SMOOTHED PARTICLE HYDRODYNAMICS: A DEVELOPER'S GUIDE

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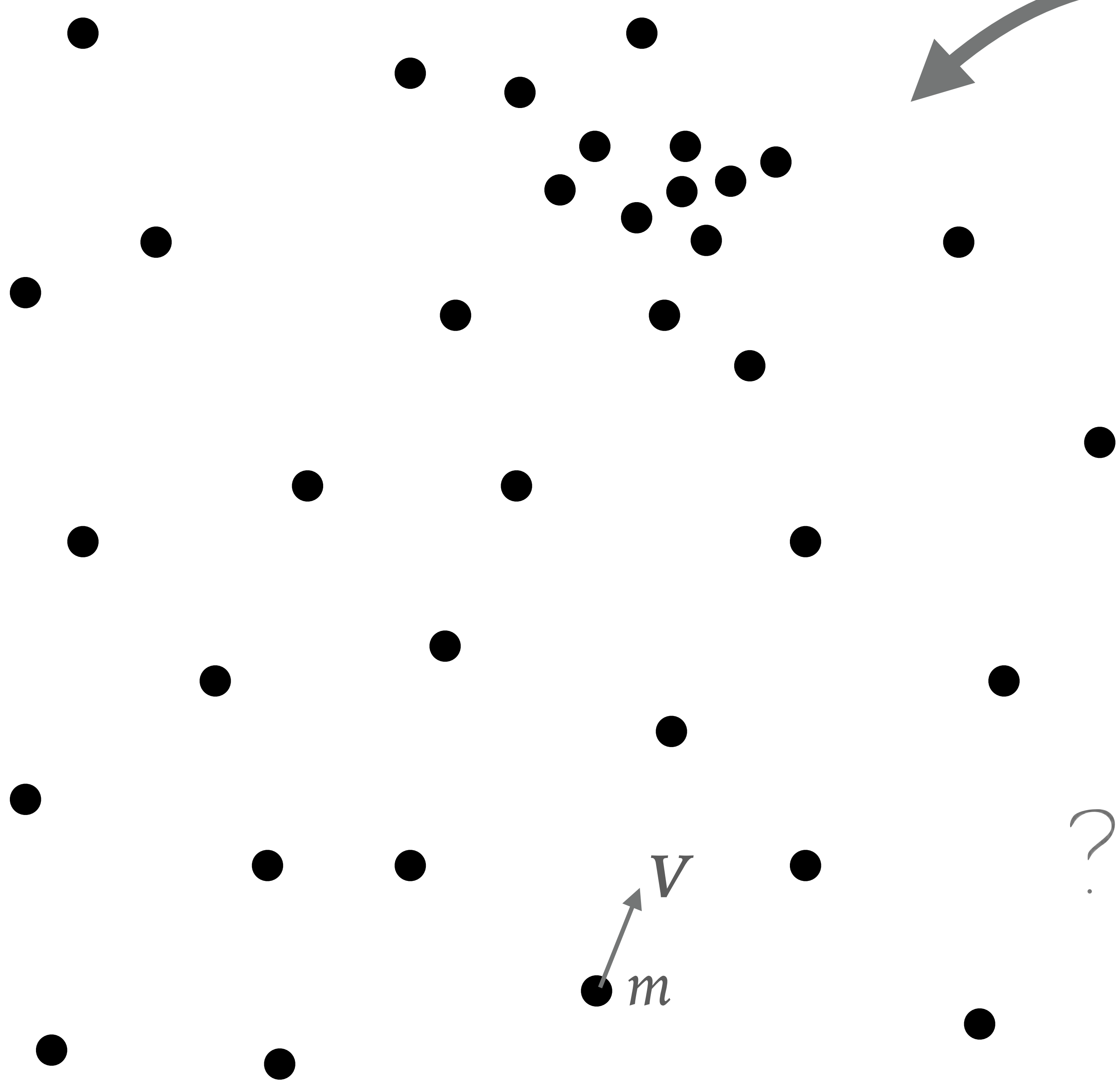
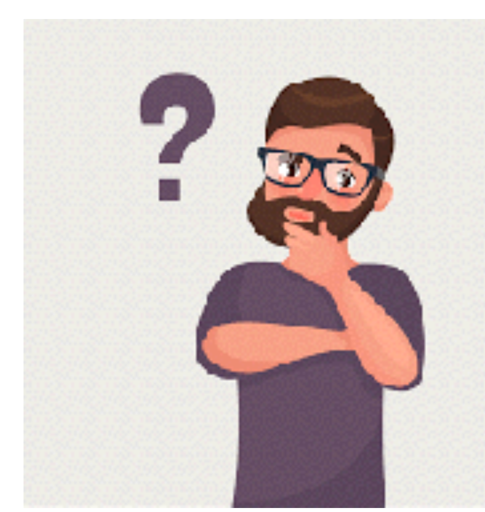
SPHERIC global pandemic SPH seminar series 2020

@danprice_astro

THE PROBLEM



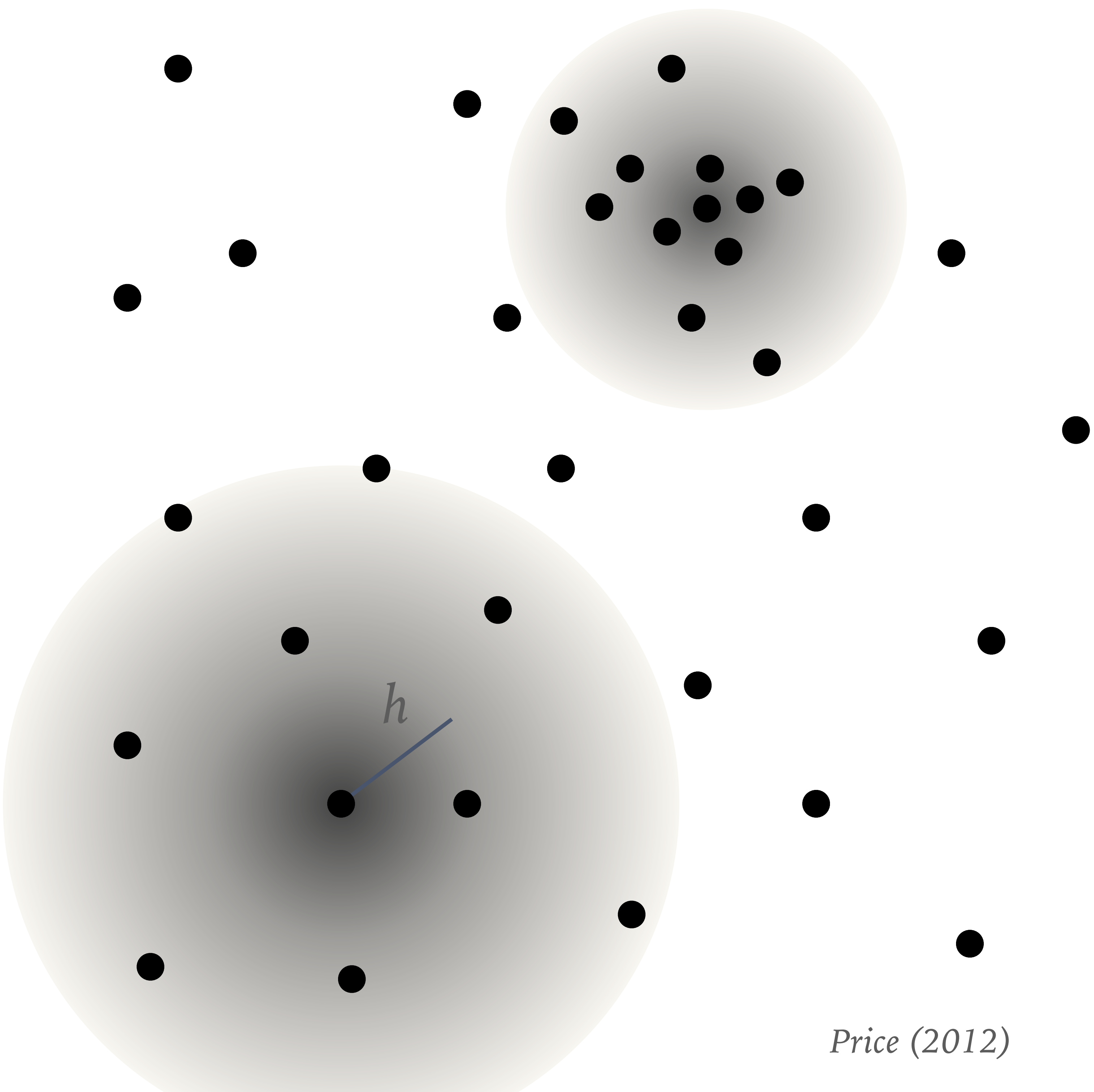
$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho = -\rho(\nabla \cdot \boldsymbol{v})$$
$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla P}{\rho}$$
$$\frac{\partial u}{\partial t} + (\boldsymbol{v} \cdot \nabla)u = -\frac{P}{\rho}(\nabla \cdot \boldsymbol{v})$$



LESSON 1:
THE DENSITY SUM IS THE BRIDGE

WHERE SPH STARTS

e.g. Lucy (1977), Gingold & Monaghan (1977), Monaghan (1992)



What is the density?

$$\rho(\mathbf{r}) = \sum_{j=1}^{N} m_j W(|\mathbf{r} - \mathbf{r}_j|, h)$$

$\rho(x, y)$ m_i, x_i, y_i

Price (2012)

EXAMPLE: SPH VISUALISATION

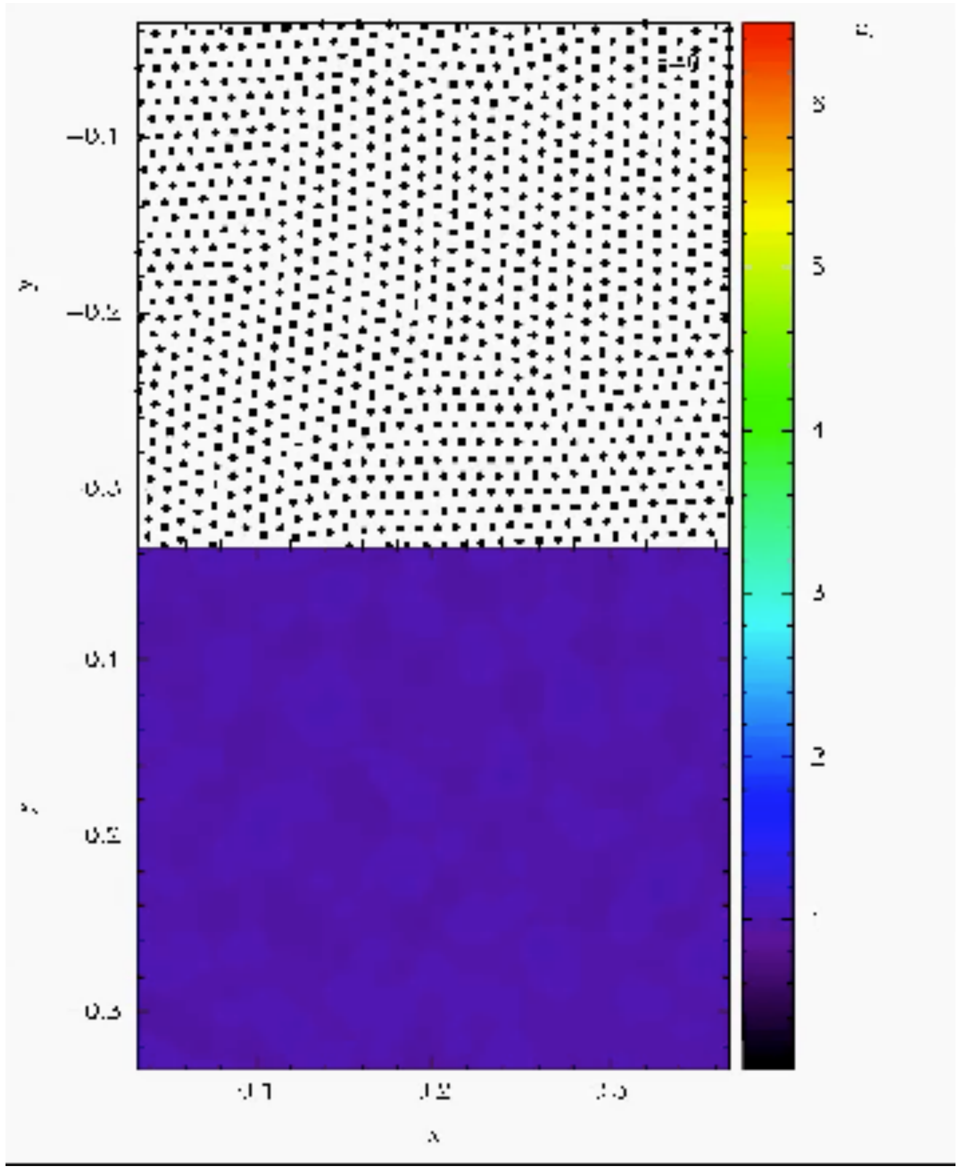
github.com/danieljprice/splash

$$A(\boldsymbol{r}) = \sum_j \frac{m_j}{\rho_j} A_j W(|\boldsymbol{r} - \boldsymbol{r}_j|, h)$$

e.g. Gingold & Monaghan (1977)

