A simple expression for the ADM mass

Leo Brewin

School of Mathematical Sciences
Monash University
Clayton 3800
Australia

20-Sep-2006

Abstract

We show by an almost elementary calculation that the ADM mass of an asymptotically flat space can be computed as a limit involving a rate of change of area of a closed 2-surface. The result is essentially the same as that given by Brown and York [1, 2]. We will prove this result in two ways, first by direct calculation from the original formula as given by Arnowitt, Deser and Misner and second as a corollary of an earlier result by Brewin for the case of simplicial spaces.

1   Introduction

It is well known that the for the Schwarzschild spacetime the ADM and the Schwarzschild masses are equal. Interestingly one can also easily recover the Schwarzschild mass by a simple calculation involving the areas of 2-spheres. Consider the time symmetric initial data for the Schwarzschild spacetime in the standard Schwarzschild coordinates

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

The area of a sphere of radius r is $A = 4\pi r^2$ and its rate of change with respect to proper radial distance is

$$\frac{dA}{ds} = 8\pi r \left(1 - \frac{2m}{r}\right)^{(1/2)}$$

For $r >> 2m$ we can expand this in powers of m/r to obtain

$$\frac{dA}{ds} = 8\pi r - 8\pi m + \mathcal{O}\left(\frac{1}{r}\right)$$

The leading term on the right hand side is exactly the flat space value for dA/ds and thus we can re-write the above in the suggestive form

$$8\pi m = \left(\frac{dA}{ds}\right)_f - \left(\frac{dA}{ds}\right)_c + \mathcal{O}\left(\frac{1}{r}\right)$$

with the subscripts 'f' and 'c' denoting flat and curved space quantities respectively. The main result of this paper is that a similar result applies to all asymptotically flat spacetimes.

The first point to address is how we might compute the curved and flat space versions of dA/ds for a general 3-metric. The procedure we shall follow is quite simple to state. We start with a given space $(\mathcal{M}, g)$ and then chose a large closed 2-surface S in $\mathcal{M}$. For later convenience we shall assume that the surface S is everywhere convex (i.e. that its intrinsic Gauss curvature is everywhere non-negative). This is not a serious constraint and should be easily satisfied provided the surface is built sufficiently far into the asymptotically flat regime of the space.

The surface S will inherit, from its embedding in $(\mathcal{M}, g)$, a unit normal n^μ and an intrinsic 2-metric $h_{\mu\nu}$. Thus we can compute not only the surface area A of S but also the surface areas for images of S obtained by dragging S a short distance along the integral curves of n^μ. This in turn will allow us to compute the rate of change of the area, dA/ds, with respect to the proper distance s off S.

Now we shall engage in a small piece of surgery. First we discard the portion of $(\mathcal{M}, g)$ outside of S (though we retain S itself). Since, by assumption, S is everywhere convex we are assured by the Weyl-Lewy embedding theorem [3, 4] that the truncated space may be extended by attaching a sub-set of flat space to S. The result is a new space which we denote by $(\tilde{\mathcal{M}}, \tilde{g}, S)$. The surface S can now be viewed as a transition surface from the curved interior to a flat exterior. We shall use a '+' subscript to denote quantities on the exterior and likewise a '−' subscript for quantities associated with the interior. Note that the surface components of the metric (i.e. the projection of $\tilde{g}$ onto S) will be at least C^0 but no such continuity can be expected for the normal components (i.e. the projection onto the normal to S). Thus we can not expect the area A to vary smoothly as we cross S. Indeed we now

claim that, in the limit $S \to i^0$, the jump in dA/ds is directly proportional to the ADM mass,

$$M = \frac{1}{8\pi} \lim_{S \to i^0} \left(\left(\frac{dA}{ds} \right)_+ - \left(\frac{dA}{ds} \right)_- \right) \tag{1}$$

We will present two very simple yet independent proofs. The first applies to smooth metrics and begins with the standard expression for the ADM mass in asymptotic Euclidian coordinates [5, 6, 7] while the second proof, applicable to simplicial spaces, follows as a simple corollary to an earlier result by Brewin [8].

