Curvature corrected estimates for geodesic arc-length.

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Abstract

We will develop simple relations between the arc-lengths of a pair of geodesics that share common end-points. The two geodesics differ only by the requirement that one is constrained to lie in a subspace of the parent manifold. We will present two applications of our results. In the first example we explore the convergence of Gaussian curvature estimates on a simple triangular mesh. The second example demonstrates an improved error estimate for the area of a Schwarz lantern.

1   Introduction

If S is an $(n-1)$-dimensional subspace of an n-dimensional Riemannian space M then S will inherit a metric from its embedding in M. Consider a pair of points p and q on S chosen sufficiently close to ensure that there exists a unique geodesic in S connecting p to q. If necessary, the points p and q could be further constrained to ensure that a unique geodesic, this time lying in M, also connects p to q. A natural question to pose would be – how are the arc-lengths of the two geodesics related? This is a simple question and as shown below rather easy to answer. We will show that the difference between the arc-lengths is (not surprisingly) controlled by the embedding of S in M, that is, the difference can be expressed solely in terms of the second fundamental form of S.

2 Geodesic arc-length

Lemma. *Let M be an n-dimensional manifold with a Riemannian metric g. Consider an $(n-1)$-dimensional subspace S of M and let h be the metric induced on S by the embedding of S in (M, g). Choose a set of coordinates on M and choose the natural coordinate basis ∂_μ for the tangent space on points of M. Then*

$$\tilde{\Gamma}^\mu{}_{\alpha\beta} = \bar{\Gamma}^\mu{}_{\alpha\beta} + n^\mu K_{\alpha\beta} \tag{2.1}$$

where $\tilde{\Gamma}^\mu{}_{\alpha\beta}$ and $\bar{\Gamma}^\mu{}_{\alpha\beta}$ are the metric compatible connection components on S and M respectively and where $K_{\alpha\beta}$ are the components of the extrinsic curvature of S.

Proof. Let v^μ be the tangent vector to a geodesic of (S, h) and let $|$ denote the covariant derivative with respect to $\tilde{\Gamma}^\mu{}_{\alpha\beta}$. Then

$$0 = v^\mu{}_{|\nu} v^\nu \tag{2.2}$$

However

$$v^\mu{}_{|\nu} = \perp (v^\mu{}_{;\nu}) = (\delta^\mu{}_\nu + n^\mu n^\nu)(v^\mu{}_{;\nu}) \tag{2.3}$$

where ; denotes the covariant derivative with respect to $\bar{\Gamma}^\mu{}_{\alpha\beta}$. Substituting (2.3) into (2.2) while also using $0 = (n_\alpha v^\alpha)_{;\beta}$ and $2K_{\alpha\beta} = \perp(n_{\alpha;\beta} + n_{\beta;\alpha})$ leads to

$$0 = v^\mu{}_{;\nu} v^\nu + n^\mu K_{\alpha\beta} v^\alpha v^\beta \tag{2.4}$$

Comparing this equation with (2.2) leads directly to (2.1). $\qquad\square$

Note that $\tilde{\Gamma}^\mu{}_{\alpha\beta}$ can be viewed as a second connection on M constructed so that it admits geodesics lying solely within S.

Theorem. *Let M be an n-dimensional manifold with a Riemannian metric g. Consider an $(n-1)$-dimensional subspace S of M and let h be the metric induced on S by the embedding of S in (M, g). Consider a pair of nearby points p and q in S chosen so that they are connected by a pair of unique geodesics, one in (S, h) and the other in (M, g). Let $\bar{L}(p, q)$ be the arc-length from p to q in (M, g) and likewise let $\tilde{L}(p, q)$ be the arc-length in (S, h). Let K be the second fundamental form for S and v the unit-tangent to the geodesic in (M, g). Then*

$$\tilde{L}^2(p, q) = \bar{L}^2(p, q) + \frac{1}{12} \left(K(v, v) \bar{L}^2(p, q) \right)^2 + \mathcal{O}\left(\bar{L}^5\right) \tag{2.5}$$