Discrete

Continuum

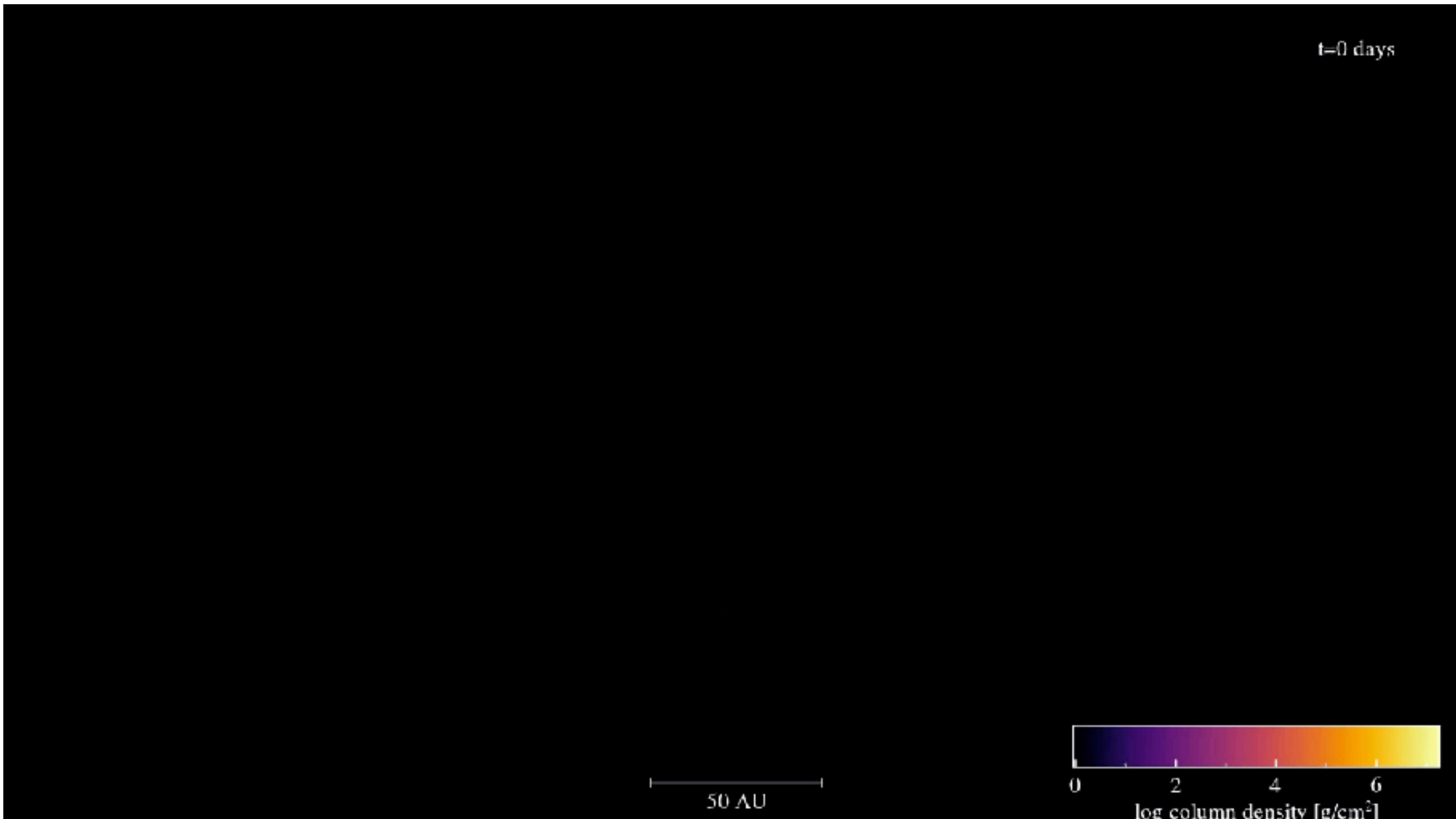


c.f. Price (2007), PASA, 24, 159

EXAMPLE: SPH VISUALISATION

github.com/danieljprice/splash

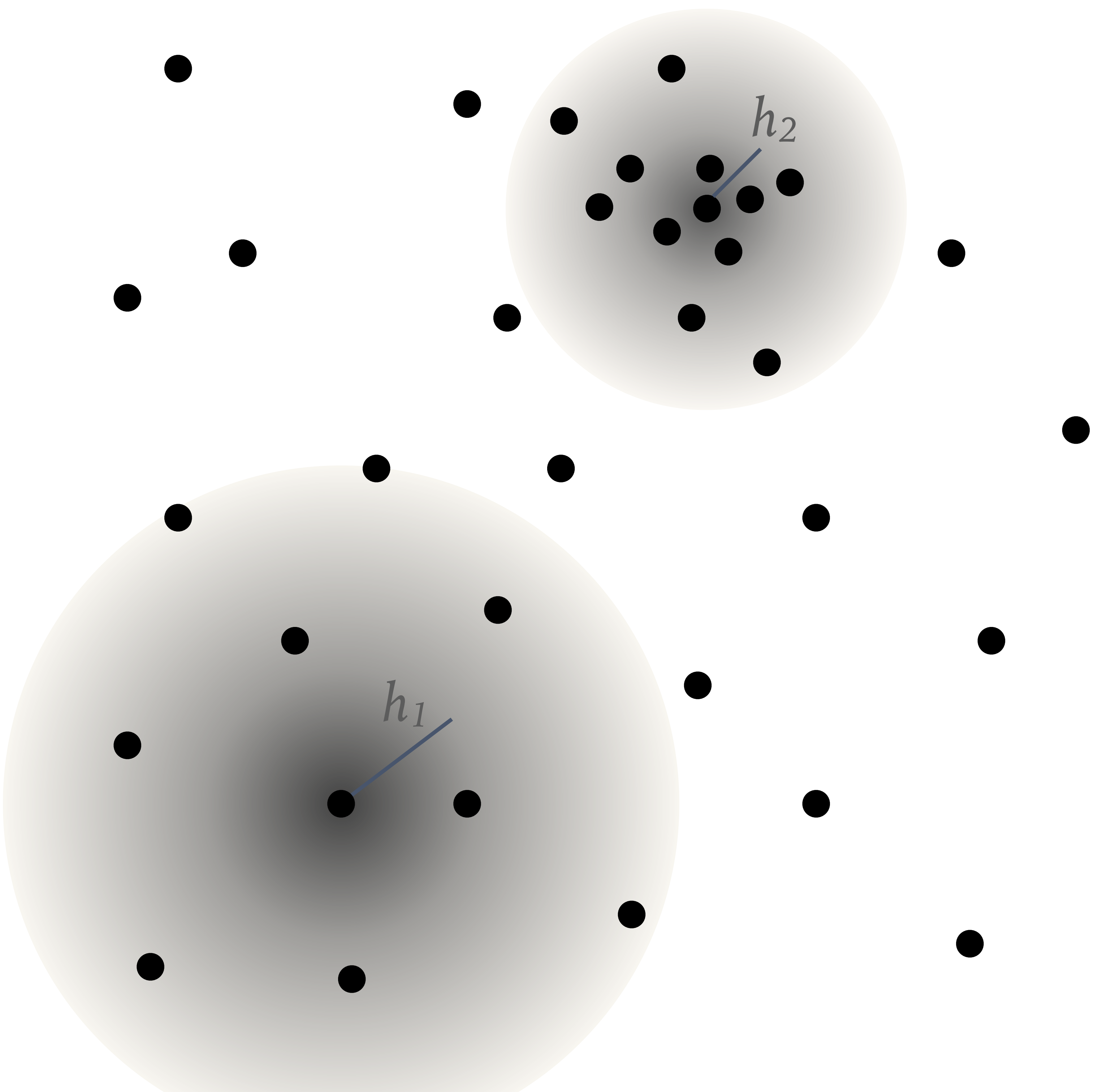
e.g. column density $\Sigma(x,y) = \int \rho(\boldsymbol{r})\mathrm{d}z = \sum_j m_j \int W(|\boldsymbol{r} - \boldsymbol{r}_j|, h)\mathrm{d}z$



Credit: David Liptai (2019)

ADAPTIVE RESOLUTION LENGTHS

e.g. Hernquist & Katz (1989), Benz et al. (1990), Springel & Hernquist (2002), Monaghan (2002), Price & Monaghan (2007)



$$h_i = \eta \left(\frac{m_i}{\rho_i} \right)^{1/n_{\rm dim}}$$

$$\rho(\boldsymbol{r}_i) = \sum_j m_j W(|\boldsymbol{r}_i - \boldsymbol{r}_j|, h_i)$$

- Simultaneous equations for h, rho
- requires iterative solution
- can solve to arbitrary precision

EXAMPLE: GRAVITATIONAL FORCES

Price & Monaghan (2007)

$$\boldsymbol{F} = -\sum_j \frac{G m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3}(\boldsymbol{r}_i - \boldsymbol{r}_j)$$

Diverges if particles come too close!

$$\nabla^2 \Phi = 4\pi G \rho(\boldsymbol{r})$$

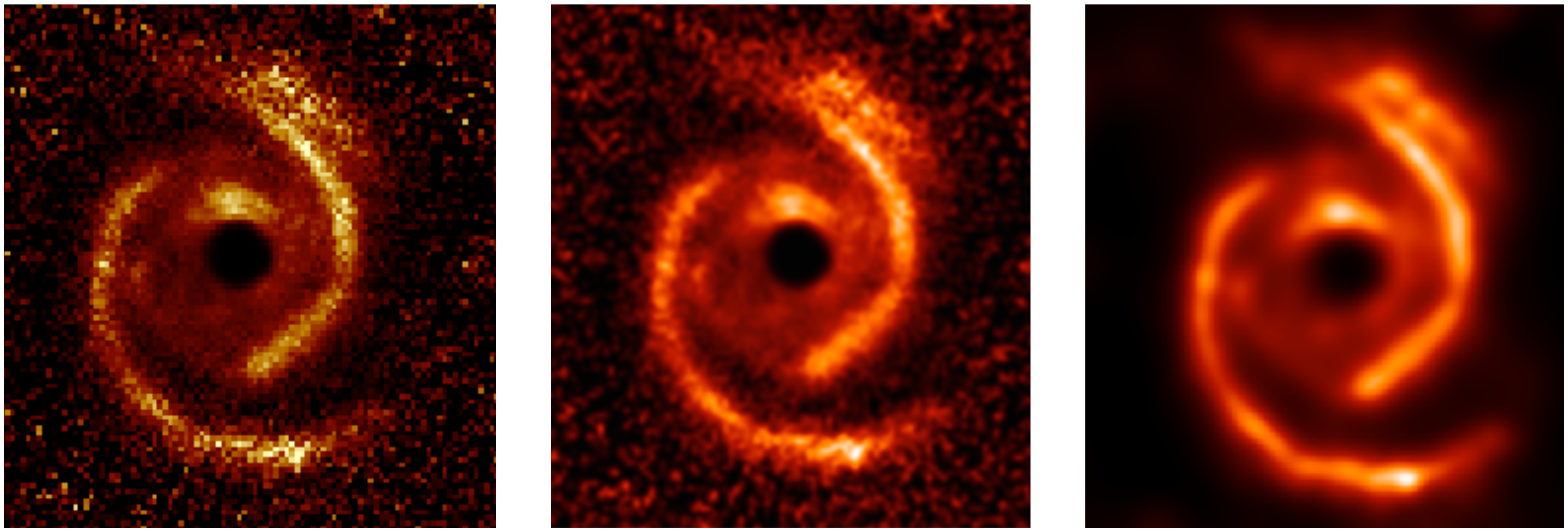
$$\rho(\boldsymbol{r}) = \sum_j m_j W(|\boldsymbol{r} - \boldsymbol{r_j}|, h)$$

$$\frac{\mathrm{d}\boldsymbol{v}_a}{\mathrm{d}t} = -G\sum_b m_b \left[\frac{\phi'_{ab}(h_a) + \phi'_{ab}(h_b)}{2}\right]\frac{\boldsymbol{r}_a - \boldsymbol{r}_b}{|\boldsymbol{r}_a - \boldsymbol{r}_b|}$$
$$-\frac{G}{2}\sum_b m_b \left[\frac{\zeta_a}{\Omega_a}\frac{\partial W_{ab}(h_a)}{\partial \boldsymbol{r}_a} + \frac{\zeta_b}{\Omega_b}\frac{\partial W_{ab}(h_b)}{\partial \boldsymbol{r}_a}\right]$$

Gravitational force "softening" with adaptive softening lengths

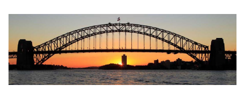
EXAMPLE: IMAGE INTERPOLATION

github.com/danieljprice/denoise



Discrete *Continuum (fixed h)* *Continuum (adaptive h)*

$$h \propto \left[\frac{1}{I(x,y)}\right]^{1/2}$$



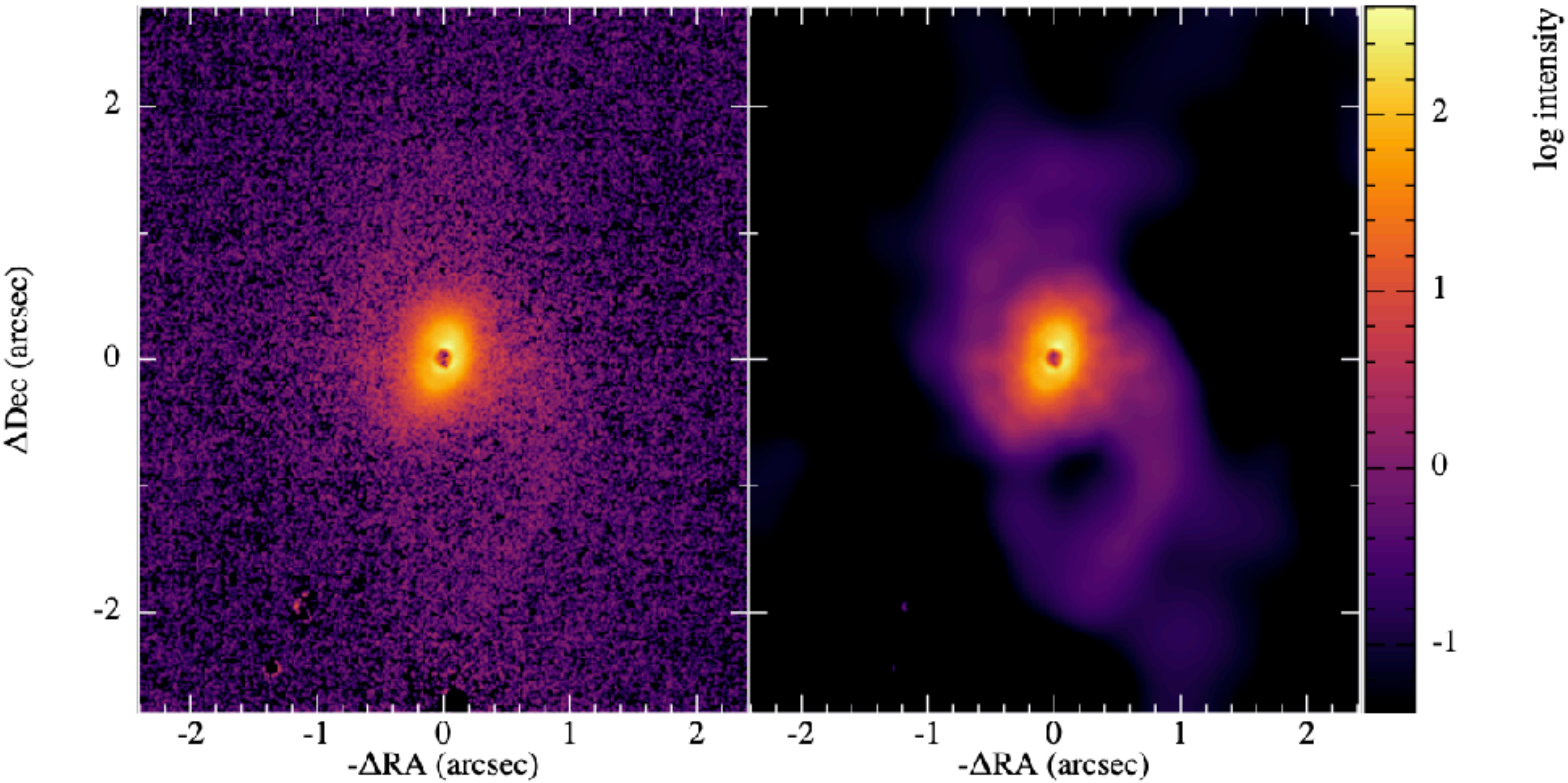
$$I(x,y) = \sum_i \sum_j I(x_i, y_j) W(|x - x_i|, |y - y_j|, h) \Delta x \Delta y$$

- Faint astronomical images: noise proportional to intensity
- Pretend pixels are SPH particles
- Interpolate to find continuum image
- Can we use adaptive smoothing lengths to reconstruct the smooth image?