Brown and York [1, 2] have produced a very similar result (with a slight change of notation),

$$M = \frac{1}{8\pi} \lim_{R \to \infty} \int (K_- - K_+) \, dS \tag{2}$$

where K_+ and K_- are the traces of the intrinsic curvatures of S_+ and S_- respectively. By noting that for any 2-surface embedded in a larger ambient 3-space, $dA/ds = -\int K dS$ it follows that two expressions (1,2) for M are equivalent.

What then is new in the present calculation? Simply that we offer an alternative, and perhaps simpler proof of the main result (1). This fits the maxim that simple results demand simple proofs.

Our calculations are adapted to just one quantity, the ADM mass. In contrast Brown and York's computations are much more extensive. They derive quasi-local estimates for the energy and momentum and they show that these reduce to the familiar ADM expressions at spacelike infinity. Their calculations are much more sophisticated than ours. In our approach we will compute the ADM mass directly from the standard integral formula.

2   The ADM mass

Given a 3-manifold $\mathcal{M}$ with an asymptotically flat positive definite 3-metric g the ADM mass of the space $(\mathcal{M}, g)$ can be defined by ([5, 6, 7])

$$M = \frac{1}{16\pi} \lim_{S \to i^0} \int_S g^{\mu\nu} \left(g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha} \right) n^\alpha \, dS \tag{3}$$

where S is a topological 2-sphere with outward pointing unit normal n^μ and area element dS. Note that the integrand is not in covariant from (indeed there is no covariant expression involving the first derivatives of the metric other than $g_{\mu\nu;\alpha} = 0$). Thus the above integral is defined only for a restricted class of coordinate systems and these are known as asymptotic Euclidian coordinates. In such a system the metric components are required to be of the form $g_{\mu\nu} = \delta_{\mu\nu} + \mathcal{O}(1/r)$.

We first prepare for a series of local calculations by sub-dividing S into a finite set of non-overlapping patches $S_i, i = 1, 2, \cdots$. We do this simply to avoid any complications that might arise from doing a single global calculation. For much of the following we will focus on a single patch which we shall denote by S_i. Our next step will be to show that a simple coordinate system can be constructed in a subset of $\mathcal{M}$ containing S_i. The construction is quite elementary. Consider a typical S_i of S and build a short cylinder C_i, with cross-section S_i, by following the integral curves of the normal vectors to S. The cylinder will extend partly into $\mathcal{M}_+$ and partly into $\mathcal{M}_-$. On the patch S_i construct a 2-dimensional set of coordinates (u, v). If the cylinder is sufficiently short then each point P in the cylinder will be threaded by exactly one of the integral curves originating from S_i. This allows us to assign coordinates (u, v, n) to P in the obvious way, namely n is the (signed) proper distance to P measured along the integral curve from S_i and (u, v) are the coordinates in S_i where the integral curve intersects S_i. We take $n > 0$ for points in $\mathcal{M}_+$ and $n < 0$ for points in $\mathcal{M}_-$. We will now make a coordinate transformation in $\mathcal{M}_+$. Since $\tilde{g}_+$ is flat we know that we can transform the coordinates in $\mathcal{M}_+$ such that $\tilde{g}_+ = diag(1, 1, 1)$. Finally we choose any smooth extension of the coordinate transformation into $\mathcal{M}_-$. By this construction we will have built a (locally) asymptotically Euclidian coordinates in the cylinder C_i thus allowing us to use the above integral (3) for the contribution from S_i to the ADM mass M.

Let A_i be the surface area of S_i. Then using standard results from differential geometry we have
$$\frac{dA_i}{ds} = \frac{1}{2} \int_{S_i} g^{\mu\nu} \mathcal{L}_n \, g_{\mu\nu} \, dS$$
where dS is the area element on S_i and n^μ is the unit normal vector to S_i. However

$$
\begin{aligned}
g^{\mu\nu} \mathcal{L}_n \, g_{\mu\nu} &= g^{\mu\nu} \left(g_{\mu\nu,\alpha} n^\alpha + g_{\alpha\nu} n^\alpha{}_{,\mu} + g_{\mu\alpha} n^\alpha{}_{,\nu} \right) \\
&= g^{\mu\nu} g_{\mu\nu,\alpha} n^\alpha + g^{\mu\nu} \left(g_{\alpha\nu} n^\alpha \right)_{,\mu} - g^{\mu\nu} g_{\alpha\nu,\mu} n^\alpha + g^{\mu\nu} g_{\mu\alpha} n^\alpha{}_{,\nu} \\
&= g^{\mu\nu} \left(g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu} \right) n^\alpha + g^{\mu\nu} n_{\mu,\nu} + n^\mu{}_{,\mu}
\end{aligned}
$$

and thus we have

$$\frac{1}{8\pi}\frac{dA_i}{ds} = \frac{1}{16\pi}\int_{S_i} g^{\mu\nu}\left(g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu}\right)n^\alpha\, dS + \frac{1}{16\pi}\int_{S_i}\left(g^{\mu\nu}n_{\mu,\nu} + n^\mu{}_{,\mu}\right)dS$$