Proof. We begin by constructing a local set of Riemann normal coordinates x^μ that covers a subset of M containing the two points p and q. The assumption that there is a unique geodesic in M that connects p and q ensures that such a set of coordinates can be constructed. We are free to locate the origin of the coordinates to be at p and also to align the coordinates axes so that one axis is parallel to the geodesic connecting p to q. Let that axis be the x^1 axis. Then the coordinates of p are $x_p = (0,0,0...)$ while for q we have $x_q = (x_q^1, 0, 0, ...)$. The x^μ do not provide a set of Riemann normal coordinates on S. However in a new set of coordinates y^μ in the neighbourhood of p given by

$$y^\mu = x^\mu + \frac{1}{2}\tilde{\Gamma}^\mu{}_{\alpha\beta}x^\alpha x^\beta + \frac{1}{6}\left(\tilde{\Gamma}^\mu{}_{\alpha\beta}\tilde{\Gamma}^\beta{}_{\theta\phi} + \tilde{\Gamma}^\mu{}_{\theta\phi,\alpha}\right)x^\alpha x^\theta x^\phi + \mathcal{O}\left(L^4\right) \qquad (2.6)$$

it is easy to verify that the connection components at p satisfy

$$0 = \Gamma^\mu{}_{\alpha\beta} \qquad (2.7)$$
$$0 = \Gamma^\mu{}_{(\alpha\beta,\rho)} \qquad (2.8)$$

and thus the y^μ, when restricted to S, serve as a set of Riemann normal coordinates on S. In the y^μ coordinates we have $y_p = (0,0,0\cdots)$ and $y_q = (y_q^1, 0, 0, ...)$. The choice of Riemann normal coordinates is motived by the following simple expression for the geodesic arc-length

$$L^2(p,q) = g_{\mu\nu}\Delta x_{pq}^\mu \Delta x_{pq}^\nu - \frac{1}{3}R_{\mu\alpha\nu\beta}x_p^\mu x_p^\nu x_q^\alpha x_q^\alpha + \mathcal{O}\left(L^5\right) \qquad (2.9)$$

where $g_{\mu\nu} = \mathrm{diag}(1,1,1,\cdots)$ and $R_{\mu\alpha\nu\beta}$ are the Riemann curvature components evaluated at p and where $\Delta x_{pq}^\mu = x_q^\mu - x_p^\mu$. There are two geodesics to be considered. They both join p to q but one uses the connection $\tilde{\Gamma}^\mu{}_{\alpha\beta}$ while the other uses $\bar{\Gamma}^\mu{}_{\alpha\beta}$. The squared arc-lengths for this pair of geodesics can be found using (2.9) leading to

$$\tilde{L}^2(p,q) = g_{\mu\nu}y_q^\mu y_q^\nu \qquad (2.10)$$
$$\bar{L}^2(p,q) = g_{\mu\nu}x_q^\mu x_q^\nu \qquad (2.11)$$

Now we can combine equations (2.1) and (2.6) and substitute the result into (2.10) to obtain

$$\bar{L}^2(p,q) = \tilde{L}^2(p,q) - \frac{1}{4}\left(K_{\mu\nu}x_q^\mu x_q^\nu\right)^2 + \frac{1}{3}K_{\mu\nu}K_{\alpha\beta}x_q^\mu x_q^\nu x_q^\alpha x_q^\beta + \mathcal{O}\left(L^5\right) \quad (2.12)$$

Now recall that $x_q = (x_q^1, 0, 0, 0\cdots)$ and thus $x_q = \tilde{L}(p,q)v$ where v is a unit vector from p to q. Thus we can re-write the previous equation as

$$\bar{L}^2(p,q) = \tilde{L}^2(p,q) + \frac{1}{12}\left(K_{\mu\nu}v^\mu v^\nu \tilde{L}^2(p,q)\right)^2 + \mathcal{O}\left(L^5\right) \qquad (2.13)$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Corollary 1. *Let $n(p)$ and $n(q)$ be the unit normal vectors on S at points p and q respectively. Then*

$$\bar{L}^2(p,q) = \tilde{L}^2(p,q) + \frac{1}{12}\left((n_\mu(p) - n_\mu(q))\Delta x^\mu_{pq}\right)^2 + \mathcal{O}\left(L^5\right) \qquad (2.14)$$

Proof. Since the connection $\bar{\Gamma}^\mu{}_{\alpha\beta}$ vanishes at p we have

$$(n_\mu(p) - n_\mu(q))v^\mu = n_{\mu;\nu}\Delta x^\nu_{pq}v^\mu + \mathcal{O}\left(L^2\right) \qquad (2.15)$$