Price, in prep

EXAMPLE: IMAGE DENOISING

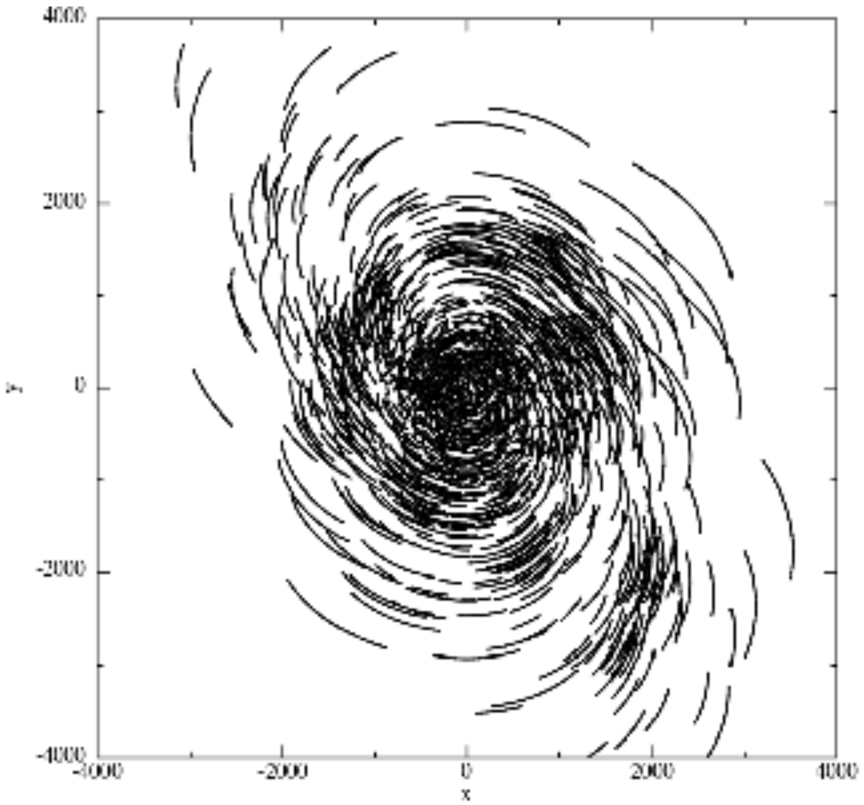
github.com/danieljprice/denoise



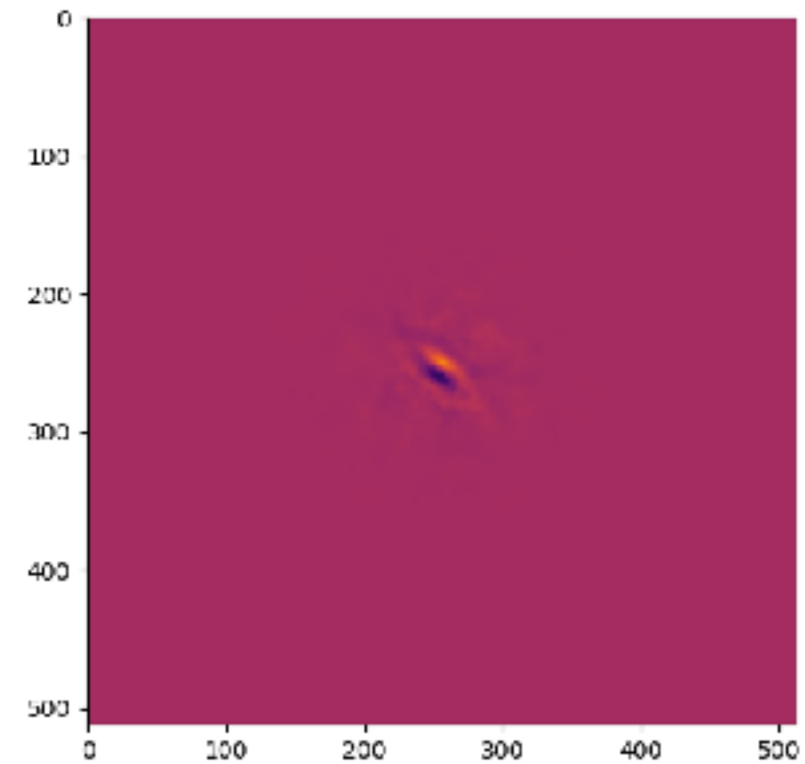
Ménard et al. (2020) arXiv:2006.02439

EXAMPLE: RADIO ASTRONOMY

github.com/danieljprice/uvsph

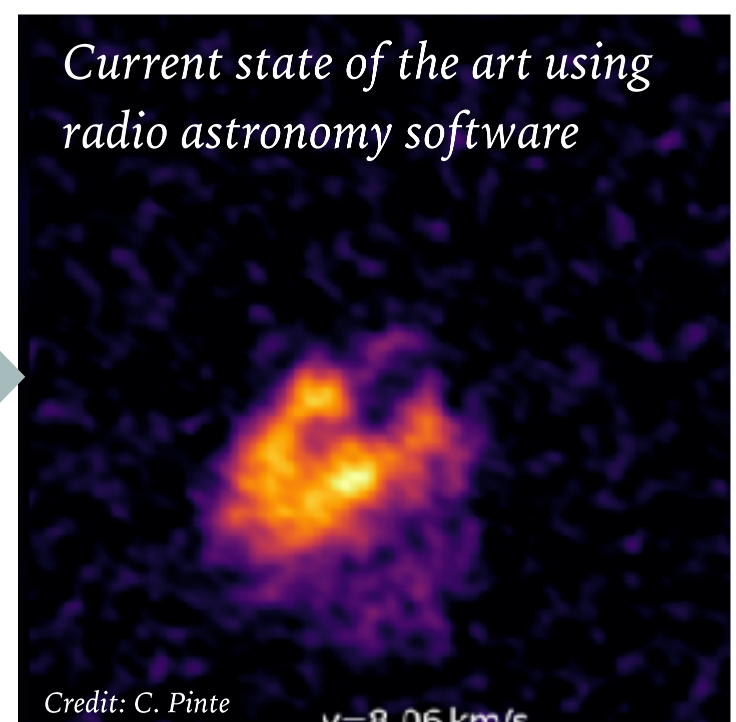


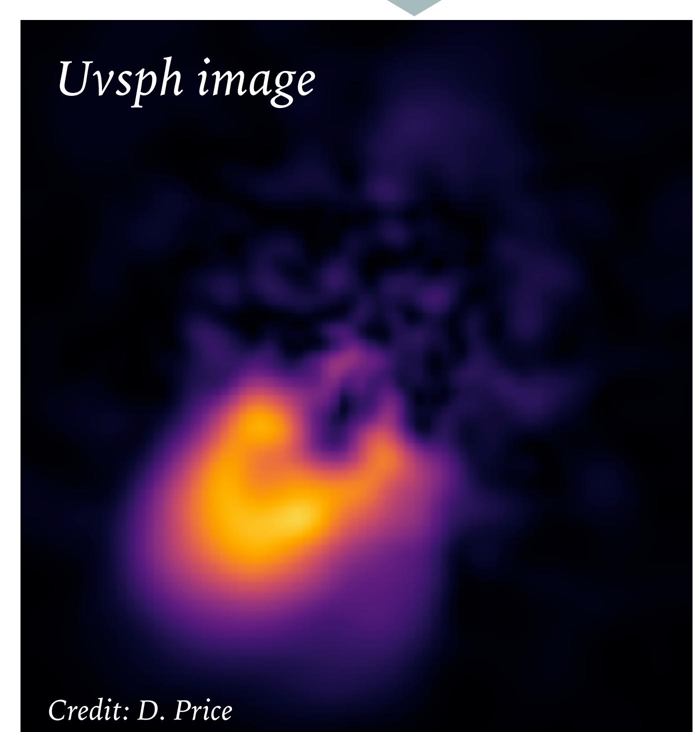
SPH interpolation



Inverse FFT

Image deconvolution





LESSON 2:
USE THE LAGRANGIAN

FROM DENSITY TO HYDRODYNAMICS

$$L_{sph} = \sum_j m_j \left[\frac{1}{2} v_j^2 - u_j(\rho_j, s_j) \right]$$ ← Lagrangian

$+$

$$\mathrm{d}u = \frac{P}{\rho^2} \mathrm{d}\rho$$ ← 1st law of thermodynamics

$+$

$$\nabla \rho_i = \sum_j m_j \nabla W_{ij}(h)$$ ← density sum

$+$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \mathbf{v}} \right) - \frac{\partial L}{\partial \mathbf{r}} = 0$$ ← Euler-Lagrange equations

$=$

$$\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = - \sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla_i W_{ij}(h)$$ ← equations of motion!

$$\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = - \frac{\nabla P}{\rho} \right)$$

EXAMPLE: RELATIVISTIC HYDRODYNAMICS

Monaghan & Price (2001)
Liptai & Price (2019)

$$L_{grsph} = -\sum_j m_j (1 + u_j) \sqrt{-g_{\mu\nu} v_j^\mu v_j^\nu}$$ ← Lagrangian

+

$$\mathrm{d}u = \frac{P}{\rho^2} \mathrm{d}\rho$$ ← 1st law of thermodynamics

+

$$\nabla \rho_i = \sum_j m_j \nabla W_{ij}(h)$$ ← density sum

+

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0$$ ← Euler-Lagrange equations

=

$$\frac{\mathrm{d}\mathbf{p_i}}{\mathrm{d}t} = -\sum_j m_j \left(\frac{\sqrt{-g_i} P_i}{\rho_i^{*2}} + \frac{\sqrt{-g_j} P_j}{\rho_j^{*2}} \right) \nabla_i W_{ij}(h)$$ ← equations of motion!

$$\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\nabla(\sqrt{-g}P)}{\rho *} \right)$$

EXAMPLE: SMOOTHED PARTICLE MAGNETOHYDRODYNAMICS

Price & Monaghan 2004a,b,2005, Price 2012, Tricco & Price (2012), Tricco, Price & Bate (2016)

$$L_{sph} = \sum_b m_b \left[\frac{1}{2} v_b^2 - u_b(\rho_b, s_b) - \frac{1}{2\mu_0} \frac{B_b^2}{\rho_b} \right]$$

$$\int \delta L \mathrm{dt} = 0$$

$$\delta \rho_b = \sum_c m_c \left(\delta \mathbf{r}_b - \delta \mathbf{r}_c \right) \cdot \nabla_b W_{bc},$$

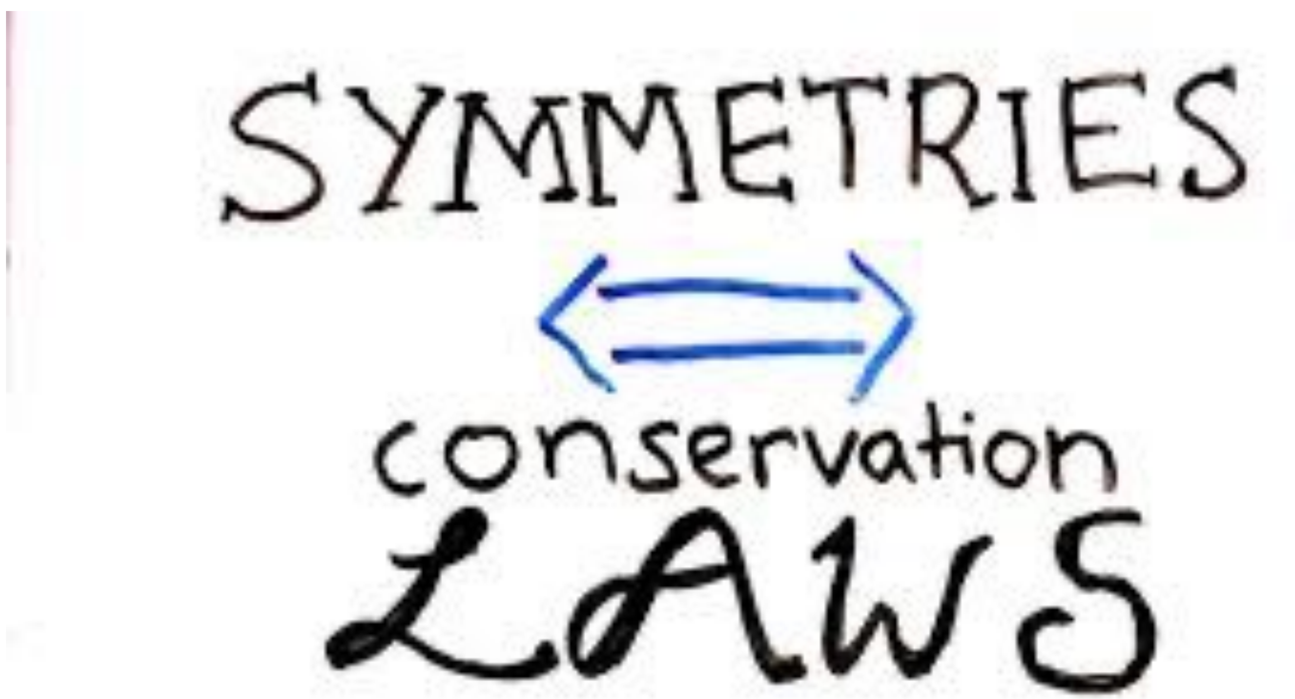
$$\delta \left(\frac{\mathbf{B}_b}{\rho_b} \right) = -\sum_c m_c (\delta \mathbf{r}_b - \delta \mathbf{r}_c) \frac{\mathbf{B}_b}{\rho_b^2} \cdot \nabla_b W_{bc}$$

$$\frac{dv_a^i}{dt} = -\sum_b m_b \left[\left(\frac{S^{ij}}{\rho^2} \right)_a + \left(\frac{S^{ij}}{\rho^2} \right)_b \right] \nabla_a^j W_{ab},$$

$$S_{ij} = \left(P + \frac{B^2}{2\mu_0} \right) \delta_{ij} - \frac{B_i B_j}{\mu_0}$$

WHAT THE LAGRANGIAN GIVES US

Noether's theorem

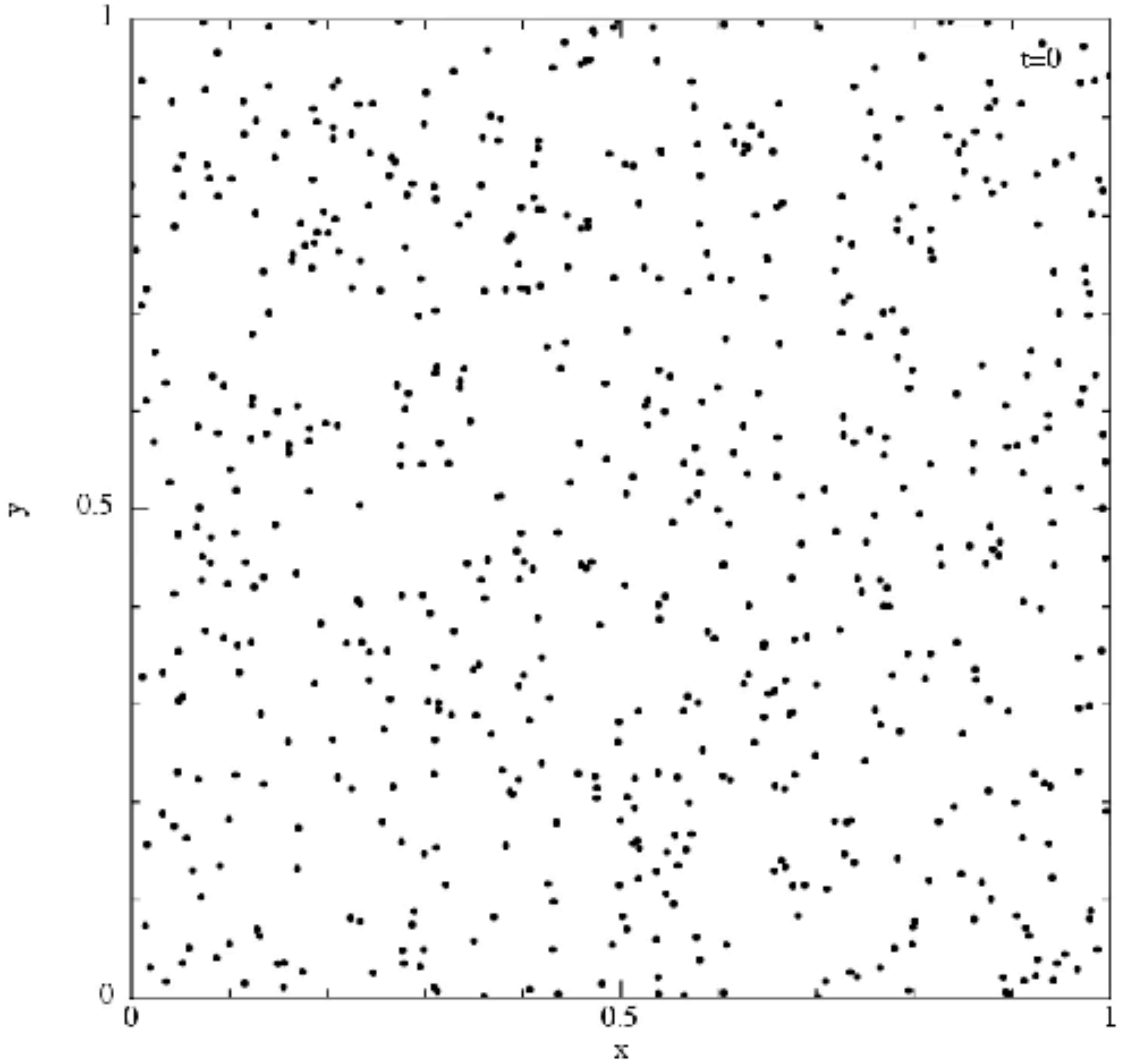


Emmy Noether 1882-1935

YouTube "the most beautiful idea in physics"

THE MINIMUM ENERGY STATE

What happens to a random particle arrangement?