The integrand has been deliberately re-arranged to isolate the terms that arise in the original definition of the ADM mass.

Let us now define $[Q]$ for any object Q to be the jump in Q across S, that is

$$[Q] = Q_+ - Q_-$$

Now consider the pair of integrals

$$
\begin{aligned}
I_1 &= \int_{S_i}\left[g^{\mu\nu}\left(g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu}\right)n^\alpha\right]dS \\
I_2 &= \int_{S_i}\left[g^{\mu\nu}n_{\mu,\nu} + n^\mu{}_{,\mu}\right]dS
\end{aligned}
$$

It is not hard to see that our claim (equation 1) is true provided we can show that the second integral I_2 vanishes in the limit $S \to i^0$. To prove this we need to carefully examine the asymptotic behaviour of the integrand. Let us first look at the relationship between $n^\mu{}_+$ and $n^\mu{}_-$. Since n^μ_+ and n^μ_- are normal to the surface defined by $n = 0$ we must have $n_{\mu+} = f_+ n_{,\mu}$ and $n_{\mu-} = f_- n_{,\mu}$ where f_+, f_- are scalar functions on S_i. Thus we find

$$n_{\mu-} = f n_{\mu+}$$

where $f = f_-/f_+$. We can also establish a similar result for the components n^μ. To this end, consider any pair of tangent vectors a^μ, b^μ to S_i. Clearly $0 = a_\mu n^\mu_+$ and $0 = b_\mu n^\mu_+$ and thus $n^\mu_+ = q_+\epsilon^{\mu\alpha\beta}_{123}a_\alpha b_\beta$ where q_+ is a function on S_i. Likewise, for n^μ_- we find $n^\mu_- = q_-\epsilon^{\mu\alpha\beta}_{123}a_\alpha b_\beta$ and thus $n^\mu_- = (q_-/q_+)n^\mu_+$. But since both normals are required to be unit vectors we must have $q_-/q_+ = 1/f$ and thus

$$n^\mu_- = \frac{1}{f}n^\mu_+$$

What can we say about the asymptotic behaviour of f? We know that in the conformal analysis of asymptotically flat spacetime the 3-metric is $C^{>0}$ at i^0. The lack of differentiability is intimately tied to the direction dependent limits at i^0. Thus it is not unreasonable to assume that the function f will vary smoothly along each spacelike geodesic ending at i^0. The limit may well depend on which geodesic is chosen but along any one geodesic the function f

5

should vary smoothly. Thus we propose that f should posses an asymptotic expansion of the form

$$f = 1 + \frac{a}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

where a may be a direction dependent quantity near i^0. Alternatively one might argue that the following analysis is applicable only to those spaces for which f obeys the above form. In either case the class of spaces with this behaviour admits at least the time symmetric initial data of the Schwarzschild spacetime.

Now let us turn our attention to the 3-metric and its behaviour across S. The integral curves through S_i provide a natural means for foliating C_i by a sequence of images of S_i. Thus we can write the 3-metric in C_i in a 2+1 form

$$g_{\mu\nu} = h_{\mu\nu} + n_\mu n_\nu$$

In this form $h_{\mu\nu}$ is the induced 2-metric on each leaf of the foliation. Continuity of the 3-metric across S_i requires

$$0 = [h_{\mu\nu}] \qquad i.e. \;\; h_{\mu\nu+} = h_{\mu\nu-}$$

All indices will be raised and lowered using the 3-metric $g_{\mu\nu}$.