But $\Delta x^\nu_{pq} = v^\nu \bar{L}(p,q)$ thus we also have

$$(n_\mu(p) - n_\mu(q))v^\mu = n_{\mu;\nu}v^\mu v^\nu \bar{L}(p,q) + \mathcal{O}\left(L^2\right) \qquad (2.16)$$

The vector v^μ, which is tangent to the geodesic in (M,g) connecting p to q, is in general not tangent to the geodesic in (S,h). It can however be written as a linear combination of a vectors parallel and perpendicular to S at p. That is

$$v^\mu = \alpha \tilde{v}^\mu + \beta n^\mu \qquad (2.17)$$

where α and β are numbers yet to be determined and $\tilde{v}^\mu$ is the unit tangent vector to the geodesic in S. It is clear that when p and q are close then $\alpha = 1 + \mathcal{O}\left(L^2\right)$ and $\beta = \mathcal{O}\left(L\right)$. Substituting this into the previous equation leads to

$$(n_\mu(p) - n_\mu(q))v^\mu = n_{\mu;\nu}\tilde{v}^\mu \tilde{v}^\nu \bar{L}(p,q) + \mathcal{O}\left(L^2\right) \qquad (2.18)$$

Now we can use

$$2K_{\mu\nu} = \perp (n_{\mu;\nu} + n_{\mu;\nu}) \qquad (2.19)$$

to obtain

$$(n_\mu(p) - n_\mu(q))v^\mu = K_{\mu\nu}\tilde{v}^\mu \tilde{v}^\nu \bar{L}(p,q) + \mathcal{O}\left(L^2\right) \qquad (2.20)$$

Multipling through by $\bar{L}(p,q)$ and using $\Delta x^\mu_{pq} = v^\mu \bar{L}(p,q)$ we see that (2.5) can be written as

$$\bar{L}^2(p,q) = \tilde{L}^2(p,q) + \frac{1}{12}\left((n_\mu(p) - n_\mu(q))\Delta x^\mu_{pq}\right)^2 + \mathcal{O}\left(L^5\right) \qquad (2.21)$$

which completes the proof. $\qquad\qquad\square$

Corollary 2. *Consider a one parameter family of hypersurfaces generated from S by dragging S along its unit normal n. This family forms a local foliation of M in which the points p and q are now viewed as functions along the integral curves of n. Then*

$$\bar{L}^2(p,q) = \tilde{L}^2(p,q) + \frac{1}{48}\left(\frac{d\bar{L}(p,q)}{dn}\right)^4 + \mathcal{O}\left(L^5\right) \qquad (2.22)$$

where n (not to be confused with the unit normal) is the arc length measured along the integral curves of the unit normal.

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Proof. The equation for the first variation of arc states that

$$\frac{d\bar{L}(p,q)}{dn} = [n_\mu v^\mu]_p^q \tag{2.23}$$

where v^μ is the unit tangent vector to the geodesic in (M, g). But in our Riemann normal coordinates we have $v_p^\mu = v_q^\mu$ and thus we also have

$$\frac{d\bar{L}(p,q)}{dn} = (n_\mu(p) - n_\mu(q)) \, v^\mu \tag{2.24}$$

which leads to

$$\frac{d\bar{L}^2(p,q)}{dn} = 2 \, (n_\mu(p) - n_\mu(q)) \, \Delta x_{pq}^\mu \tag{2.25}$$

Combining this with the previous corollary completes the proof. $\qquad \square$

3 Examples

3.1 Estimating Gaussian curvature

It is common practice in computer graphics to model a smooth 2-dimensional surfaces such as a sphere, a torus or even teapots by a finite collection of connected triangles. The vertices of the triangles are taken as sample points of the smooth surface while the legs are taken as geodesics of the flat 3-dimensional space in which the surface resides. This discrete approximation to the smooth surface is commonly known as a triangulation.

One of the more important quantities associated with any 2-dimensional surface is the Gaussian curvature. This is usually computed by taking various derivatives on a smooth surface. Yet that is clearly not possible on a triangulation (as a smooth function) since the local metric is at best piecewise constant. Nonetheless it seems reasonable to expect that where a triangulation closely approximates a smooth surface then the curvature on the triangulation should be close to the curvature of the smooth surface. How then can such a curvature on a triangulation be computed? Various methods ([1, 2, 3, 4, 5, 6]) have been developed over the years that broadly speaking divide into two approaches. In one approach a smooth surface is interpolated through the vertices which in turn allows the curvature to be computed using standard methods (see [2] for an extensive review). The other approach uses area weighted sums to estimate the local curvature (see [3]).

