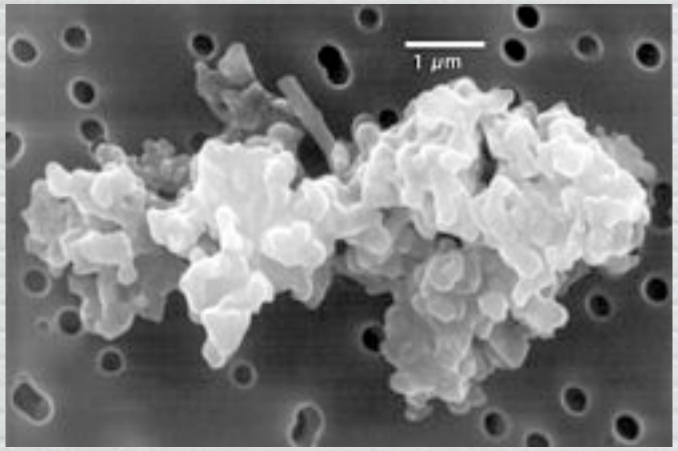

en.wikipedia.org

Dust+gas in SPH

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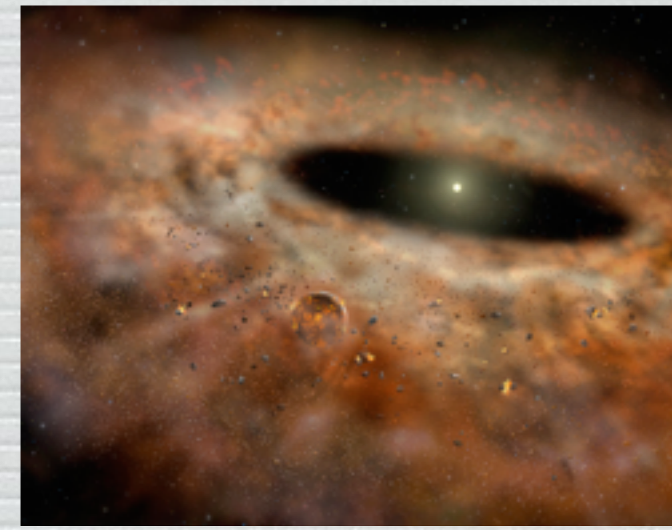


Image: Gemini Observatory/
AURA Artwork by Lynette Cook

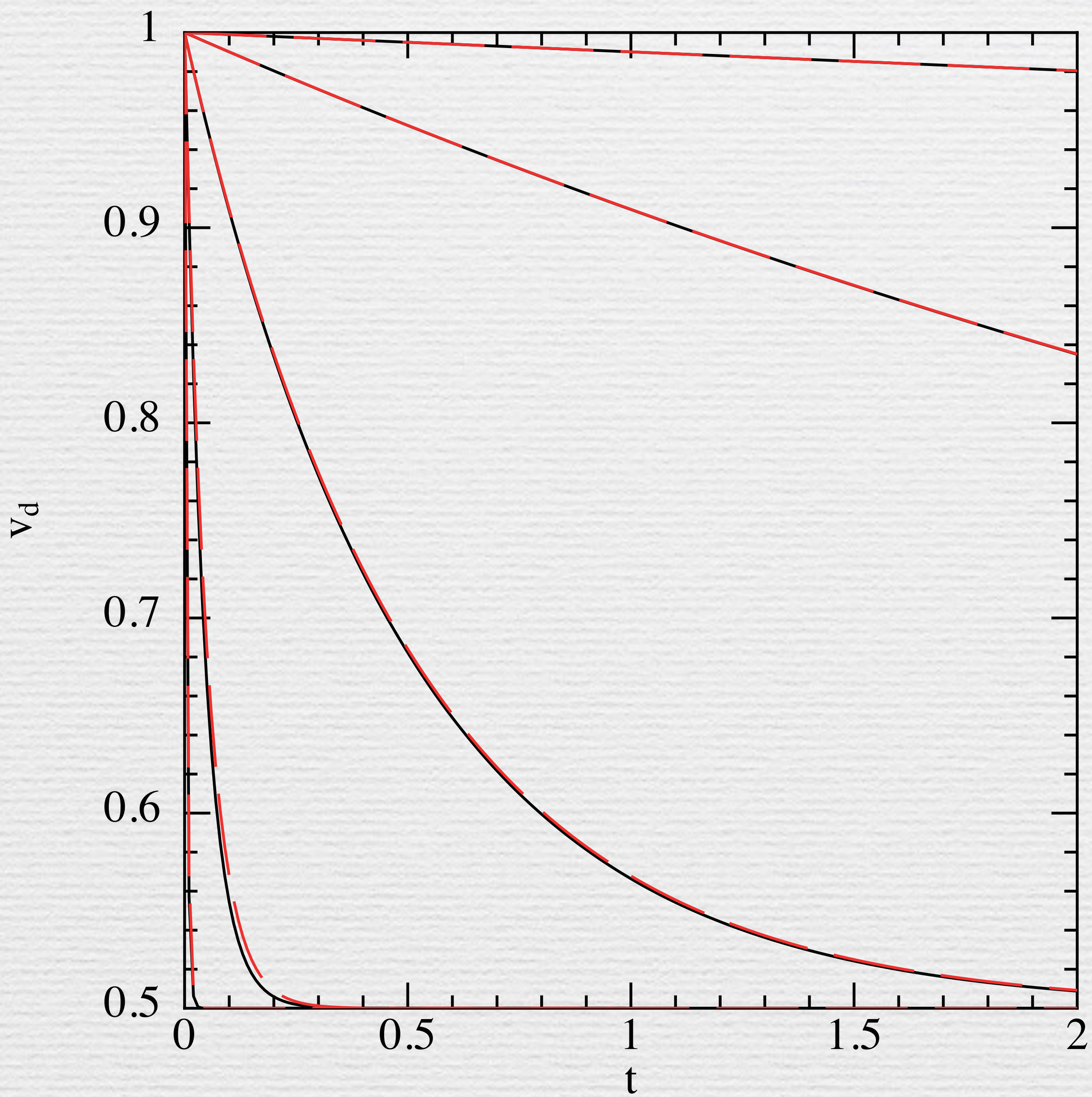
10th SPHERIC meeting, 16th-18th June 2015, Parma, Italy,