We can now express all of the terms in I_2 solely in terms of quantities based in $\mathcal{M}_+$. Thus we obtain

$$
\begin{aligned}
\left[g^{\mu\nu}n_{\mu,\nu}\right] &= \left(h^{\mu\nu} + n^\mu n^\nu\right) n_{\mu,\nu} - \left(h^{\mu\nu} + \frac{1}{f^2}n^\mu n^\nu\right)(f n_\mu)_{,\nu} \\
&= (1-f)h^{\mu\nu}n_{\mu,\nu} - \frac{1}{f^2}f_{,\mu}n^\mu \\
\left[n^\mu{}_{,\mu}\right] &= n^\mu{}_{,\mu} - \left(\frac{1}{f}n^\mu\right)_{,\mu} \\
&= \left(1 - \frac{1}{f}\right)n^\mu{}_{,\mu} + \frac{1}{f^2}f_{,\mu}n^\mu
\end{aligned}
$$

where all terms on the right hand side are understood to be evaluated in $\mathcal{M}_+$ and thus, for simplicity, the $+$ subscripts have been dropped. Note that we have also made use of the fact that the metric in $\mathcal{M}_+$ is flat and thus in the Euclidian coordinates (which we chose earlier) $h^{\mu\nu}n_{\mu,\nu} = g^{\mu\nu}n_{\mu,\nu} = n^\mu{}_{,\mu}$ and $0 = n^\mu n^\nu n_{\mu,\nu}$. Combining the above we obtain

$$\left[g^{\mu\nu}n_{\mu,\nu}\right] + \left[n^\mu{}_{,\mu}\right] = -\frac{1}{f}\left(1 - f\right)^2 h^{\mu\nu}n_{\mu,\nu}$$

Notice that the term $h^{\mu\nu}n_{\mu,\nu}$ is proportional to the trace of the intrinsic curvature on S_i and thus varies as $\mathcal{O}(1/r)$. Finally, since $f - 1 = \mathcal{O}(1/r)$ we see that the integrand of I_2 varies as $\mathcal{O}(1/r^3)$ and thus I_2 will vanish in the limit as $S_i \to i^0$. This completes the proof, that I_2 vanishes. The last step is to form a sum over all of the $S_i, i = 1, 2, \cdots$ yielding the formula as stated above (equation 1).

3   Simplicial lattices

Lattice theories such as the Regge Calculus (see [9, 10, 11]) have enjoyed some success as a tool for numerical relativity though they have more commonly been used as a model for discrete theories of gravity. A popular example is the Regge calculus in which a simplicial lattice is built from a collection of non-intersecting blocks (which in 3-d are usually tetrahedral), each with a flat internal metric, and an assignment of leg lengths to each leg in the lattice. Though the metric is flat in each tetrahedron, it is not flat everywhere. In fact the curvature must be treated as a distribution with support on the legs of the lattice. This complicates the study of such lattices but progress can be made. Indeed as an example of the use of standard distribution theory we showed [8] that the ADM mass of a simplicial lattice could be computed as[1]

$$M = -\frac{1}{8\pi} \sum_S \theta L \qquad (4)$$

where the sum includes all the legs in the (triangulated) 2-surface S, θ is the defect angle on the leg and L is the length of that leg. The defect angle θ is defined as the $\theta = 2\pi - \sum_i \theta_i$ where θ_i are the angles subtended at the leg by the blocks attached to the leg. In our case we have four triangular cylinders attached to each leg on the surface S, with a pair of cylinders on either side of S. The defect angle can then be computed as $\theta = -\theta_+ - \theta_-$ where θ_+ and θ_- are the contributions from the respective pairs of cylinders in $\mathcal{M}_+$ and $\mathcal{M}_-$. Finally, by inspection of Fig. 1, we can see that $(\theta_+)L = (dA/ds)_+$ and $(\theta_-)L = (dA/ds)_-$. Upon substituting these into (4) we recover (1).

Equally we could have started with equation (1) and used it to derive equation (4).

---

[1]Note that there was a typographical error in Brewin's paper [8] . The above corrected formula differ's from Brewin's by a factor of -1/2

References

[1] J. David Brown and James W. York Jr. Quasilocal energy and conserved charges derived from the gravitational action. *Phys.Rev. D*, 47:1407–1419, 1993.

[2] J. David Brown, Stephen R. Lau, and James W. York Jr. Energy of isolated systems at retarded times as the null limit of quasilocal energy. *Phys.Rev. D*, 55:1977–1984, 1997.