It is well known that for the case where 4 triangles meet at a vertex the estimated Gaussian curvature need not converge to the correct value (as the triangulation is refined towards a continuum limit). Here we shall demonstrate that failure for the simple case of four identical triangles on a 2-sphere. We will also show that the correct convergent estimate of the curvature can be recovered by using an adjusted set of leg lengths given by the main theorem.

The Gaussian curvature on a unit 2-sphere S^2 in E^3 is 1 everywhere on S^2. Consider now any point p on S^2 enclosed by 4 equally spaced points a, b, c and d also on S^2. This set of points can be connected to form 4 triangles attached to p as indicated in figure (1). The Gaussian curvature at p will be estimated by solving the coupled system of equations

$$L_{ij}^2 = g_{\mu\nu}\Delta x_{ij}^\mu \Delta x_{ij}^\nu - \frac{1}{3}R_{\alpha\mu\beta\nu}x_i^\alpha x_j^\beta x_i^\mu x_j^\nu \tag{3.1}$$

for the Riemann normal coordinates x_i^μ for each vertex $i = p, a, b, c, d$ and the Riemann components $R_{\alpha\mu\beta\nu}$ at p. The L_{ij}^2 are the squared arc-lengths between vertices i and j. In the first instance we will take the L_{ij} to be the Euclidian arc-length given by the embedding of the vertices in E^3. Later we will adjust the L_{ij} by using equation (2.22).

The symmetry of the 2-sphere allows us to choose all triangles to be identical and to also choose the Cartesian and Riemann coordinates of each vertex as per table (1). With this choice of coordinates the equations (3.1) can be reduced to just two equations, namely

$$\bar{L}_{pa}^2 = \bar{x}^2 + (\bar{z} - 1)^2 = \tilde{x}^2 \tag{3.2}$$

$$\bar{L}_{ab}^2 = 2\bar{x}^2 = 2\tilde{x}^2 - \frac{1}{3}K\tilde{x}^4 \tag{3.3}$$

where $K = R_{1212}$ is the Gaussian curvature at p. The constraint that the points lie on the unit sphere leads to just one equation

$$1 = \bar{x}^2 + \bar{z}^2 \tag{3.4}$$

Thus we have three equations for four unknowns $\bar{x}$, $\bar{z}$, $\tilde{x}$ and K. Clearly we can choose $\bar{x}$ as a free parameter which leads to the following solution for K

$$K = \frac{3}{2} + \mathcal{O}\left(\bar{x}^2\right) \tag{3.5}$$

This shows clearly that as the set of points converge to p the estimate for K converges to the incorrect value of $3/2$. This is a well known result and is not germain to the use of Riemann normal coordinates (see [7, 8]).

Vertex	Coordinates	
	Cartesian	Riemann
p	$(\ \ 0,\ \ 0,\ \ 1)$	$(\ \ 0,\ \ 0)$
a	$(\ \ \bar{x},\ \ 0,\ \ \bar{z})$	$(\ \ \tilde{x},\ \ 0)$
b	$(\ \ 0,\ \ \bar{x},\ \ \bar{z})$	$(\ \ 0,\ \ \tilde{x})$
c	$(-\bar{x},\ \ 0,\ \ \bar{z})$	$(-\tilde{x},\ \ 0)$
d	$(\ \ 0,-\bar{x},\ \ \bar{z})$	$(\ \ 0,-\tilde{x})$

Table 1: The Cartesian and Riemann normal coordinates of the 5 vertices. This choices makes full use of the known symmetries of the 2-sphere. The size of the triangles is controlled by the freely chosen coordinate $\bar{x}$ while $\bar{z}$ is set by the contraint that the points lie on the unit 2-sphere. The Riemann normal coordinate $\tilde{x}$ and the Gaussian curvature can then be computed from the leg-lengths as described in the text.