Dust + Gas: A simple example of a two-fluid mixture

- Two fluids coupled by a drag term

$$
\begin{aligned}
\frac{\partial \rho_{\rm g}}{\partial t} + \nabla . \left(\rho_{\rm g} \mathbf{v}_{\rm g}\right) &= 0, \\
\frac{\partial \rho_{\rm d}}{\partial t} + \nabla . \left(\rho_{\rm d} \mathbf{v}_{\rm d}\right) &= 0, \\
\frac{\partial \mathbf{v}_{\rm g}}{\partial t} + \left(\mathbf{v}_{\rm g} . \nabla\right) \mathbf{v}_{\rm g} &= -\frac{\nabla P_{\rm g}}{\rho_{\rm g}} + \frac{K}{\rho_{\rm g}}\left(\mathbf{v}_{\rm d} - \mathbf{v}_{\rm g}\right) + \mathbf{f}, \\
\frac{\partial \mathbf{v}_{\rm d}}{\partial t} + \left(\mathbf{v}_{\rm d} . \nabla\right) \mathbf{v}_{\rm d} &= -\frac{K}{\rho_{\rm d}}\left(\mathbf{v}_{\rm d} - \mathbf{v}_{\rm g}\right) + \mathbf{f},
\end{aligned}
$$

Stopping time



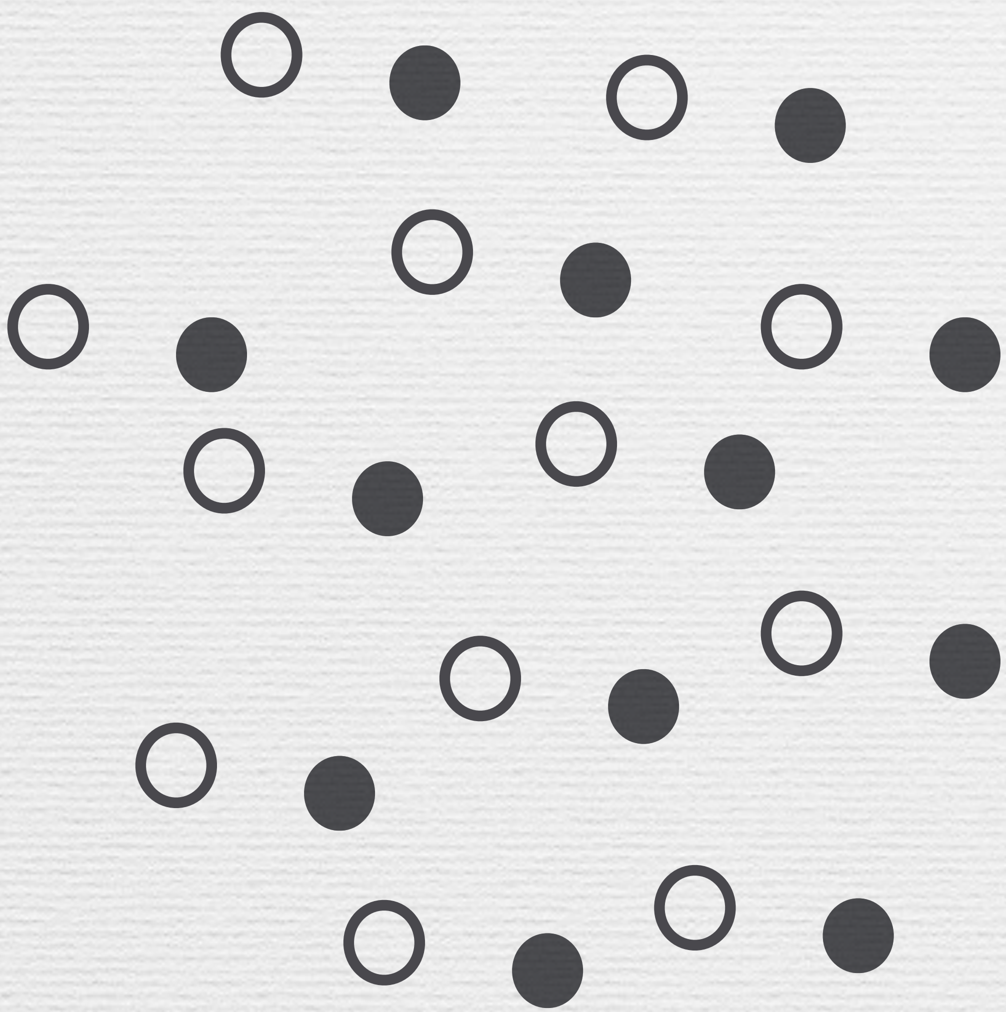
K=0.01

K=100

$$t_{\rm stop} \equiv \frac{\rho_{\rm d}\rho_{\rm g}}{K(\rho_{\rm d}+\rho_{\rm g})}$$

Two fluid dust+gas in SPH

Monaghan & Kocharyan (1995), Monaghan (1997),
Maddison et al. (2003), Laibe & Price (2012a,b MNRAS)



$$\rho_{\rm g}^a = \sum_b m_b W(|\mathbf{r}_a - \mathbf{r}_b|, h_a),$$

$$\rho_{\rm d}^i = \sum_j m_j W(|\mathbf{r}_i - \mathbf{r}_j|, h_i),$$

$$\frac{{\rm d}\mathbf{v}_{\rm g}^a}{{\rm d}t} = -\sum_b m_b \left[\frac{P_a}{\Omega_a \rho_{{\rm g},a}^2} \nabla_a W_{ab}(h_a) + \frac{P_a}{\Omega_b \rho_{{\rm g},b}^2} \nabla_a W_{ab}(h_b) \right] - \nu \sum_j m_j \frac{\mathbf{v}_{aj} \cdot \hat{\mathbf{r}}_{aj}}{(\rho_a + \rho_j) t_{aj}^{\rm s}} D_{aj}(h_a),$$

$$\frac{{\rm d}\mathbf{v}_{\rm d}^i}{{\rm d}t} = -\nu \sum_b m_b \frac{\mathbf{v}_{ib} \cdot \hat{\mathbf{r}}_{ib}}{(\rho_i + \rho_b) t_{ib}^{\rm s}} D_{ib}(h_b$$

$$\frac{{\rm d}u}{{\rm d}t} = -\frac{P_a}{\rho_{{\rm g},a}^2} \sum_b m_b \mathbf{v}_{ab} \cdot \nabla_a W_{ab}(h$$

