The coclass graph associated with p-groups of maximal class

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Maximal class and coclass

A p-group of order p^n has **maximal class** if its nilpotency class is $n-1$.

- Blackburn [1] classified the 2- and 3-groups of maximal class by finitely many *parametrised presentations*.
- The 5-groups of maximal class are investigated in detail by Leedham-Green & McKay [6], Newman [7], and Dietrich, Eick & Feichtenschlager [2]. It is conjectured that these groups can be classified in a similar way.
- For $p \geq 7$, such a classification is open.

Nowadays, the p-groups of maximal class are an important special case in **coclass theory**.

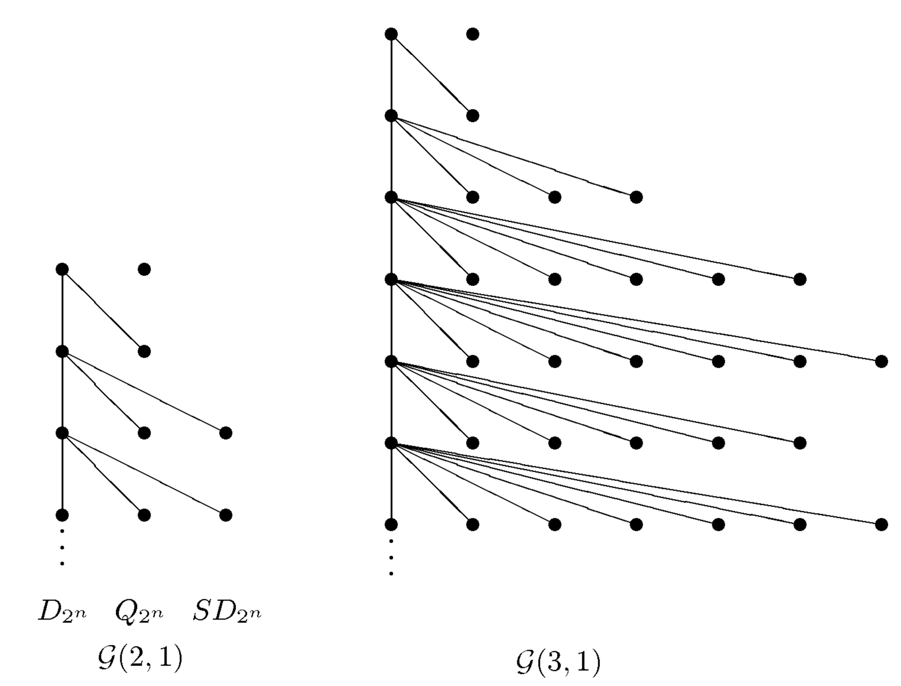
The **coclass** of a p-group of order p^n and nilpotency class c is $n-c$, and the p-groups of coclass 1 are exactly the p-groups of maximal class. New approaches and results on the classification of p-groups by coclass have been developed since 1999; a fundamental tool is the **coclass graph**.

Coclass graphs

The **coclass graph $\mathcal{G}(p,r)$** associated with p-groups of coclass r is defined as the graph with

vertices: isomorphism type representatives of finite p-groups of coclass r, and
edges: $H \to G$ if and only if $H \cong G/\gamma_c(G)$ where c is the nilpotency class of G.

By Blackburn, the graphs $\mathcal{G}(2,1)$ and $\mathcal{G}(3,1)$ have the following structure where groups of smaller order are drawn above groups of larger order:



In general, the coclass graph $\mathcal{G}(p,r)$ is much more complicated and its detailed structure is not known. It is a deep result in coclass theory that, up to finitely many groups, $\mathcal{G}(p,r)$ is the union of finitely many trees such that every tree has exactly one maximal infinite path. These trees are the **coclass trees** of $\mathcal{G}(p,r)$. The unique maximal infinite path in a coclass tree, $S_k \to S_{k+1} \to \ldots$ with $|S_i| = p^i$ say, is its **mainline**. The **n-th branch $\mathcal{B}_n$** of a coclass tree is the subtree induced by the groups which have S_n as a quotient but not S_{n+1}. An important conjecture in coclass theory is

Conjecture (**Central Conjecture**).
(i) *The coclass graph $\mathcal{G}(p,r)$ can be described by a finite subgraph and periodic patterns.*
(ii) *The groups in $\mathcal{G}(p,r)$ can be classified by finitely many parametrised presentations.*

The graph $\mathcal{G}(2,r)$

Newman & O'Brien [8] investigated coclass trees for $p=2$ and conjectured that their branches are isomorphic with some periodicity. In 2001, du Sautoy [3] proved this conjecture. His proof is *non-constructive* as no detailed information about the periodicity or the structure of the groups is obtained. Eick & Leedham-Green [4] provided a *constructive proof* based on an explicit group theoretic construction: **The Central Conjecture is proved for $p=2$.** du Sautoy and Eick & Leedham-Green did not only consider $p=2$ but also **shaved coclass graphs**.