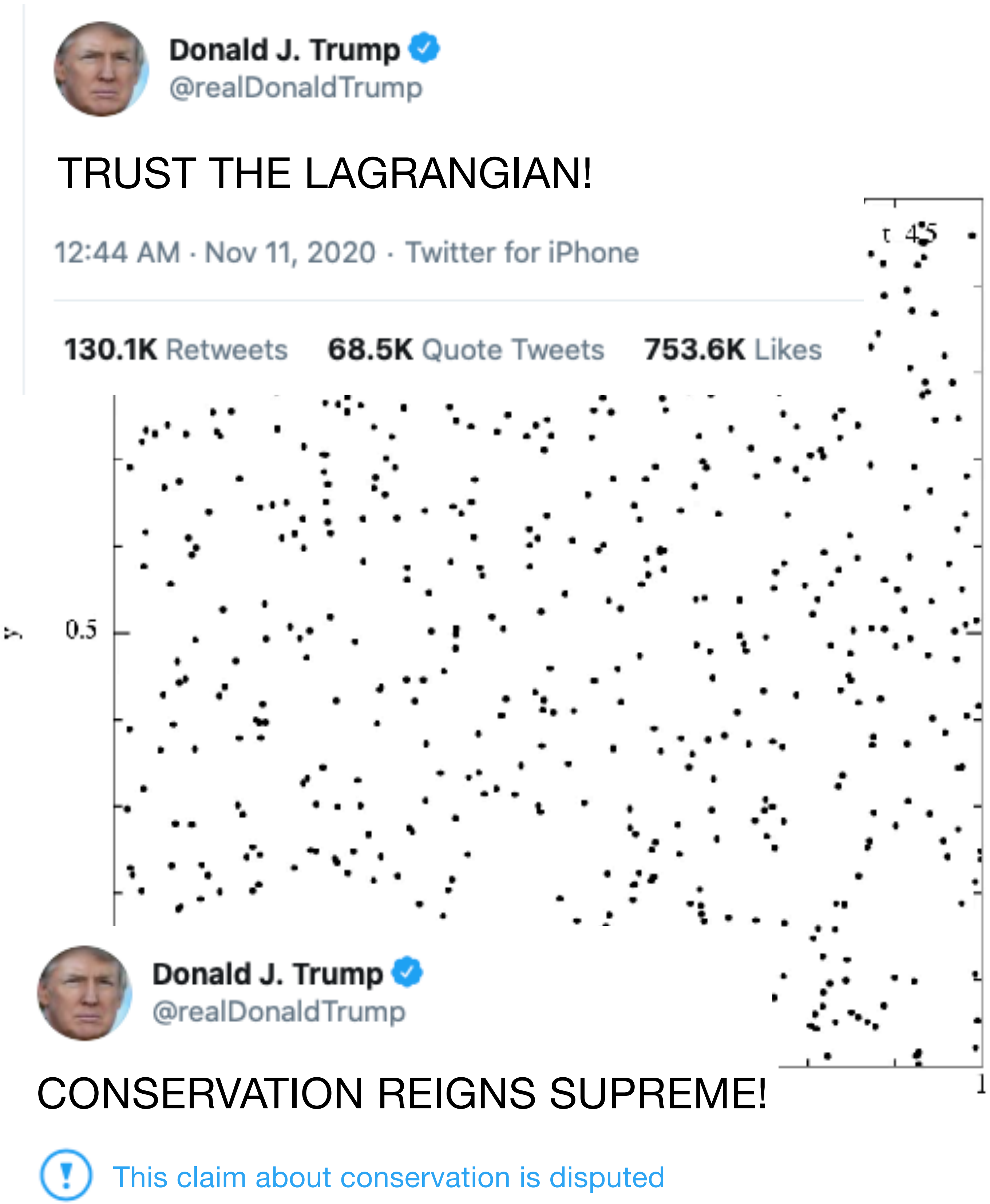


$$\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla_i W_{ij}$$

SPH particles know how to stay regular

e.g. Price (2012), see also Oger+2016 and Lind+(2016) for particle shifting techniques for incompressible flow

WHY BETTER GRADIENTS ARE A BAD IDEA



Abel 2010, Price 2012

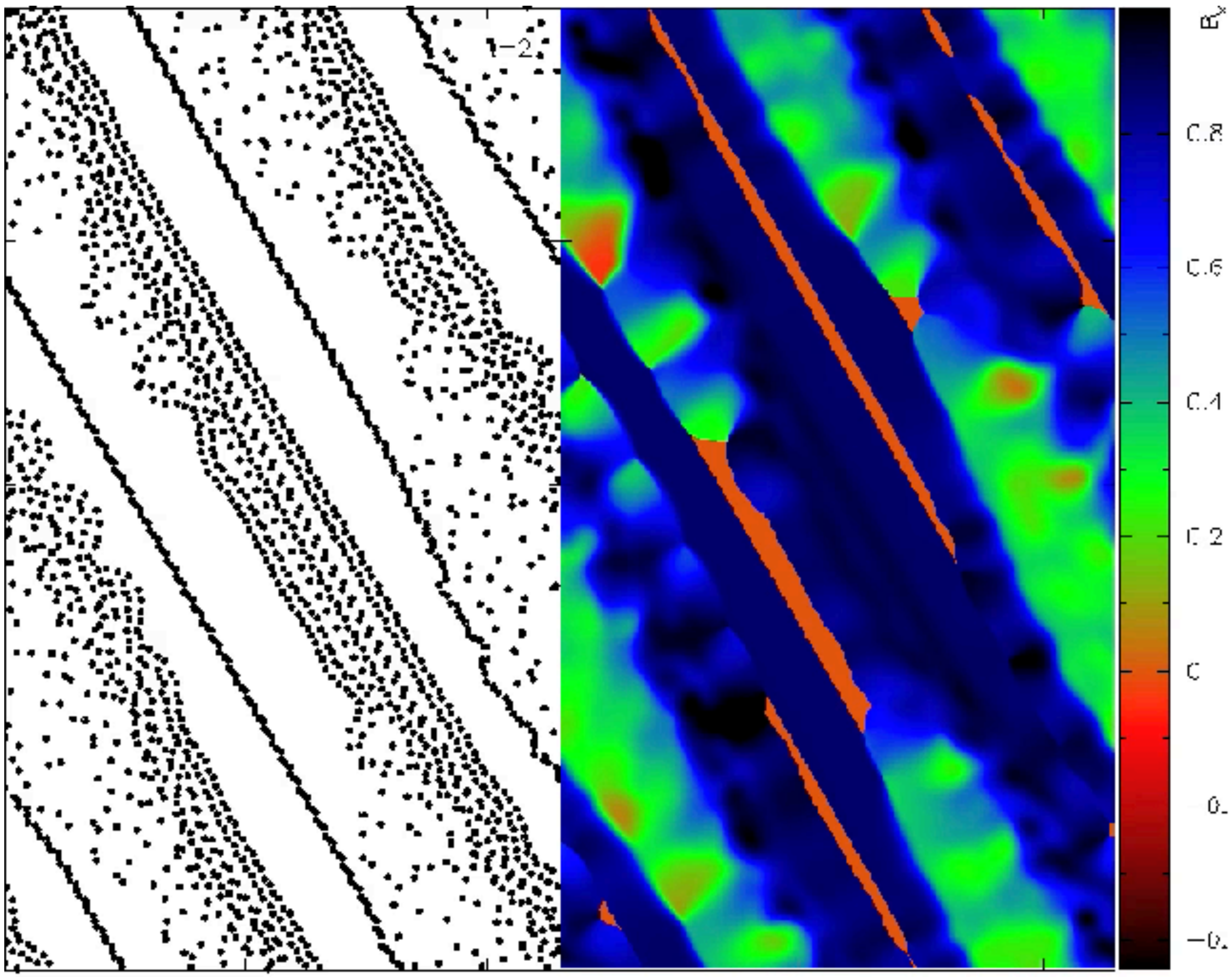
$$\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = \sum_j m_j \left(\frac{P_i - P_j}{\rho_j^2} \right) \nabla_i W_{ij}$$

Improving the gradient operator leads to WORSE results

Corollary: Better to use a worse gradient operator but conserve momentum

LESSON 3: ERRORS GO INTO THE PARTICLE DISTRIBUTION

TENSILE INSTABILITY: NEED POSITIVE PRESSURES



MHD

$$S_{ij} = \left(P + \frac{B^2}{2\mu_0}\right)\delta_{ij} - \frac{B_i B_j}{\mu_0}$$

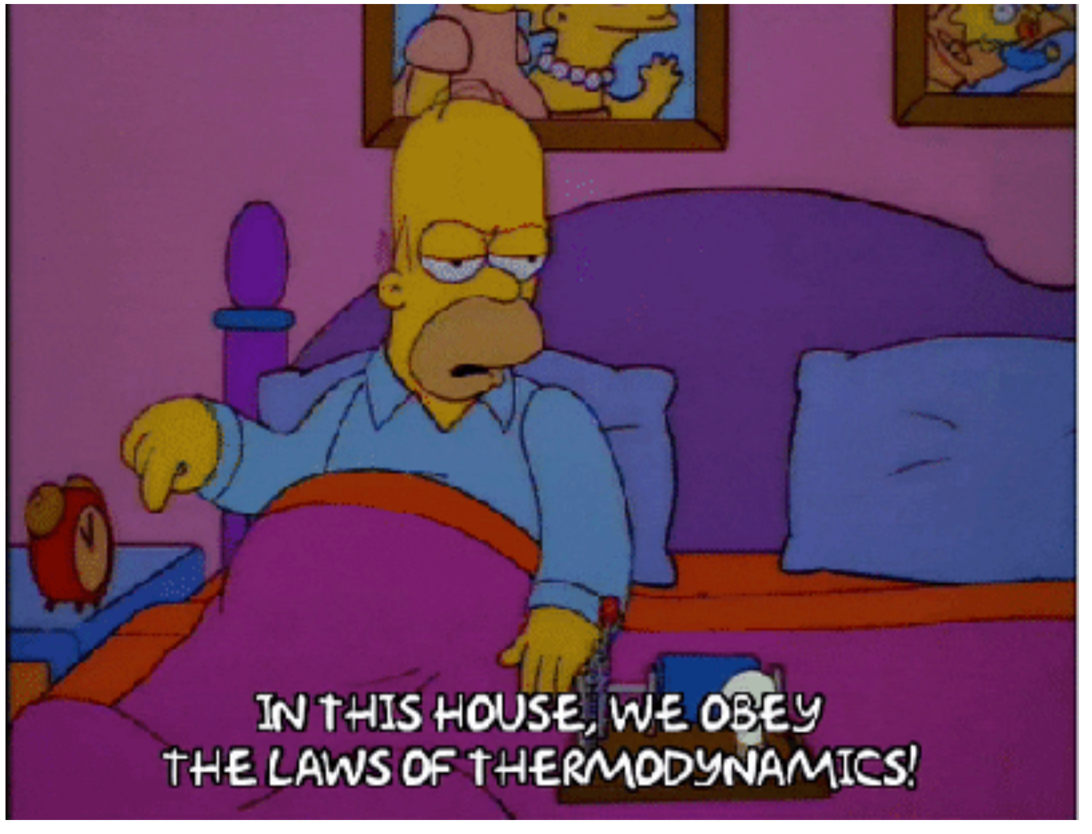
This is known as the tensile instability in SPH: occurs when net stress is negative

LESSON 3: ERRORS GO INTO THE PARTICLE DISTRIBUTION

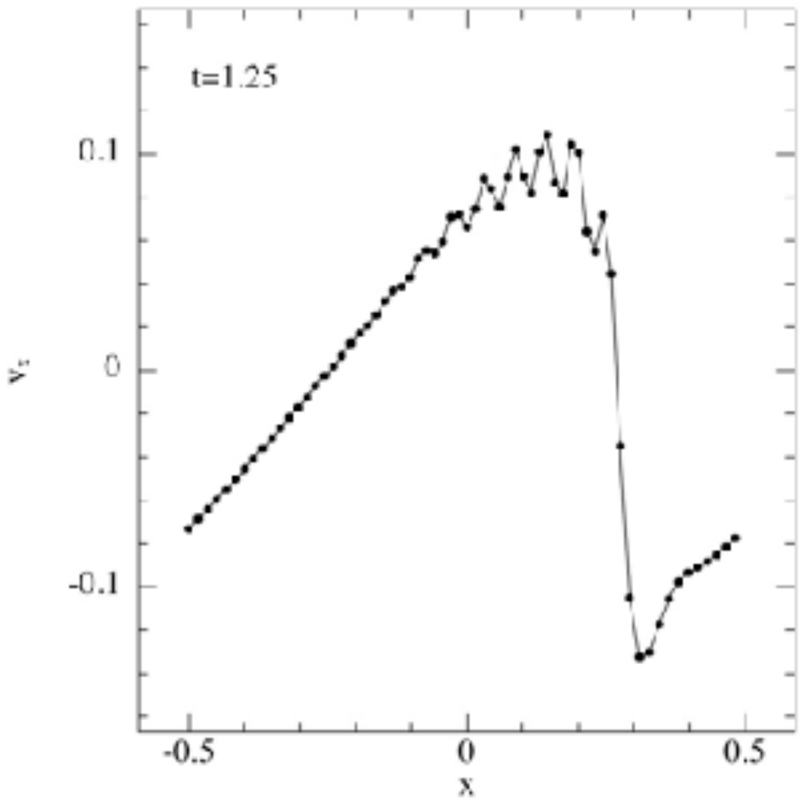
- Calculations keep going, even when they're screwed up…