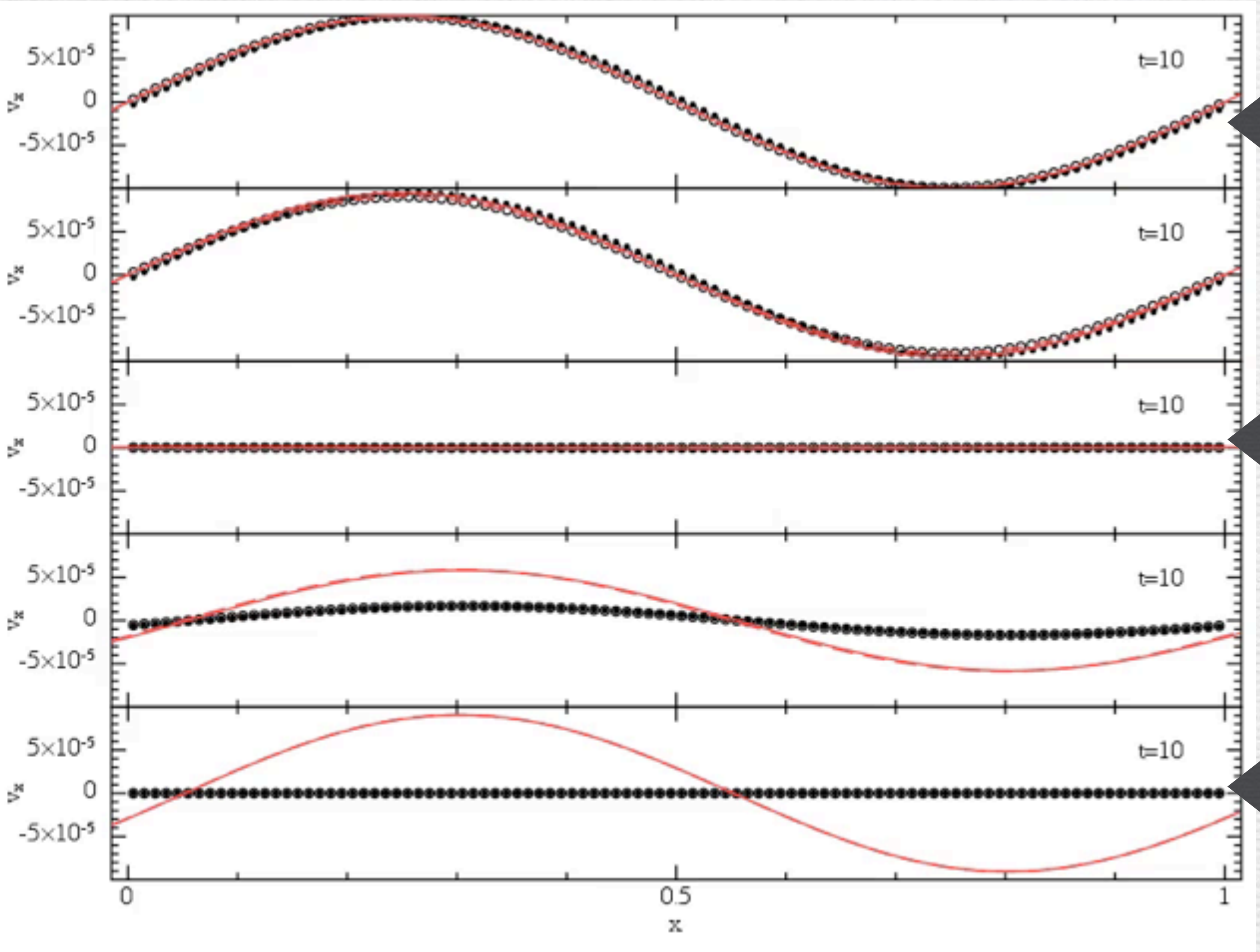
$$\nu \sum_j m_j \frac{(\mathbf{v}_{aj} \cdot \hat{\mathbf{r}}_{aj})^2}{(\rho_a + \rho_j) t_{aj}^{\rm s}} D_{aj}(h_a),$$

Double
hump kernel

Two sets of particles coupled by drag terms

Two problems with two fluids

1) Overdamping problem



No drag=no damping
SPH=exact

Intermediate drag =
strong damping in
both SPH + exact

High drag =
no damping
but SPH strongly damped

Red=analytic solution for dust/gas waves derived by Laibe & Price (2011) MNRAS 418, 1491

Overdamping problem: Resolution Criterion

Laibe & Price, 2012, MNRAS 420, 2345

Temporal: $\Delta t < t_{\rm stop}$ (can be fixed with implicit timestepping methods)