Shaved coclass graphs and the graph $\mathcal{G}(p)$

If $\mathcal{B}_n$ is a branch of a coclass tree $\mathcal{T}$ of $\mathcal{G}(p,r)$, then the **shaved branch $\mathcal{B}_n(k)$** is the subtree of $\mathcal{B}_n$ induced by the groups of distance at most k from its root S_n. The **shaved coclass graph $\mathcal{G}_k(p,r)$** is the subgraph of $\mathcal{G}(p,r)$ induced by *all* shaved branches $\mathcal{B}_n(k)$ in $\mathcal{G}(p,r)$.

By du Sautoy and Eick & Leedham-Green, there exist integers d and f depending on k and $\mathcal{T}$ only such that $\mathcal{B}_i(k) \cong \mathcal{B}_{i+d}(k)$ for all $i \geq f$. The proof of Eick & Leedham-Green shows: **The Central Conjecture is proved for the shaved coclass graphs $\mathcal{G}_k(p,r)$.**

It is known: The difference graph of $\mathcal{G}(p,r)$ and $\mathcal{G}_k(p,r)$ is finite for some k only if $p=2$ or both $p=3$ and $r=1$. The periodicity described above does not suffice to describe $\mathcal{G}(p,r)$.

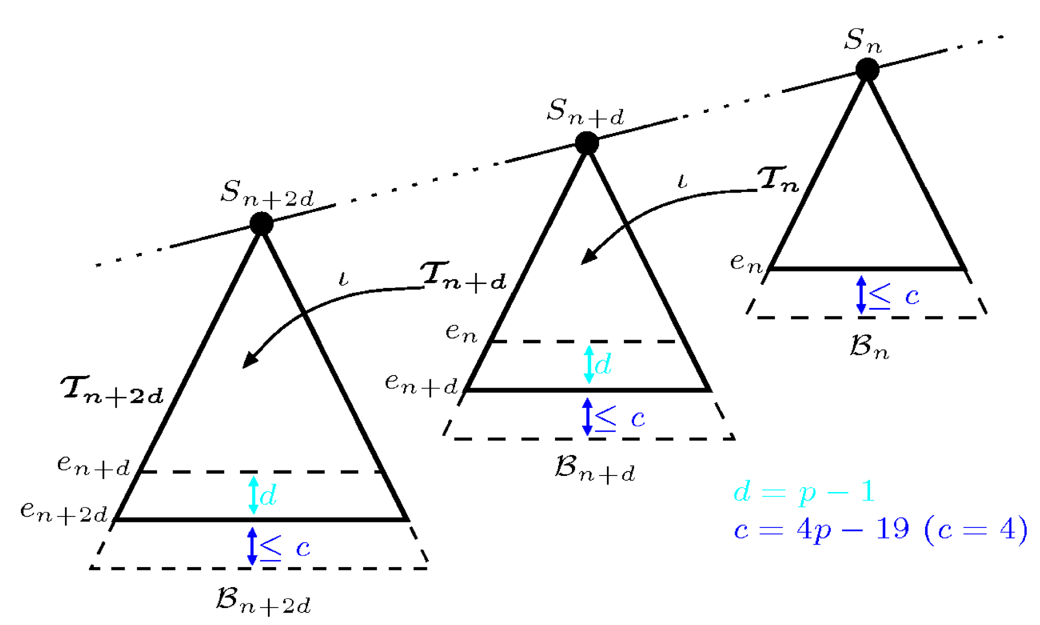
Coclass 1 seems to be the easiest case for which the Central Conjecture is still open. While in general the number of coclass trees in a coclass graph grows rapidly, the graph $\mathcal{G}(p) = \mathcal{G}(p,1)$ has only one coclass tree $\mathcal{T}$ with mainline $S_2 \to S_3 \to \ldots$ If $p \geq 5$, then there exists no bound on the depths of the (finite) branches $\mathcal{B}_2, \mathcal{B}_3, \ldots$

Aim of this research project: **Test the graph $\mathcal{G}(p)$ for periodic patterns.**

Maximal class: Periodicity I

Let $\mathcal{T}$ be the coclass tree of $\mathcal{G}(p)$ with branches $\mathcal{B}_2, \mathcal{B}_3, \ldots$ and let $d = p-1$ and $n > p+1$. Let $e_n = \max\{0, n-2p+8\}$ if $p \geq 7$ and $e_n = \max\{0, n-4\}$ if $p=5$ and define the **n-th body $\mathcal{T}_n$** of $\mathcal{T}$ as the shaved branch $\mathcal{T}_n = \mathcal{B}_n(e_n)$. We prove the periodicity of type I defined as follows:

Theorem (**Periodicity I**).
(i) *There is an embedding $\iota = \iota_n \colon \mathcal{T}_n \hookrightarrow \mathcal{B}_{n+d}$ such that $\mathcal{T}_n \cong \mathcal{B}_{n+d}(e_n)$.*
(ii) *The depth of $\mathcal{T}_n$ is e_n.*
(iii) *The depth of $\mathcal{B}_n$ is bounded by $e_n + c$ with $c = 4p-19$ if $p \geq 7$ and $c = 4$ if $p=5$.*



The periodicity I shows that a **significant part** of $\mathcal{G}(p)$ can be described by periodic patterns. Computer examples indicate that one cannot embed the whole branch $\mathcal{B}_n$ into $\mathcal{B}_{n+d}$. The periodicity I shows that it is sufficient to cut off at most c levels of groups to find an embedding.