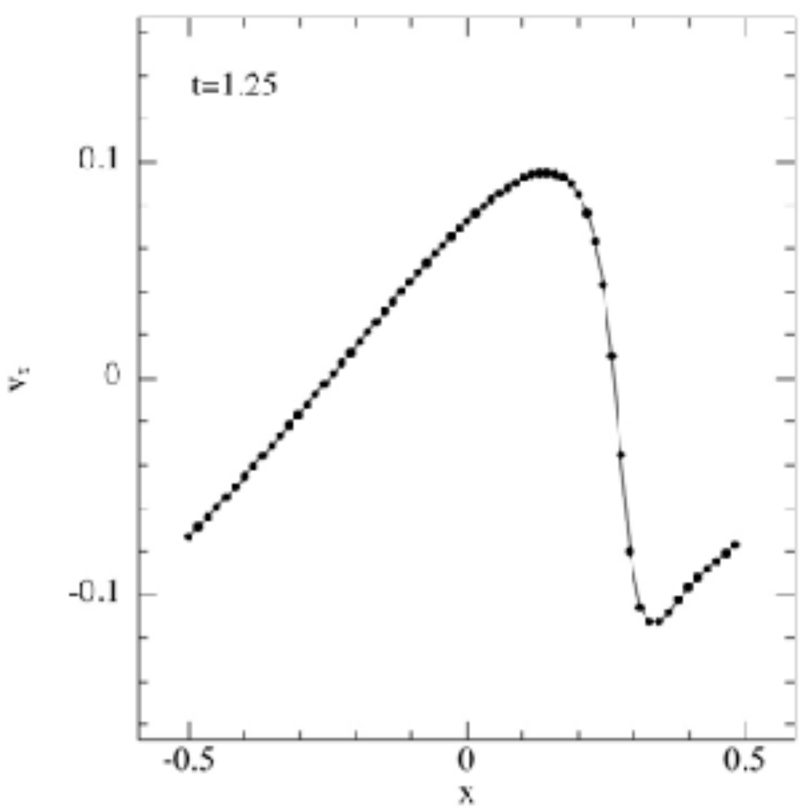
Orszag-Tang Vortex in MHD (c.f. Price & Monaghan 2005, Rosswog & Price 2007, Price 2010)

LESSON 4:
ENTROPY MUST INCREASE



EXAMPLE: SHOCK CAPTURING TERMS





- Lagrangian implies no dissipation
- Shock jump conditions imply entropy increase at discontinuities

$$\frac{\mathrm{d}\boldsymbol{v}_a}{\mathrm{d}t} = -\sum_b m_b \left[\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} - \frac{\alpha v_{\rm sig}(\boldsymbol{v}_{ab}\cdot\hat{\boldsymbol{r}}_{ab})}{\bar{\rho}_{ab}}\right]\nabla_a W_{ab}$$

$$\frac{\mathrm{d}e_a}{\mathrm{d}t} = -\sum_b m_b \left[\frac{P_a\boldsymbol{v}_b}{\rho_a^2} + \frac{P_b\boldsymbol{v}_a}{\rho_b^2} - \frac{\alpha v_{\rm sig}(e_a^* - e_b^*)\hat{\boldsymbol{r}}_{ab}}{\bar{\rho}_{ab}}\right]\cdot\nabla_a W_{ab}$$

$$e_a^* = \frac{1}{2}(\boldsymbol{v}_a\cdot\hat{\boldsymbol{r}}_{ab})^2 + u_a$$

$$T_a\frac{\mathrm{d}s_a}{\mathrm{d}t} = \frac{\mathrm{d}u_a}{\mathrm{d}t} - \frac{P_a}{\rho_a^2}\frac{\mathrm{d}\rho_a}{\mathrm{d}t}$$

First law of thermodynamics

$$= \frac{\mathrm{d}e_a}{\mathrm{d}t} - \boldsymbol{v}_a\cdot\frac{\mathrm{d}\boldsymbol{v}_a}{\mathrm{d}t} - \frac{P_a}{\rho_a^2}\frac{\mathrm{d}\rho_a}{\mathrm{d}t}$$

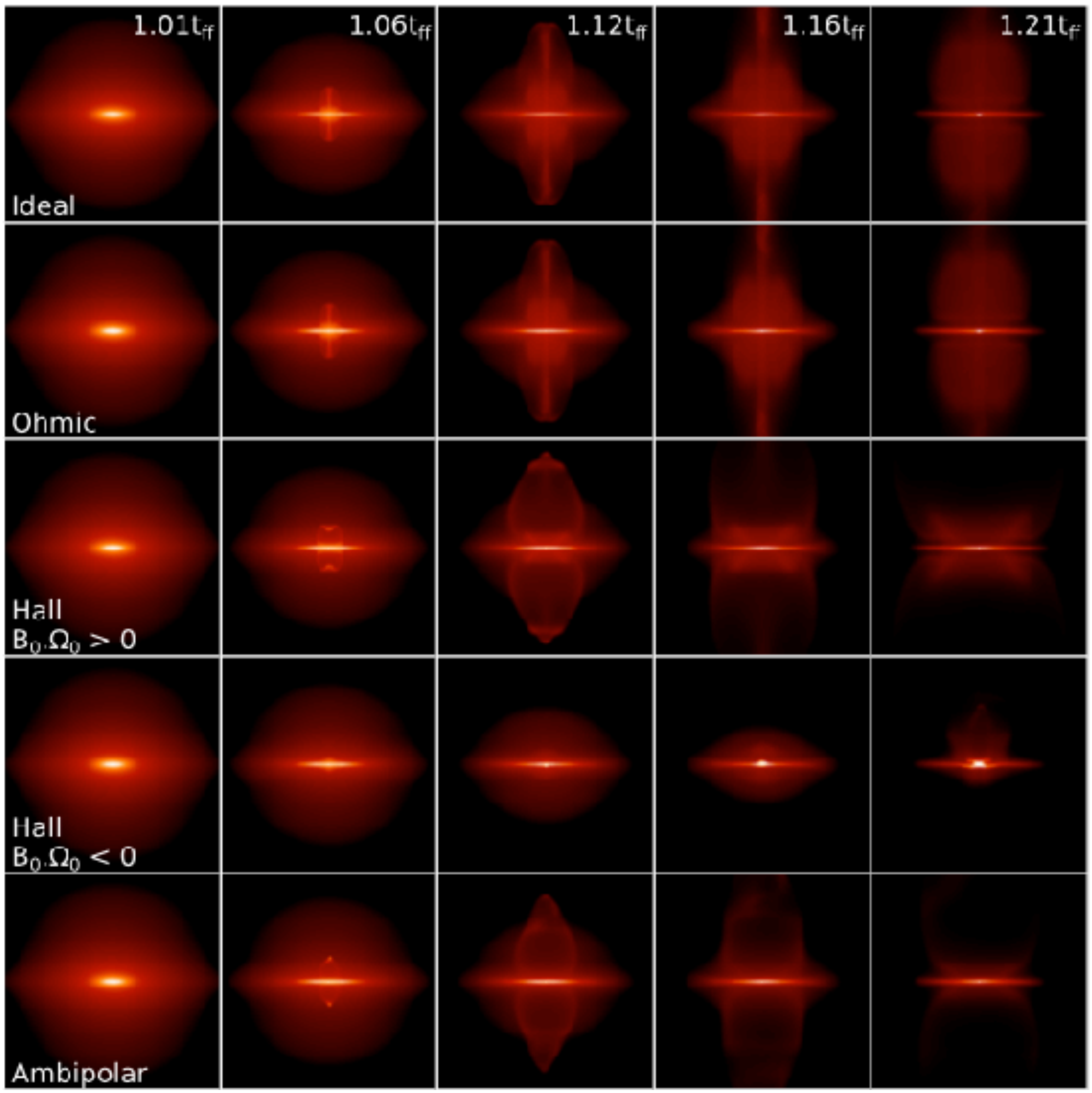
$$= \sum_b m_b \frac{\alpha v_{\rm sig}\frac{1}{2}(\boldsymbol{v}_{ab}\cdot\hat{\boldsymbol{r}}_{ab})^2}{\bar{\rho}_{ab}}$$

Entropy change is positive definite!

EXAMPLE: NON-IDEAL MAGNETOHYDRODYNAMICS

- Partially ionised plasmas (ions, electrons, neutrals)

Wurster, Price & Ayliffe (2014), Wurster, Price & Bate (2016)



Ohmic resistivity — *Hall effect* — *Ion-neutral drift*

$$\left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_{\mathrm{NI}} = -\nabla \times \underbrace{\left[\frac{\boldsymbol{J}}{\sigma} + \frac{\boldsymbol{J}\times\boldsymbol{B}}{en_e} - \frac{(\boldsymbol{J}\times\boldsymbol{B})\times\boldsymbol{B}}{\gamma_{\mathrm{AD}}\rho_i}\right]}_{\boldsymbol{D}}$$

$$\boldsymbol{J}_a = \frac{1}{\Omega_a\rho_a}\sum_b m_b(\boldsymbol{B}_a - \boldsymbol{B}_b)\times\nabla_a W_{ab}(h_a)$$

$$\left(\frac{\mathrm{d}\boldsymbol{B}_a}{\mathrm{d}t}\right)_{\mathrm{NI}} = \rho_a\sum_b m_b\left[\frac{\boldsymbol{D}_a}{\Omega_a\rho_a^2}\times\nabla_a W_{ab}(h_a) + \frac{\boldsymbol{D}_b}{\Omega_b\rho_b^2}\times\nabla_a W_{ab}(h_b)\right]$$

$$\sum_a m_a\frac{\mathrm{d}u_a}{\mathrm{d}t} = -\sum_a m_a\frac{\boldsymbol{B}_a}{\rho_a}\cdot\frac{\mathrm{d}\boldsymbol{B}_a}{\mathrm{d}t} \geq 0$$

Entropy guaranteed to increase => stable

ENTROPY MUST INCREASE!

EXAMPLE: ANISOTROPIC DIFFUSION

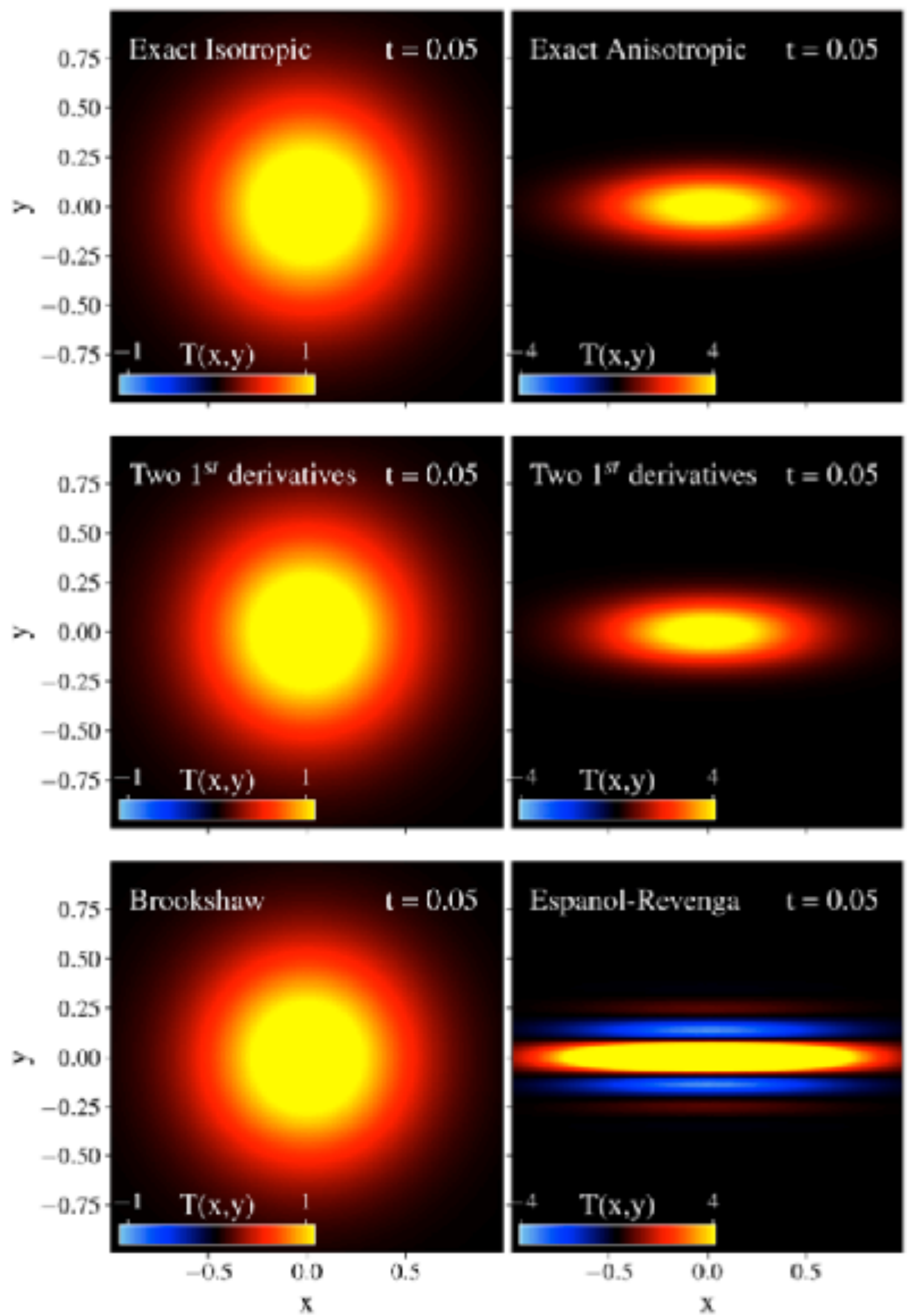
Biriukov & Price (2019), MNRAS 483, 4901