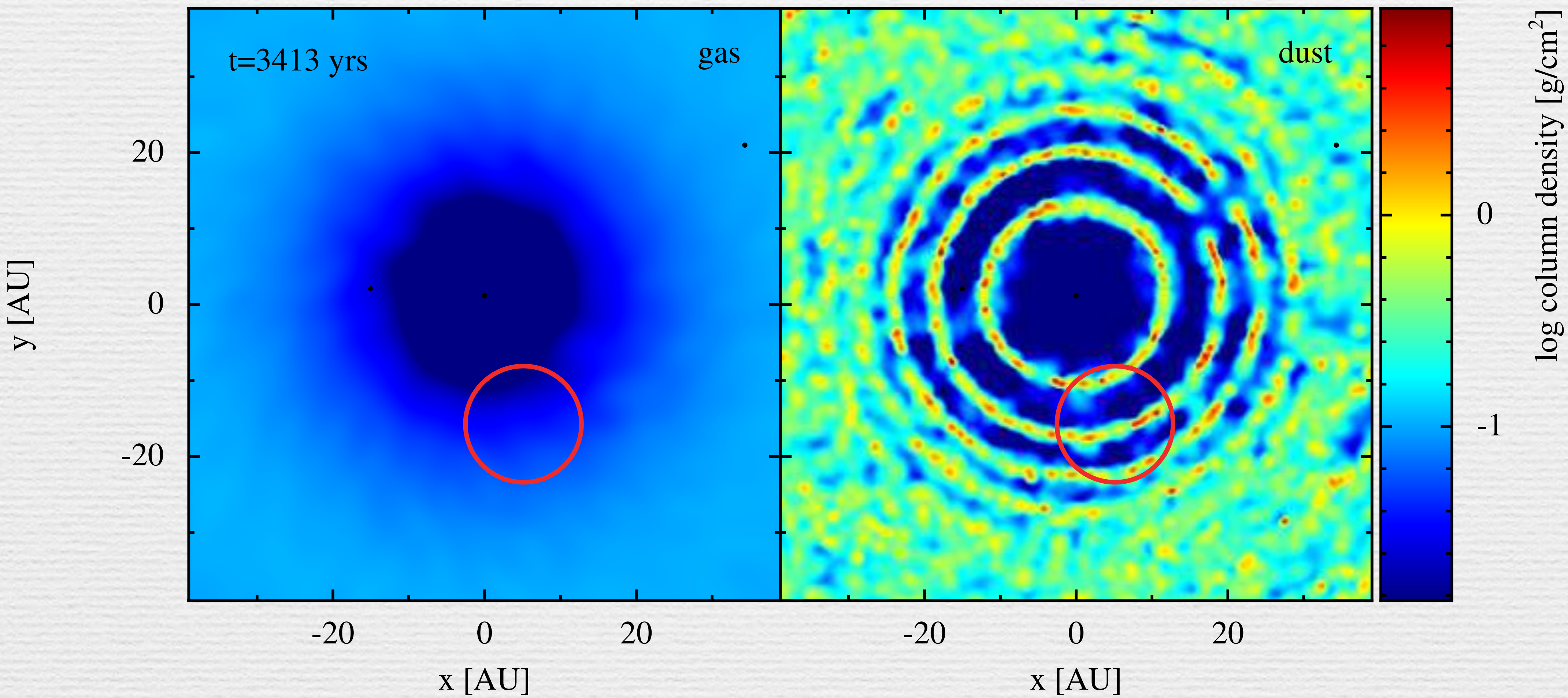
Spatial: $h < t_{\rm stop} c_{\rm s}$ (much more difficult to fix)

