a subsequent elimination of $\hat{e}$ leads to

$$\frac{\sqrt{6}}{\pi}\left(\frac{2\pi - P_2 \mathrm{Cos}^{-1}(\frac{1}{3})}{6 - P_2}\right) \leq \frac{\hat{L}(t)}{\hat{L}_{max}} \leq 1$$

where $\hat{L}_{max}$ is the maximum value of $\hat{L}(t)$ (ie. the value of $\hat{L}$ at the moment of time symmetry). This clearly shows that $\hat{L}(t)$ is bounded below by a *non-zero* number which is exactly the same behaviour as exhibited in the discrete model.

There is a rather simple explanation as to why the evolution of the models must terminate. In the later stages of the evolution of the model the relative velocity of any pair of neighbouring vertices rapidly increases. The terminal point arises when this relative velocity equals that of light. At this point the worldlines of the vertices have become null. An equivalent way of looking at this same phenomenon is to look at a pair of Cauchy surfaces S_n and S_{n+1} and to examine whether S_n is or is not contained in the past domain of dependence of S_{n+1} . This test requires the construction of all possible backward pointing non-spacelike curves from S_{n+1} to S_n. If there exists one curve for every point of S_n then S_n is in the past domain of dependence of S_{n+1} . If S_n was not contained in the past domain of dependence of S_{n+1} then the complete set of observers, uniformly distributed across S_{n+1} , could not be associated by a one-to-one map with a similar set of observers on S_n. This is clearly unphysical and therefore it is not suprising that the evolution of the model terminates.

It is possible to make a purely mathematical rearrangement of the field equations so as to avoid the terminal point. The idea is to obtain an analytic continuation of the solution by rewriting the equations in terms of $\hat{L}(s)$ rather than $\hat{L}(t)$. The resulting equations are

$$0 = \hat{\alpha}\hat{L}\frac{ds}{dh} - \frac{4\pi G}{3c^4}\hat{e}P_2\ ,$$

$$0 = \hat{\alpha}\frac{dh}{ds} + \frac{1}{8}\frac{d}{ds}\left(\hat{\alpha}\hat{L}\frac{d\hat{L}}{dh}\right) + \frac{P_2}{\sqrt{3}}\hat{L}\frac{d\hat{\beta}}{ds}\ ,$$

with

$$\hat{\alpha} = 2\pi - P_2 \mathrm{Cos}^{-1}\left(\frac{8+\eta^2}{24-\eta^2}\right) ,$$

$$\hat{\beta} = \mathrm{Sinh}^{-1}\left(\frac{\eta}{(24-\eta^2)^{1/2}}\right) ,$$

$$\frac{dh}{ds} = \left(1 - \frac{1}{8}\eta^2\right)^{\frac{1}{2}} ,$$

$$\eta = \frac{d\hat{L}}{ds} .$$

and their solutions are represented in Figs (13.1a,13.1b). It is clear that the extended solution has the same general properties as the smooth Einstein solution and in particular that the terminal point, which now occurs as $d\hat{L}/ds \to \sqrt{8}$ and $\hat{L} \to 0$, develops when the radius of the universe has shrunk to zero. These solutions coincide exactly with the previous solutions in the region prior to the development of the terminal point. It is not clear though that these new solutions are physically meaningful beyond the terminal point.
The development of the terminal point cannot be attributed to the form of the symmetries of the spacetimes since this is a local phenomena which arises whenever any of the struts tips over to the null cone. This behaviour can be expected to occur whenever the local curvature becomes too large.

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