Isotropic diffusion *Anisotropic diffusion*

Exact solution

Two first derivatives

Direct second derivatives



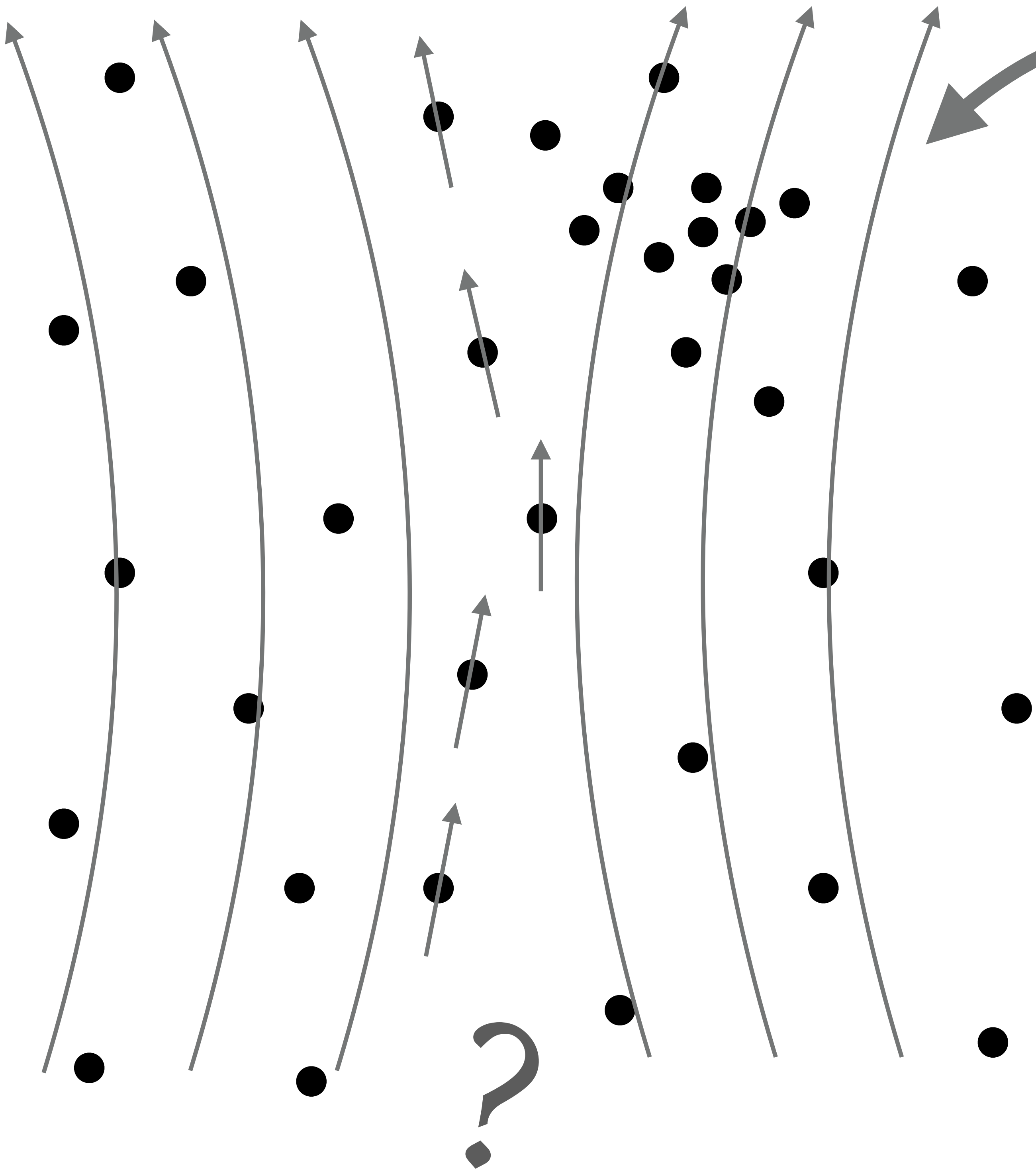
$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{\rho}\nabla\cdot(\boldsymbol{\kappa}\nabla T)$$

Entropy guaranteed to increase => stable

Entropy not guaranteed to increase => unstable

LESSON 5:
PARTICLES ARE NOT REAL

MAGNETIC FIELDS WITH SPH?

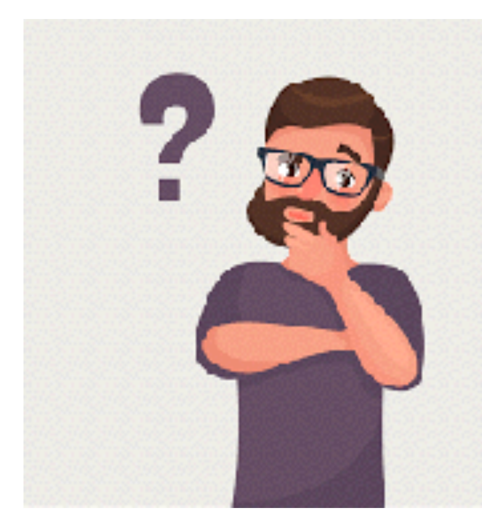


$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho = -\rho(\nabla \cdot \boldsymbol{v})$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla P}{\rho} + \frac{\boldsymbol{J} \times \boldsymbol{B}}{\rho}$$

$$\frac{\partial u}{\partial t} + (\boldsymbol{v} \cdot \nabla)u = -\frac{P}{\rho}(\nabla \cdot \boldsymbol{v})$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$$



SMOOTHED PARTICLE MAGNETOHYDRODYNAMICS

Gravitational collapse of a rotating, magnetised cloud
Price, Tricco & Bate (2012)

- Use the Lagrangian!
- Obtain discretised MHD equations
- Better to think in terms of partial differential equations, not particles

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho} + \frac{\boldsymbol{J}\times\boldsymbol{B}}{\rho}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\cdot\nabla\right)\boldsymbol{v}$$

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\sum_b m_b\left[\left(\frac{S^{ij}}{\rho^2}\right)_a + \left(\frac{S^{ij}}{\rho^2}\right)_b\right]\nabla_a^j W_{ab}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{B}_a}{\rho_a}\right) = -\sum_b m_b(\boldsymbol{v}_a - \boldsymbol{v}_b)\frac{\boldsymbol{B}_a}{\rho_a^2}\cdot\nabla_a W_{ab}$$

EXAMPLE: “ISSUE” WITH KELVIN-HELMHOLTZ INSTABILITIES

Mon. Not. R. Astron. Soc. **380**, 963–978 (2007)

doi:10.1111/j.1365-2966.2007.12183.x

Fundamental differences between SPH and grid methods

Oscar Agertz,[1★] Ben Moore,[1] Joachim Stadel,[1] Doug Potter,[1] Francesco Miniati,[2] Justin Read,[1] Lucio Mayer,[2] Artur Gawryszczak,[3] Andrey Kravtsov,[4] Åke Nordlund,[5] Frazer Pearce,[6] Vicent Quilis,[7] Douglas Rudd,[4] Volker Springel,[8] James Stone,[9] Elizabeth Tasker,[10] Romain Teyssier,[11] James Wadsley[12] and Rolf Walder[13]

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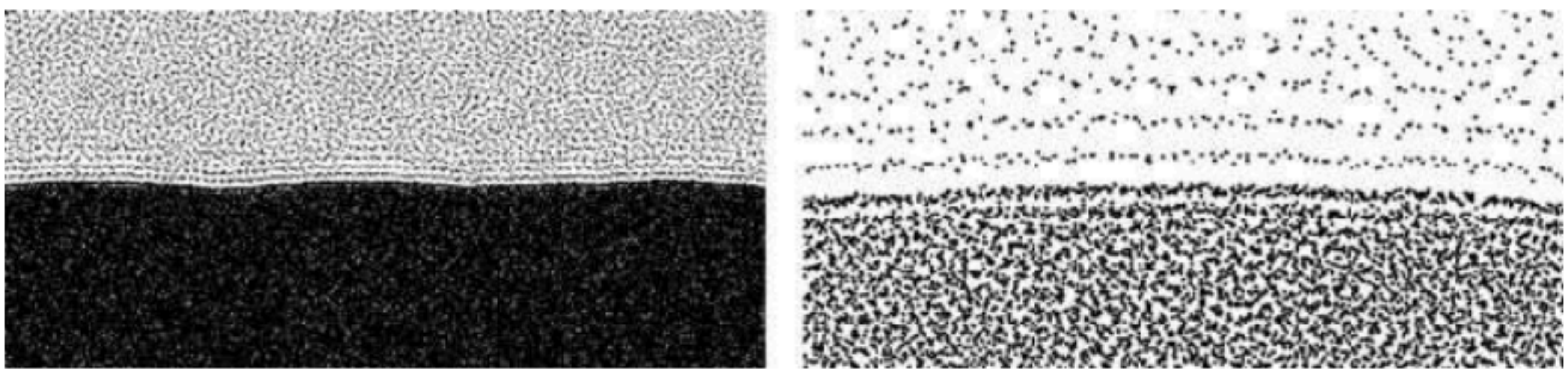
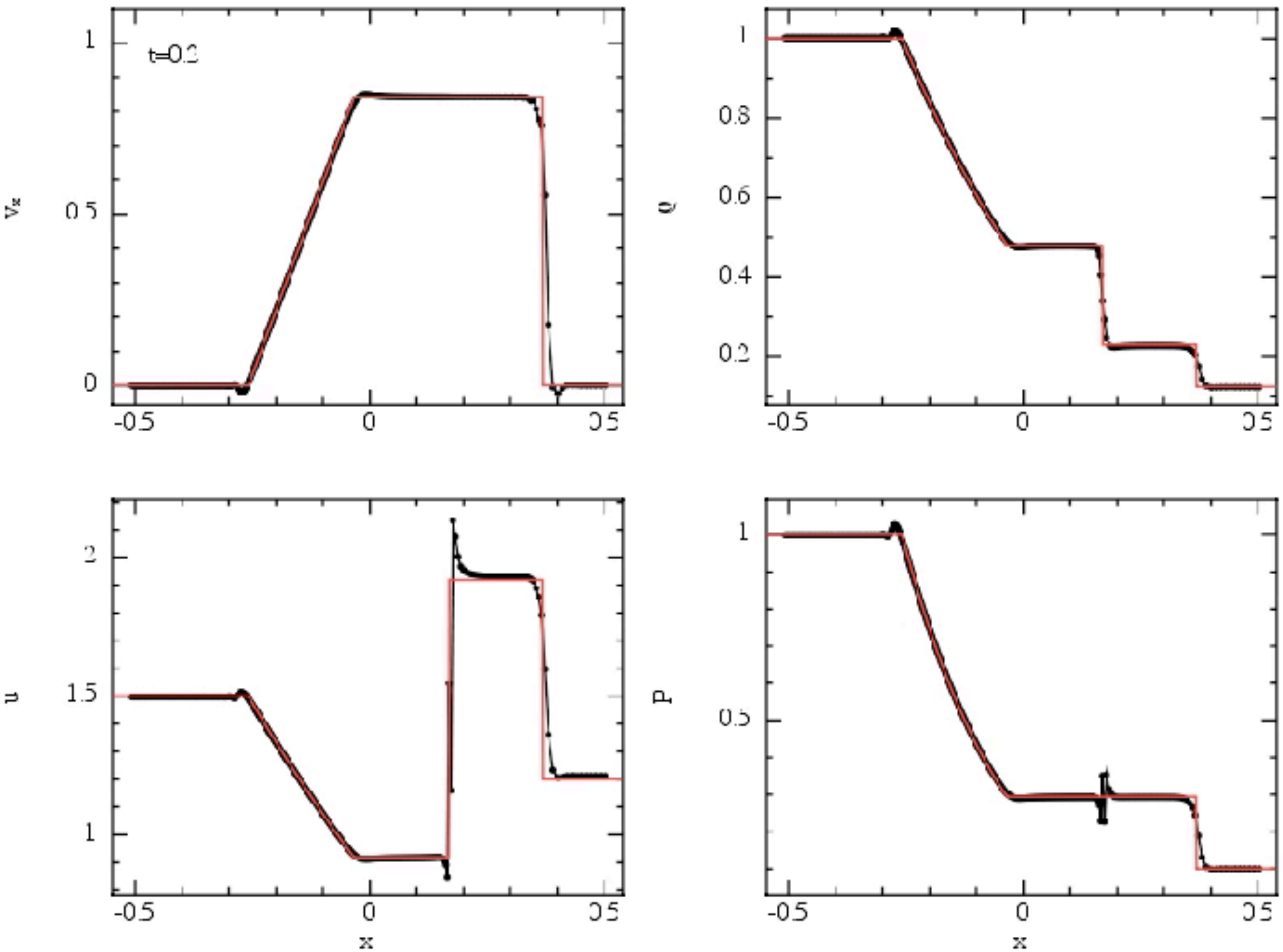


Figure 14. A close up view of the SPH particles at the boundaries between the shearing layers (left) and closer zoom in (right) for SPH3 at τ_{KH}. We can clearly see empty layers formed through erroneous pressure forces due to improper density calculations at density gradients. Even though the two fluids are moving relative to each other, the gap is so large that proper fluid interaction is severely decreased or even absent.

Lesson: Don’t look at particle plots!

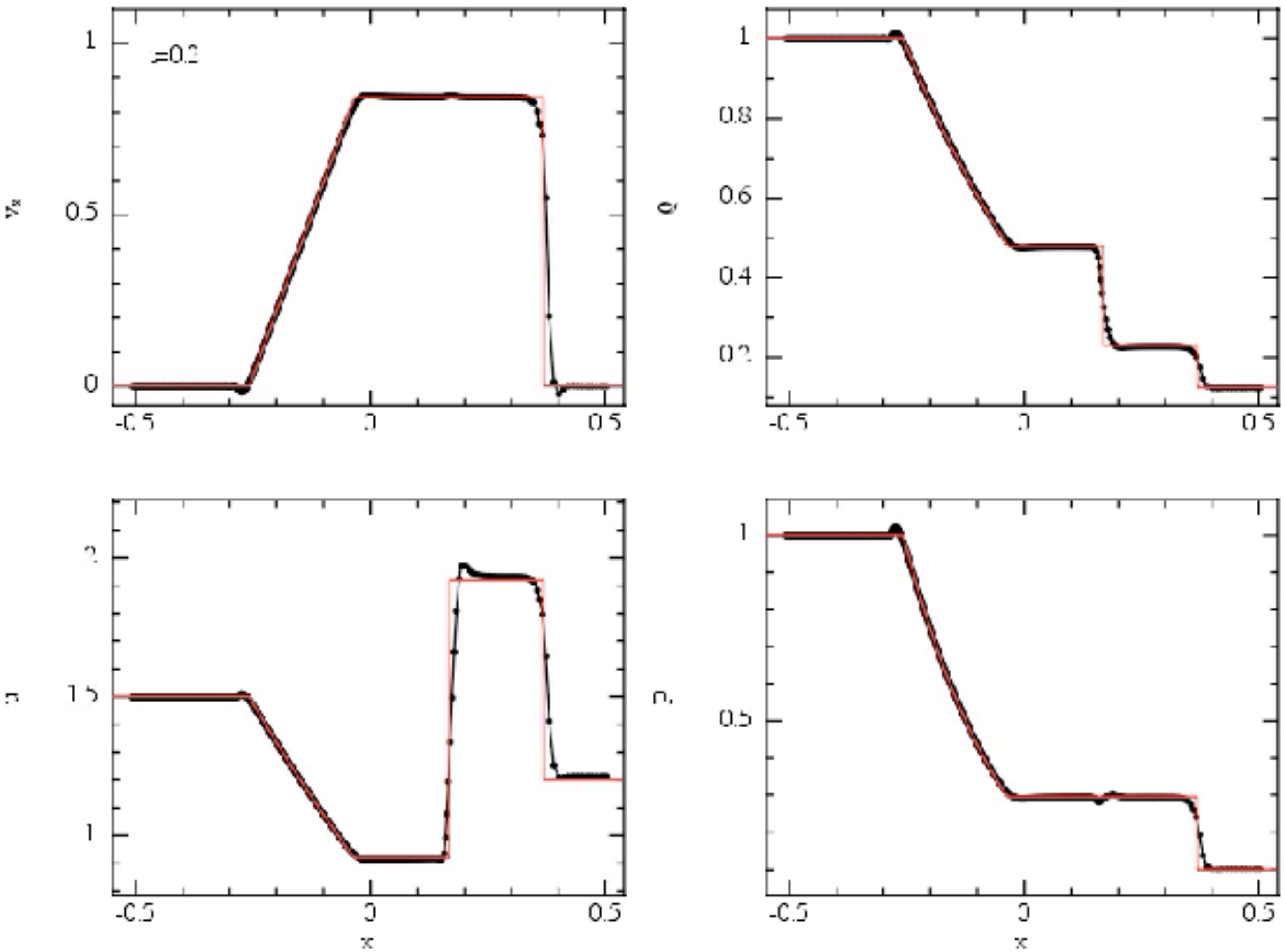
THE RIGHT WAY TO THINK ABOUT IT



1D Sod shock tube with artificial viscosity

- Shock capturing dissipation terms required at discontinuities
- Artificial viscosity applied at shock
- What about the contact discontinuity?