$t_{\rm stop} \to 0$ implies $\Delta t \to 0$

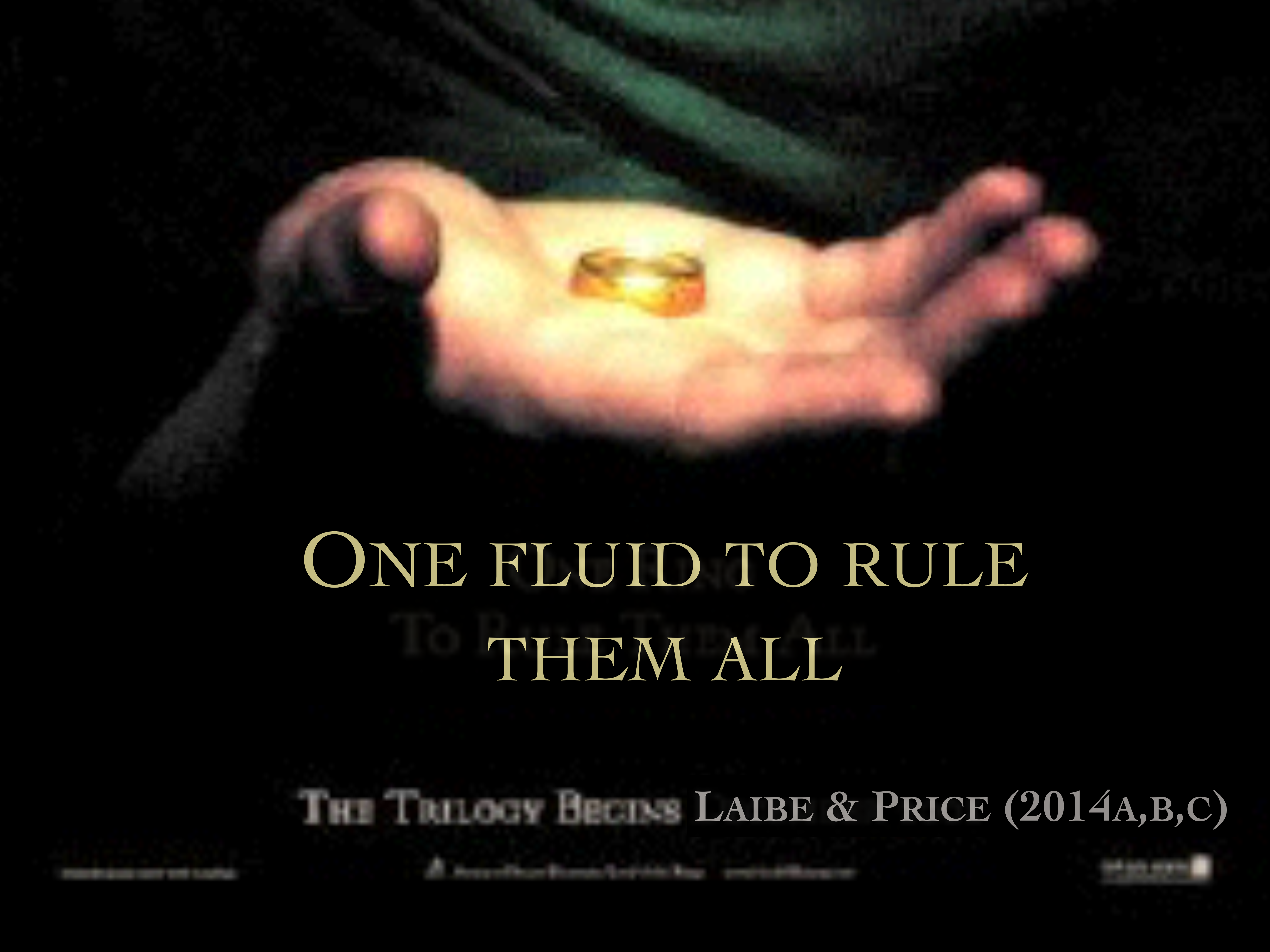
$(K \to \infty)$ $\Delta x \to 0$

- Require infinite timesteps AND infinite resolution in the obvious limit of perfect coupling!

2) Dust trapping problem



- Dust particles feel no pressure, can become `trapped' if they fall below the resolution length of the gas, forming artificial structures



One fluid to rule them all

Laibe & Price (2014a,b,c)

TWO BECOME ONE

A phoenix from the ashes

- One mixture with a differential velocity

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho (\nabla \cdot \mathbf{v}),$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\frac{1}{\rho}\nabla \cdot \left[\epsilon (1-\epsilon)\rho \Delta \mathbf{v}\right],$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho} - \frac{1}{\rho}\nabla \cdot \left[\epsilon (1-\epsilon)\rho \Delta \mathbf{v}\Delta \mathbf{v}\right] + \mathbf{f},$$