We now repeat the computations but this time using estimates for the geodesic arc-length on the unit sphere as given by equation (2.22). For the unit sphere it is easy to see that

$$\frac{dL}{dn} = L \tag{3.6}$$

and thus from (2.22) we find

$$\tilde{L}_{ij}^2 = \bar{L}_{ij}^2 + \frac{1}{12}\bar{L}_{ij}^4 + \mathcal{O}\left(L^5\right) \tag{3.7}$$

Using this equation to estimate $\tilde{L}_{pa}^2$ and $\tilde{L}_{ab}^2$ leads to the following adjusted Riemann normal equations

$$\tilde{L}_{pa}^2 = \bar{L}_{pa}^2 + \frac{1}{12}\bar{L}_{pa}^4 = \tilde{x}^2 \tag{3.8}$$

$$\tilde{L}_{ab}^2 = \bar{L}_{ab}^2 + \frac{1}{12}\bar{L}_{ab}^4 = 2\tilde{x}^2 - \frac{1}{3}K\tilde{x}^4 \tag{3.9}$$

where $\bar{L}_{pa}^2$ and $\bar{L}_{ab}^2$ should be considered as functions of $\bar{x}$ and $\bar{z}$ given by equations (3.2,3.3). Once again we have three equations for four unknowns. This system can be solved in exactly the same manner as before to obtain

$$K = 1 + \mathcal{O}\left(\bar{x}^2\right) \tag{3.10}$$

which clearly gives the correct result as $\bar{x} \to 0$.
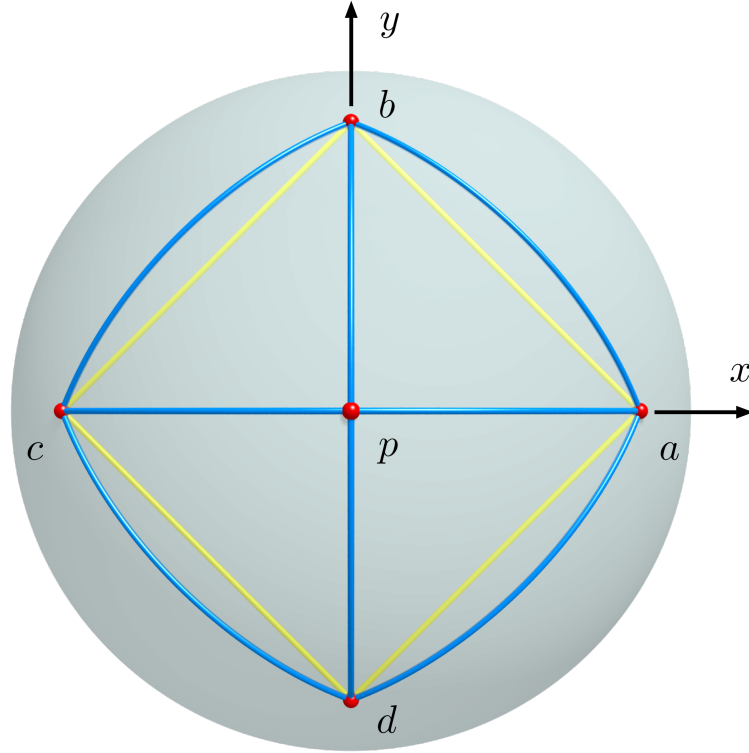
Figure 1: A (not so small) patch of triangles on a unit 2-sphere. This is a view looking down the $z-$axis onto the north pole. The vertices p, a, b, c and d lie on the the various coordinate planes with Cartesian and Riemann normal coordinates as listed in table (1). The yellow edges are the geodesics in E^3 while the blue edges are the geodesics on the sphere.

A reasonable objection to this approach is that in obtaining equation (3.10) we have made use of known properties of the unit sphere. Thus it should be no surprise that we get a better result. However we could easily propose a hybrid scheme in which the method described by Meyer et al. ([3]) would be used to estimate the normals at the vertices which in turn would allow us to use equation (2.14) to estimate $\tilde{L}_{ij}^2$. Numerical experiments on such a hybrid scheme indicates that it can offer improvements over standard methods. The results will be reported elsewhere.

3.2 The Schwarz lantern

There are many ways to triangulate a cylinder such as the Schwarz lantern shown in figure (2). This particular triangulation was chosen by Schwarz (see [9, 10]) to provide a simple counter example to a claim that if all the points of a triangulation converge to a smooth surface (i.e., by creating by more and more triangles while decreasing their size to zero) then the surface area of the triangulation would converge to the area of the surface.