ANALOGY WITH RIEMANN SOLVERS

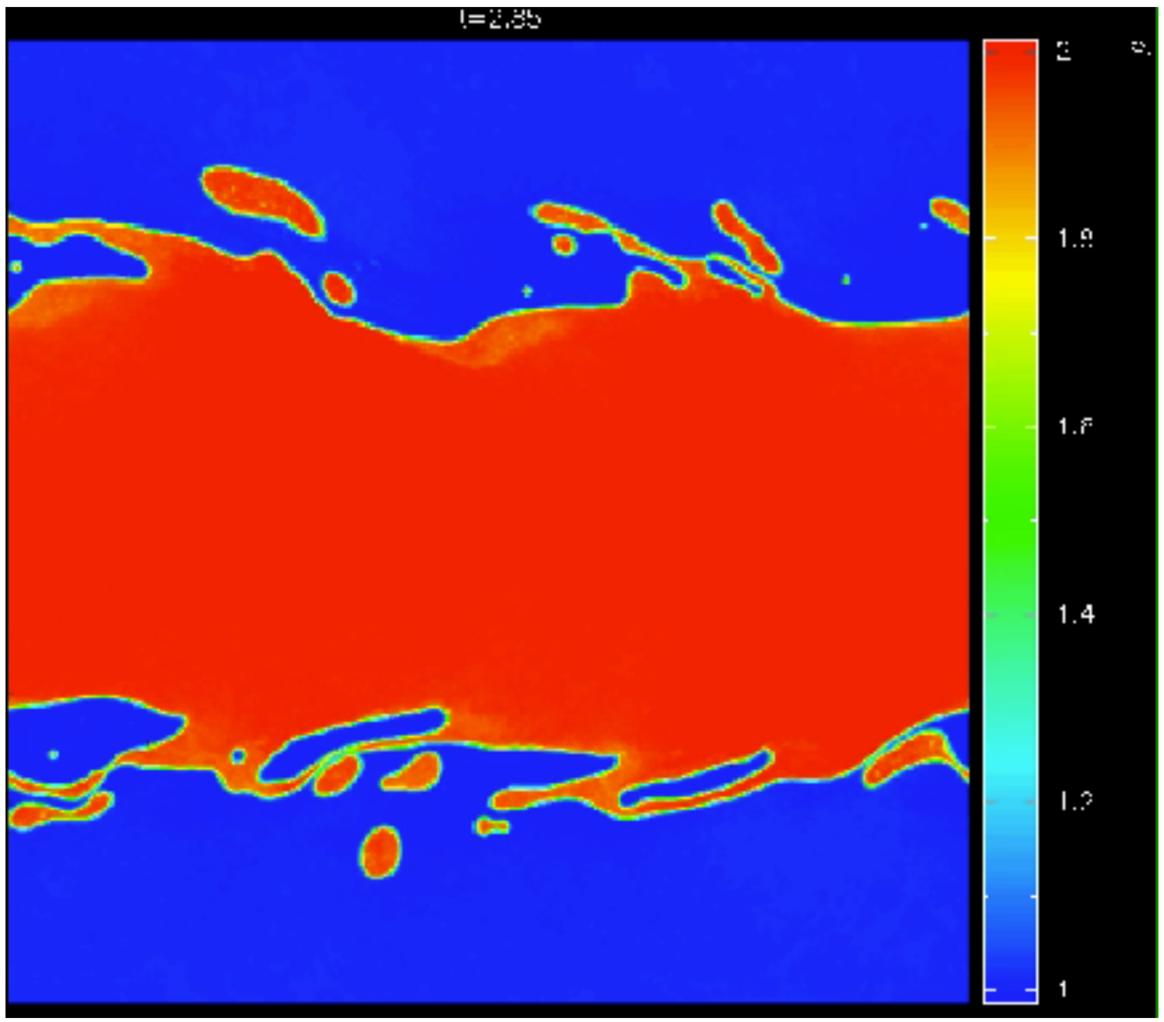


- Godunov-type solvers would add conductivity at the contact discontinuity (Monaghan 1997)
- Add analogous “artificial conductivity” term to ensure smooth pressure across discontinuous jumps in density and temperature (Chow & Monaghan 1997, Price 2008)

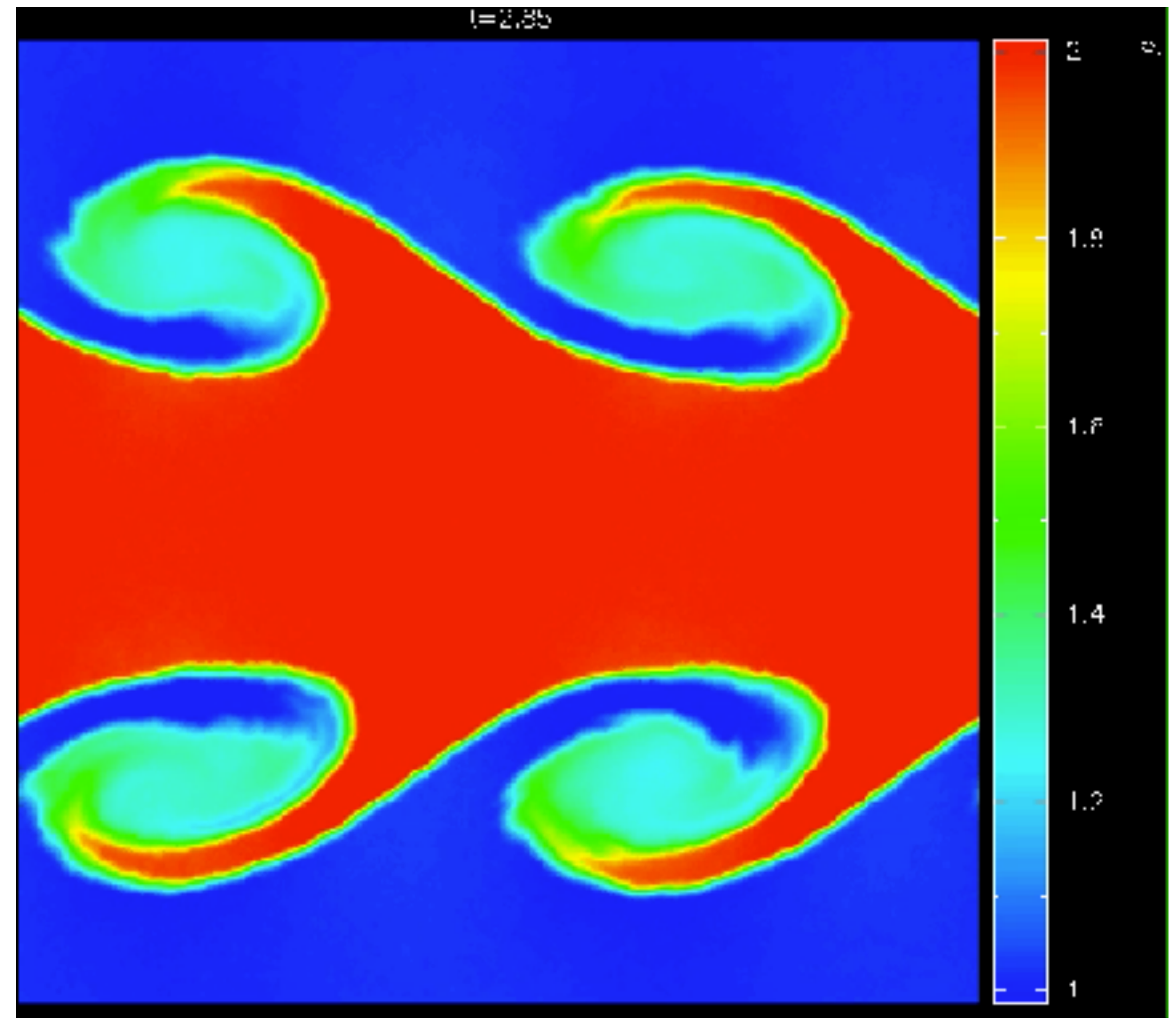
1D Sod shock tube with artificial conductivity

MUST TREAT DISCONTINUITIES PROPERLY

Price (2008)



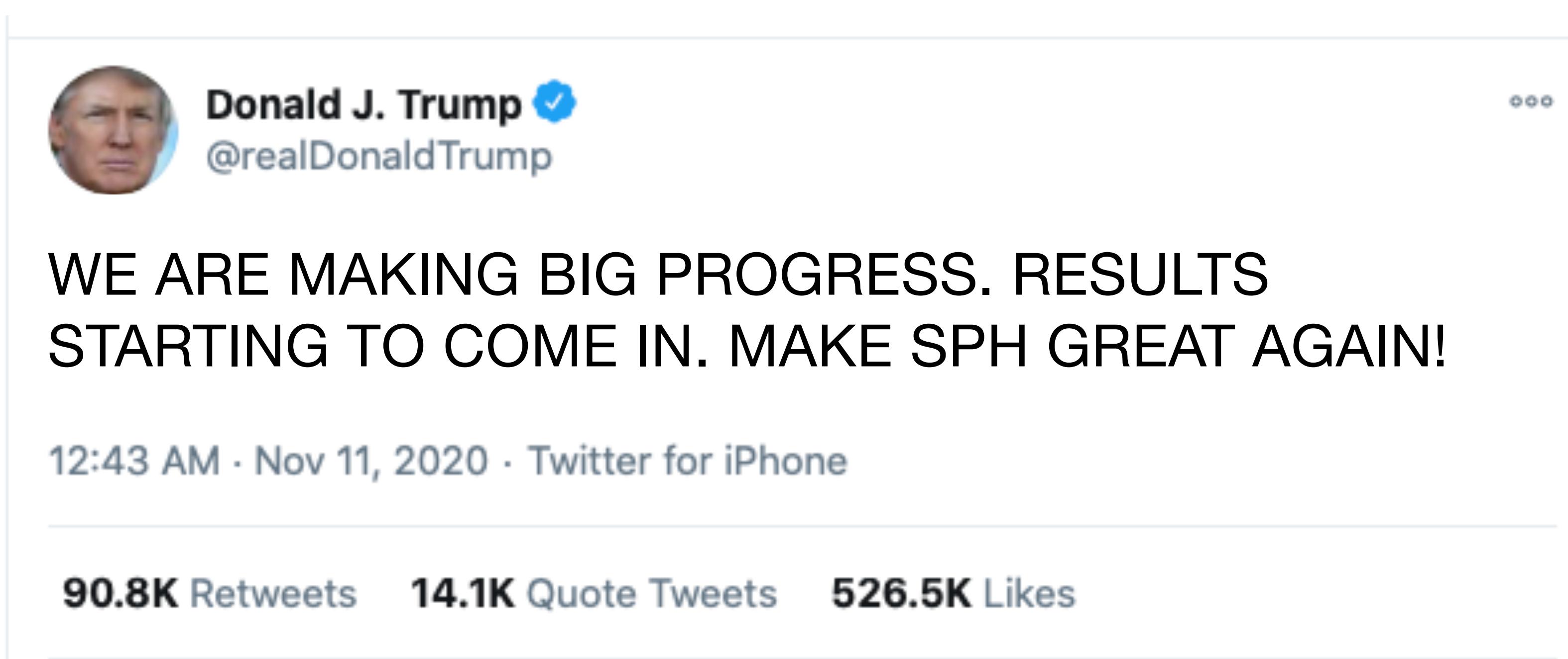
Without conductivity



With conductivity

This issue has nothing to do with the Kelvin-Helmholtz instability!

LESSON 6: USE WISDOM FROM ALL NUMERICAL METHODS



ARTIFICIAL VISCOSITY AS A GODUNOV-TYPE SCHEME

Monaghan (1997), Chow & Monaghan (1997)

- Finite volume method

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \left[\mathbf{F}(\mathbf{u})\right] = 0 \quad \longrightarrow \quad \frac{\mathbf{u}_i^{n+1} - \mathbf{u}_i^{n}}{\Delta t} = -\left[\frac{\mathbf{F}^*_{i+\frac{1}{2}} - \mathbf{F}^*_{i-\frac{1}{2}}}{\Delta x}\right]$$

$$\mathbf{F}^* = \frac{1}{2}\left[\mathbf{F}(\mathbf{u}_L) + \mathbf{F}(\mathbf{u}_R)\right] - \frac{v_{\rm sig}}{2}(\mathbf{u}_R - \mathbf{u}_L)$$

Local Lax-Friedrichs flux

- SPH

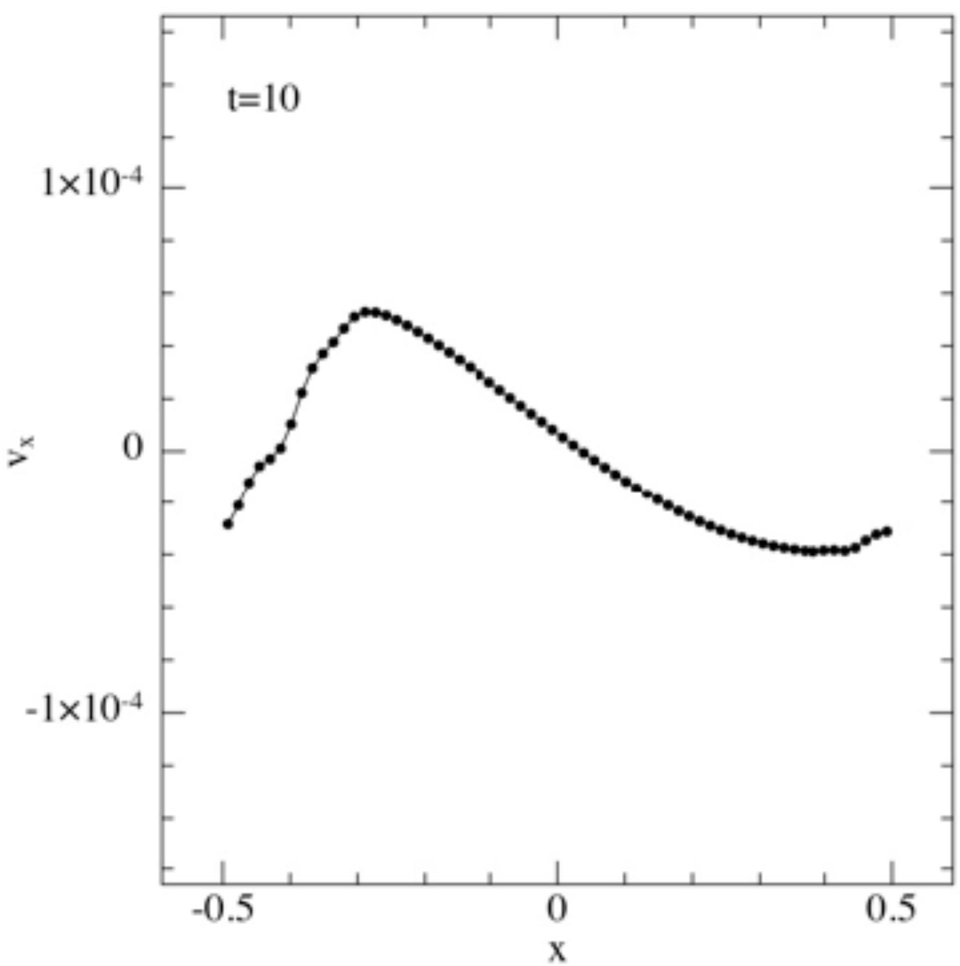
$$\frac{{\rm d}\mathbf{u}}{{\rm d}t} = -\frac{\nabla \cdot \mathbf{F}(\mathbf{u})}{\rho} \quad \longrightarrow \quad \frac{{\rm d}\mathbf{u}_a}{{\rm d}t} = -\sum_b m_b \left[\frac{\mathbf{F}_a}{\Omega_a \rho_a^2} \cdot \nabla_a W_{ab}(h_a) + \frac{\mathbf{F}_b}{\Omega_a \rho_a^2} \cdot \nabla_a W_{ab}(h_b)\right]$$

$$\mathbf{u} = [\mathbf{v}, e]^T \qquad -\sum_b \frac{m_b}{\overline{\rho}_{ab}} \overline{v}_{\rm sig}(\mathbf{u}_a - \mathbf{u}_b)\hat{\mathbf{r}}_{ab} \cdot \overline{\nabla_a W_{ab}},$$