$$\frac{\mathrm{d}\Delta \mathbf{v}}{\mathrm{d}t} = -\frac{\Delta \mathbf{v}}{t_{\mathrm{s}}} + \frac{\nabla P}{\rho_{\mathrm{g}}} - (\Delta \mathbf{v}\cdot \nabla)\mathbf{v} + \frac{1}{2}\nabla\left[(2\epsilon - 1)\Delta \mathbf{v}^2\right].$$

- Two fluids coupled by a drag term

$$\frac{\partial \rho_{\mathrm{g}}}{\partial t} + \nabla \cdot (\rho_{\mathrm{g}} \mathbf{v}_{\mathrm{g}}) = 0,$$

$$\frac{\partial \rho_{\mathrm{d}}}{\partial t} + \nabla \cdot (\rho_{\mathrm{d}} \mathbf{v}_{\mathrm{d}}) = 0,$$

$$\frac{\partial \mathbf{v}_{\mathrm{g}}}{\partial t} + (\mathbf{v}_{\mathrm{g}}\cdot\nabla)\mathbf{v}_{\mathrm{g}} = -\frac{\nabla P_{\mathrm{g}}}{\rho_{\mathrm{g}}} + \frac{K}{\rho_{\mathrm{g}}}(\mathbf{v}_{\mathrm{d}} - \mathbf{v}_{\mathrm{g}}) + \mathbf{f},$$

$$\frac{\partial \mathbf{v}_{\mathrm{d}}}{\partial t} + (\mathbf{v}_{\mathrm{d}}\cdot\nabla)\mathbf{v}_{\mathrm{d}} = -\frac{K}{\rho_{\mathrm{d}}}(\mathbf{v}_{\mathrm{d}} - \mathbf{v}_{\mathrm{g}}) + \mathbf{f},$$