If G is a group in the body $\mathcal{T}_n$, then its **periodicity class** is defined as the infinite set $\{G, \iota(G), \iota^2(G), \ldots\}$. We prove that the embeddings ι can be chosen such that

Theorem. *The groups in a periodicity class can be described by a single parametrised presentation with one integer parameter.*

This strongly supports the Central Conjecture for coclass 1.

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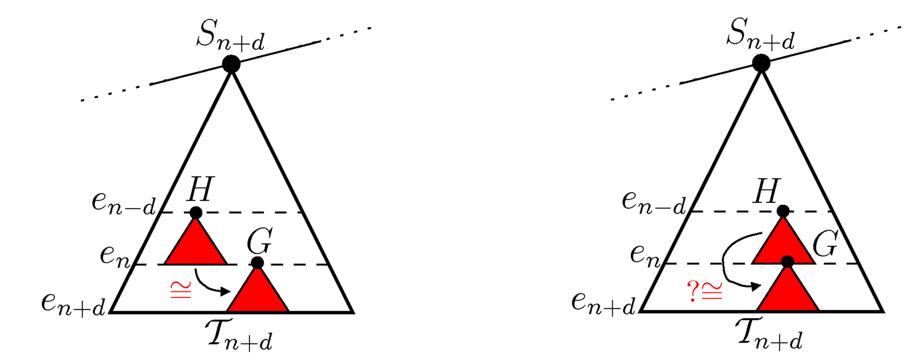
Maximal class: Periodicity II

Eick, Leedham-Green, Newman & O'Brien [5] conjectured that the growth of the branches can be described by another periodic pattern. We claim a weaker version of this conjecture as follows. If G is a group in $\mathcal{T}_n$, then $\mathcal{D}(G)$ is the subtree of $\mathcal{T}_n$ induced by the descendants of distance at most $d = p-1$ from G.

Conjecture (**Periodicity II**). *If n is large enough and G is a group at depth e_n in $\mathcal{T}_{n+d}$, then there exists a group H at depth $e_n - d$ in $\mathcal{T}_{n+d}$ such that $\mathcal{D}(G) \cong \mathcal{D}(H)$.*



periodic parent *d-step parent*

If this conjecture holds, then the bodies of $\mathcal{T}$ can be described by a finite subgraph and the periodic patterns. The main difficulty is to find a **periodic parent** H with $\mathcal{D}(G) \cong \mathcal{D}(H)$. We prove the following theorem; here G is **capable** if it is a central quotient of a group in $\mathcal{G}(p)$.

Theorem. *Let $p \equiv 5 \bmod 6$. If n is large enough and G is a capable group at depth e_n in $\mathcal{T}_{n+d}$, then the d-step parent H is a periodic parent if* $\mathrm{Aut}(H)$ *is a p-group.*

There is computational and theoretical evidence that the d-step parent is a periodic parent for *most* groups in $\mathcal{G}(p)$, but we also suggest that there are infinitely many exceptions. We conjecture an alternative construction of periodic parents and prove it in a special case.

In summary, a **significant part** of $\mathcal{G}(p)$ can be described by periodic patterns.

The **idea of the proof** is to describe the groups as group extensions and to use cohomology theory to obtain a proof on a group theoretic level. The action of *compatible pairs* is used to solve the isomorphism problem; this translates to a problem in p-adic number theory.

There are additional problems in the underlying number theory if $p \equiv 1 \bmod 6$. Though, computations for $p=7$ suggest that the situation is similar to $p \equiv 5 \bmod 6$, that is, in most cases the d-step parent is a periodic parent but there also seem to be infinitely many exceptions.

Maximal class: The graph $\mathcal{G}(5)$

As an application, we prove the following for $p=5$:

Theorem. *The bodies $\mathcal{T}_n$, $n \geq 2$, of $\mathcal{G}(5)$ can be described by a finite subgraph and the periodicities of type I and II.*

Theorem. *The groups in the bodies of $\mathcal{G}(5)$ can be described by finitely many parametrised presentations with at most two integer parameters.*

In general, the difference graph of $\mathcal{B}_n$ and $\mathcal{T}_n$ consists of 4 levels of groups. This shows that the above results are **close to a classification** of the 5-groups of maximal class by finitely many parametrised presentations. They support the conjectures made by Newman [7] and Dietrich, Eick & Feichtenschlager [2].

References

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