[3] Hans Lewy. On the existence of a closed convex surface realizing a given riemannian metric. *Proc. Natl. Acad. Sci. USA*, 24(2):104–106, Feb 1938.

[4] Louis Nirenberg. The weyl and minkowski problems in differential geometry in the large. *Comm. Pure Appl. Math.*, 6:337–394, 1953.

[5] R. Arnowitt, S. Deser, and C. W. Misner. The dynamics of general relativity. In L. Witten, editor, *Gravitation: An Introduction to Current Research*, pages 227–265. Wiley, 1962.

[6] Charles W. Misner, Kip S. Thorne, and John A. Wheeler. *Gravitation*. W. H. Freeman, 1973.

[7] Robert M. Wald. *General Relativity*. University of Chicago Press, 1984.

[8] Leo Brewin. Anowitt–deser–misner energy and three-momentum for a simplicial space. *Phys.Rev.D*, 39(8):2258–2262, 1989.

[9] Tullio Regge. General Relativity without coordinates. *Il Nuovo Cimento*, XIX(3):558–571, 1961.

[10] R.M. Willimas and P.A. Tuckey. Regge calculus : A bibliography and a brief review. *Classical and Qunatum Gravity*, 9:1409–1422, 1992.

[11] Adrian P. Gentle. Regge calculus: a unique tool for numerical relativity. *Gen. Rel. Grav.*, 34:1701–1718, 2002.
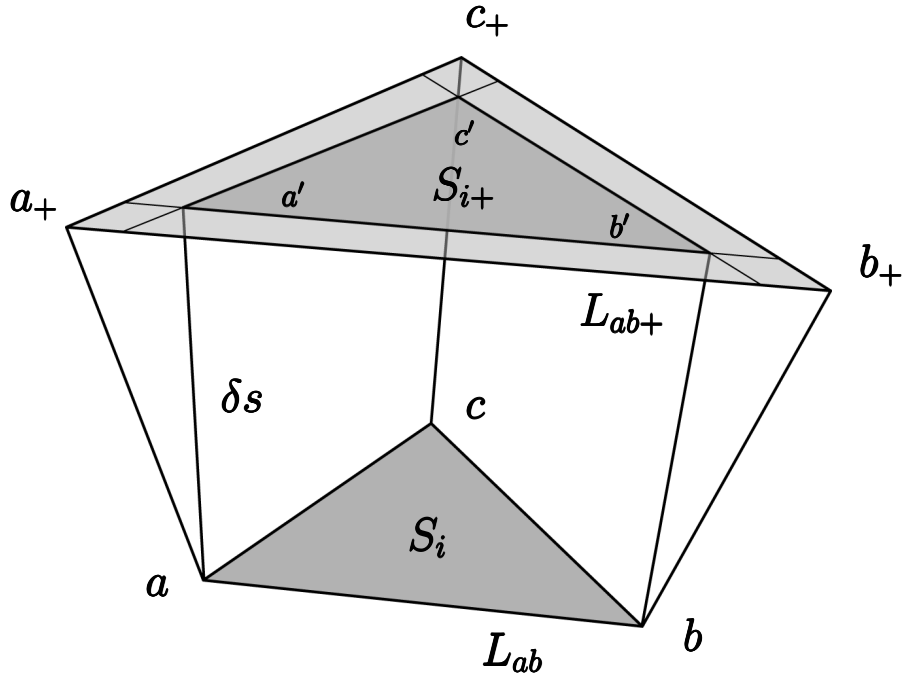
This image contains two cylinders each with a triangular crossection. The outer cylinder, based on the vertices (a, b, c) and $(a, b, c)_+$ is a cylinder of the simplicial lattice. The inner cylinder, based on the vertices (a, b, c) and $(a, b, c)'$, is built from the simplicial lattice by dragging the lower triangle (a, b, c) outward along its normal a distance δs. The areas of the two grey triangles are thus equal and thus the change in area from S_i to S_{i+} equals the sum of the areas of the three (approximate) rectangles plus the three diamonds on S_{i+}. These later areas vary as δs^2 and thus, to leading order, the contribution to δA from the leg (a, b) will be $\delta A_{ab} = \theta_{ab} L_{ab} \delta s$ where θ_{ab} is the angle between the faces (a, b, a', b') and (a, b, a_+, b_+).