$$\rho = \rho_{\mathrm{d}} + \rho_{\mathrm{g}}$$

$$\epsilon = \rho_{\mathrm{d}}/\rho$$

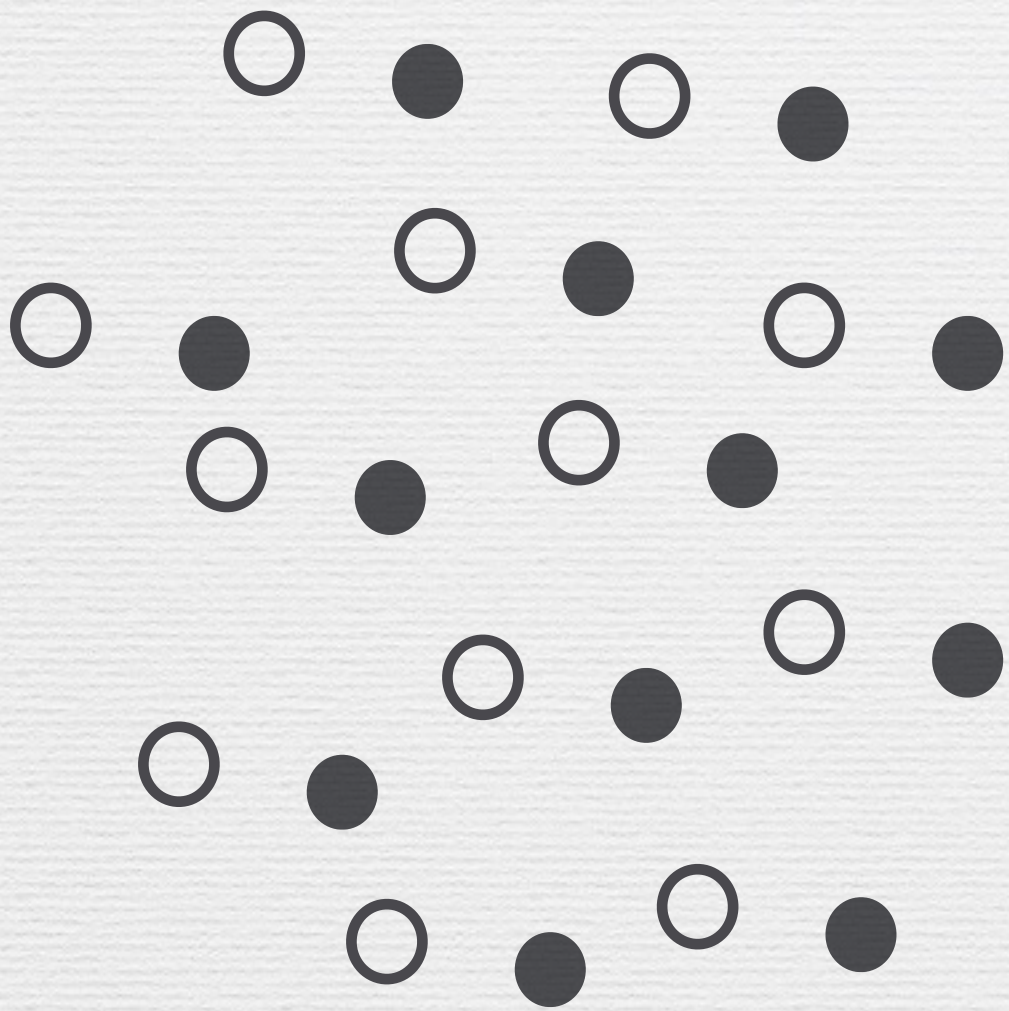
$$\mathbf{v} = \frac{\rho_{\mathrm{d}}\mathbf{v}_{\mathrm{d}} + \rho_{\mathrm{g}}\mathbf{v}_{\mathrm{g}}}{\rho}$$

$$\Delta \mathbf{v} = \mathbf{v}_{\mathrm{d}} - \mathbf{v}_{\mathrm{g}}$$

No approximations!

Laibe & Price (2014) MNRAS

SPH one fluid method



$$\rho_a = \sum_b m_b W_{ab}(h_a), \tag{26}$$

$$\frac{\mathrm{d}\epsilon_a}{\mathrm{d}t} = -\sum_b m_b \left[\frac{\epsilon_a (1-\epsilon_a)}{\Omega_a \rho_a} \Delta\mathbf{v}_a \cdot \nabla_a W_{ab}(h_a) + \frac{\epsilon_b (1-\epsilon_b)}{\Omega_b \rho_b} \Delta\mathbf{v}_b \cdot \nabla_a W_{ab}(h_b) \right], \tag{27}$$

$$\frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} = -\sum_b m_b \left[\frac{P_a}{\Omega_a \rho_a^2} \nabla W_{ab}(h_a) + \frac{P_b}{\Omega_b \rho_b^2} \nabla W_{ab}(h_b) \right] - \sum_b m_b \left[\frac{\epsilon_a (1-\epsilon_a) \Delta\mathbf{v}_a}{\Omega_a \rho_a} \Delta\mathbf{v}_a . \nabla W_{ab}(h_a) + \frac{\epsilon_b (1-\epsilon_b) \Delta\mathbf{v}_b}{\Omega_b \rho_b} \Delta\mathbf{v}_b \cdot \nabla W_{ab}(h_b) \right] + \mathbf{f}_a, \tag{28}$$