We will do the standard computation that establishes error bounds for the area. We will then repeat the computation but this time using the adjusted arc-lengths given by (2.14) yielding an improved estimate for the error bounds.
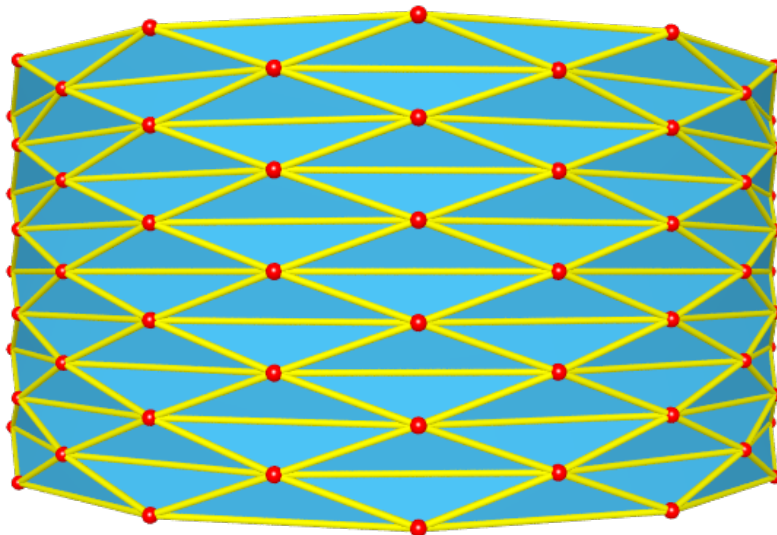

Figure 2: An example of a Schwarz lantern built from 220 identical triangles. In the general case the cylinder is divided into $2M$ horizontal slices and $2N$ vertical slices for a total of $4NM$ triangles. The example shown here has $N = 11$ and $M = 5$.

We begin by first orienting the Cartesian coordinate axes so that the $z-$axis runs up the centre of the cylinder (see figure (3)) while the $x-$axis passes through a vertex p of a typical triangle (recall that all triangles are identical to each other modulo reflections in the $xy-$plane). The coordinates of the

Vertex	Cartesian Coordinates
p	$(\quad 1, \quad\quad 0, \quad 0)$
q	$(\quad \cos\frac{\pi}{N}, \quad \sin\frac{\pi}{N}, \quad \frac{1}{2M})$
r	$(\quad \cos\frac{2\pi}{N}, \quad \sin\frac{2\pi}{N}, \quad 0)$

Table 2: The Cartesian coordinates of the 3 vertices on the Schwarz lantern.

three vertices are shown in table (2). Using standard Euclidian geometry it is easy to show that

$$\bar{L}^2_{pr} = 4\sin^2\left(\frac{\pi}{N}\right) \tag{3.11}$$

$$\bar{L}^2_{pq} = 4\sin^2\left(\frac{\pi}{2N}\right) + \frac{1}{4M^2} \tag{3.12}$$

$$A^2_{pqr} = \left(\frac{\pi}{2NM}\right)^2 \tag{3.13}$$

$$\bar{A}^2_{pqr} = \frac{1}{16}\bar{L}^2_{pr}\left(4\bar{L}^2_{pq} - \bar{L}^2_{pr}\right) \tag{3.14}$$

where A_{pqr} is the exact area (i.e., the area of a triangle drawn entirely on the cylinder) and $\bar{A}_{pqr}$ is the area of the flat triangle with vertices p, q and r.

It is not hard to show the fractional error in using $\bar{A}^2_{pqr}$ as an approximation to A^2_{pqr} is subject to the following bounds

$$\frac{\pi^2}{3N^2} - \frac{\pi^4}{45N^4}\left(2 + 45M^2\right) < \frac{A^2_{pqr} - \bar{A}^2_{pqr}}{A^2_{pqr}} < \frac{\pi^2}{3N^2} \tag{3.15}$$

But since the total area of the triangulation $\bar{S}$ is given by $\bar{S} = 4NM\bar{A}_{pqr}$ while the total area of the cylinder is $S = 4NMA_{pqr} = 2\pi$ we see that

$$\frac{4\pi^4}{3N^2} - \frac{4\pi^6}{45N^4}\left(2 + 45M^2\right) < S^2 - \bar{S}^2 < \frac{4\pi^4}{3N^2} \tag{3.16}$$

This clearly shows that for the total error to vanish we need not only $N \to \infty$ but also $(M/N^2) \to 0$. If these two conditions are not satisfied then the total area of the triangulation need not converge to that of the cylinder (and can even diverge to infinity).