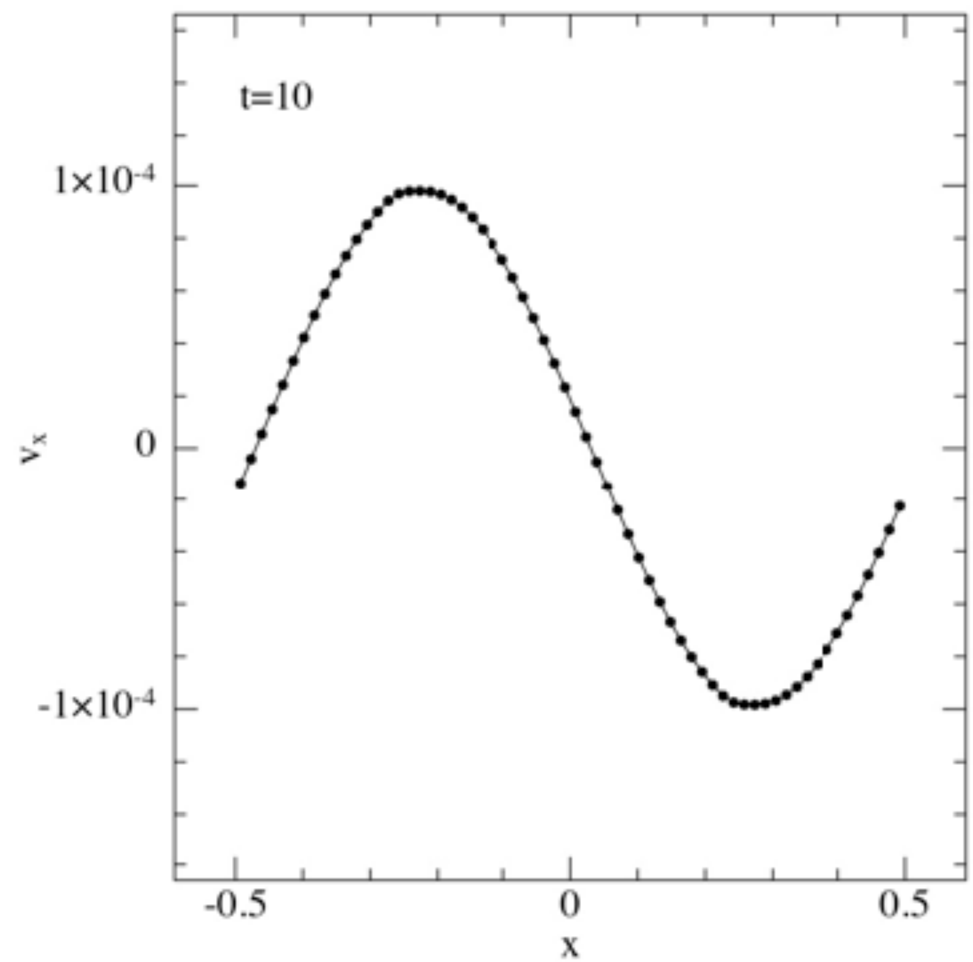
Implies both artificial viscosity AND artificial conductivity

c.f. Chow & Monaghan (1997), Price (2008)

EXAMPLE: SHOCK CAPTURING WITH RECONSTRUCTION + SLOPE LIMITERS



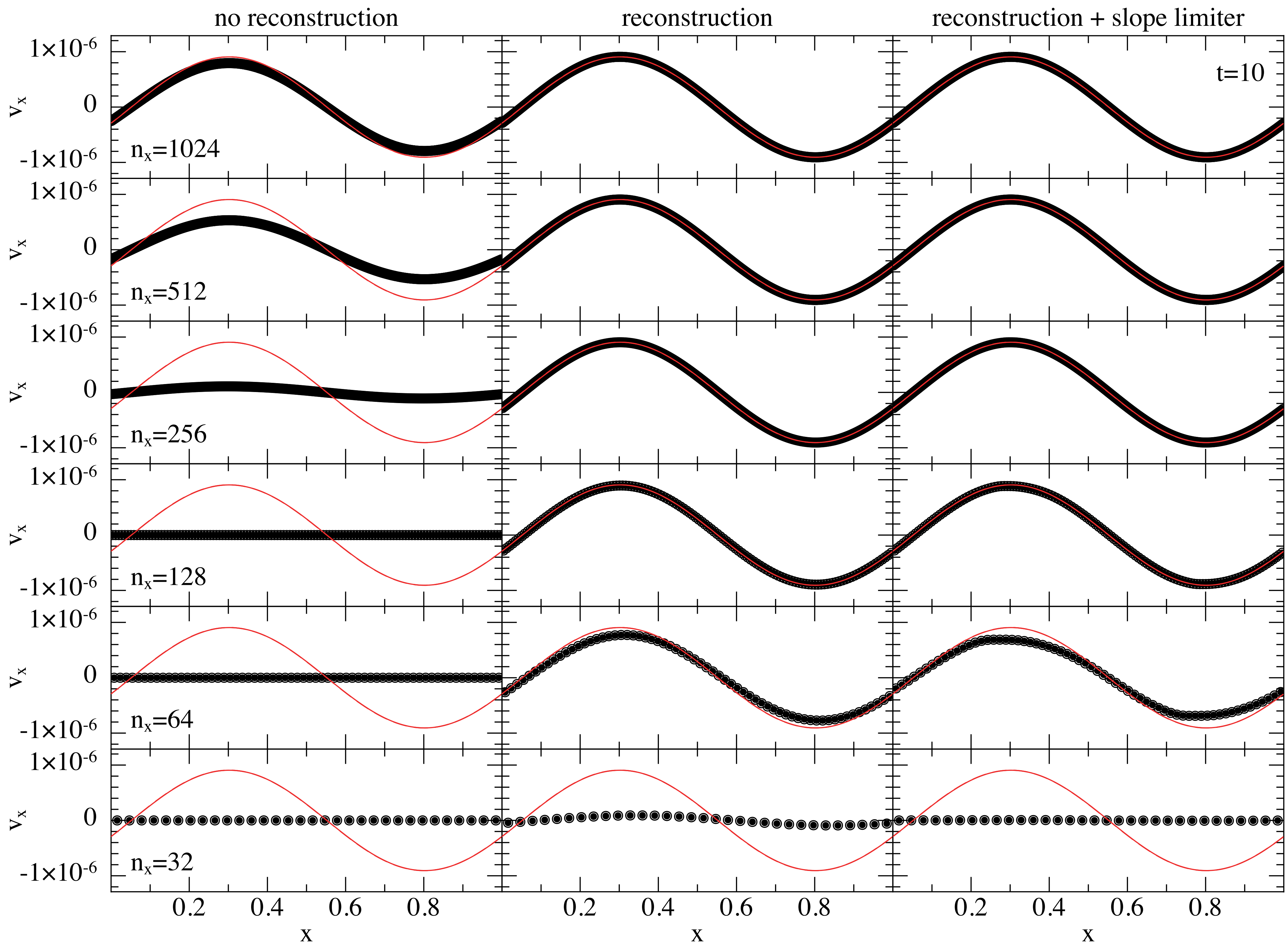
artificial viscosity,
no slope limiters



AV+ slope limiters

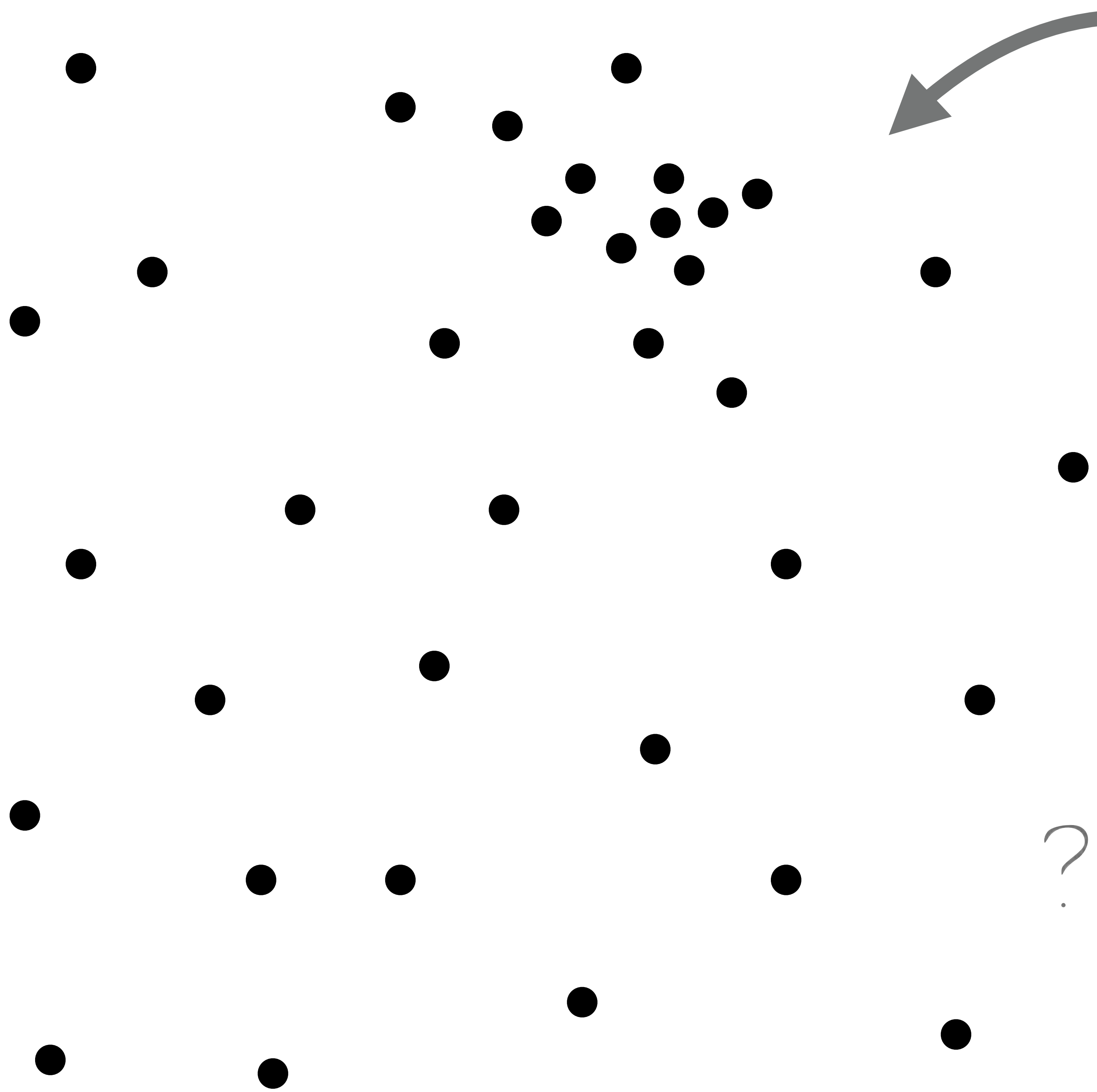
e.g. Inutsuka (2002), Cha & Whitworth (2003), Price @ SPHERIC 2019

EXAMPLE: OVERDAMPING OF WAVES IN DUST-GAS MIXTURES

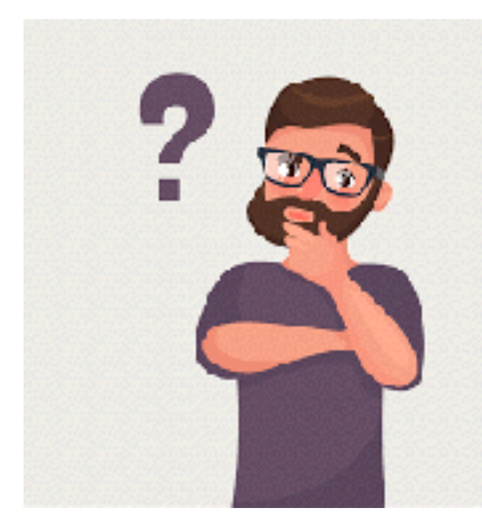


Price & Laibe (2020), also 2018 SPHERIC proceedings

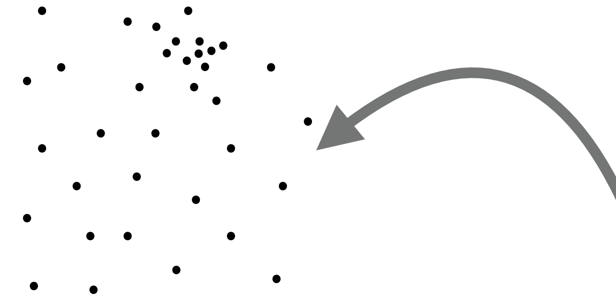
SUMMARY: THE PROBLEM



$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho = -\rho(\nabla \cdot \boldsymbol{v})$$
$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla P}{\rho}$$
$$\frac{\partial u}{\partial t} + (\boldsymbol{v} \cdot \nabla)u = -\frac{P}{\rho}(\nabla \cdot \boldsymbol{v})$$



THE SOLUTION: 6 LESSONS IN SPH



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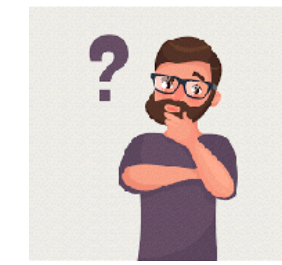

1. Use the density sum to bridge discrete and continuum
2. The Lagrangian helps you satisfy conservation laws
3. Particle noise is good heuristic for SPH going wrong
4. Ensure stability by positive definite entropy
5. Don’t think of particles as “real”, instead think about discretising partial differential equations with the Lagrangian
6. Use wisdom from other methods