We will now repeat the above calculation but this time using an adjusted set of L^2_{ij} given by (2.14). The normal vector for any point on a cylinder of unit
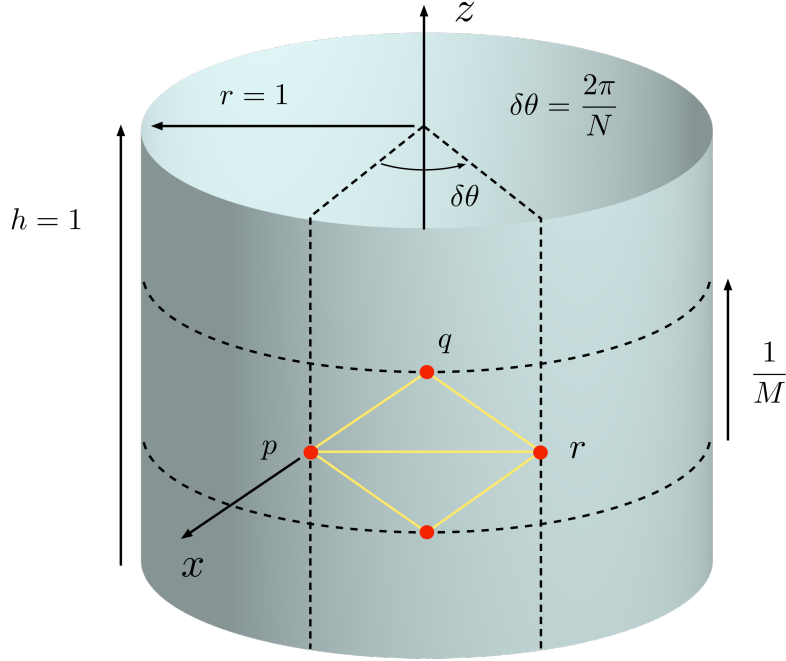
Figure 3: A typical pair of triangles in the Schwarz lantern. In the text we compute the area A_{pqr} of the triangle based on the vertices p, q and r. The total area of the smooth cylinder is $S = 2\pi$ while the sume of the areas over the triangulation is $4NMA_{pqr}$.

radius is easily computed. For example, for any point on the cylinder with coordinates (x, y, z) the unit normal vector at that point has components $(x, y, 0)$. This allows us to easily apply equation (2.14) to estimate the $\tilde{L}_{ij}$ on the cylinder. Note also that since the cylinder has zero Gaussian curvature we can use the standard Euclidian formula for the area of triangle. Thus we have

$$\tilde{L}_{pr}^2 = \bar{L}_{pr}^2 + \frac{1}{12}\left(n_\mu(p) - n_\mu(r)\right)\Delta x_{pr}^\mu\right)^2 \tag{3.17}$$

$$\tilde{L}_{pq}^2 = \bar{L}_{pq}^2 + \frac{1}{12}\left(n_\mu(p) - n_\mu(q)\right)\Delta x_{pq}^\mu\right)^2 \tag{3.18}$$

$$\tilde{A}_{pqr}^2 = \frac{1}{16}\tilde{L}_{pr}^2\left(4\tilde{L}_{pq}^2 - \tilde{L}_{pr}^2\right) \tag{3.19}$$

With these estimates for $\tilde{L}_{ij}^2$ we find the following error bounds for the total area

$$\frac{32\pi^6}{45N^4} - \frac{8\pi^8}{189N^6}\left(6 + 63M^2\right) < S^2 - \tilde{S}^2 < \frac{32\pi^6}{45N^4} \tag{3.20}$$

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where $\tilde{S}$ is the total area of the triangulation (using the adjusted arc-lengths). As with the previous example, $\tilde{S}$ will converge to S only when $N \to \infty$ while $M/N^2 \to 0$. But note that when $M/N^2 \to 0$ the errors bounds are of order $\mathcal{O}\left(N^{-4}\right)$ whereas in the previous example (using Euclidian arc-lengths) the errors were of order $\mathcal{O}\left(N^{-2}\right)$. This is a considerable improvement.

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