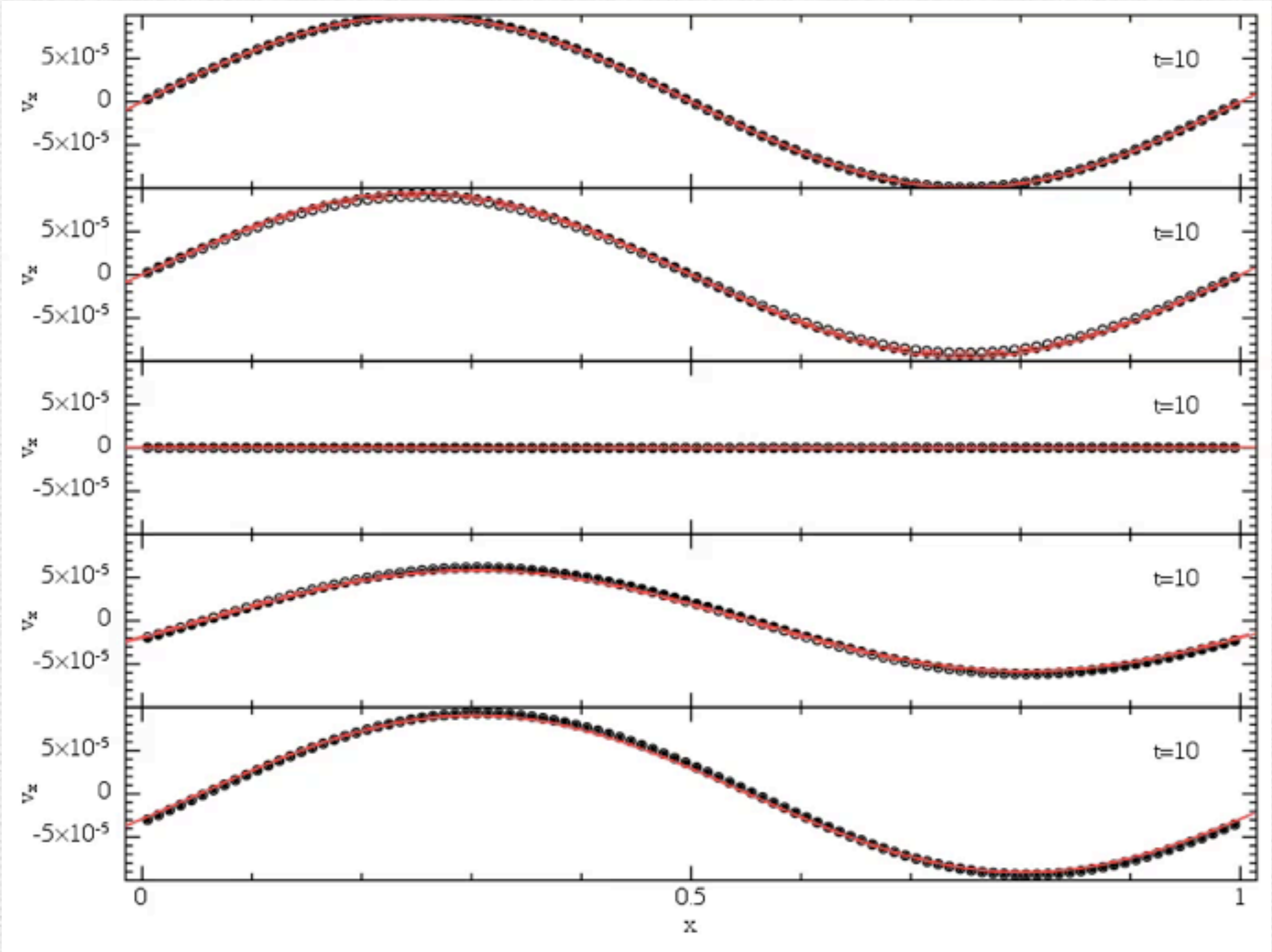
$$\frac{\mathrm{d}\Delta\mathbf{v}_a}{\mathrm{d}t} = -\frac{\Delta\mathbf{v}_a}{t_{\mathrm{s,a}}} - \frac{\rho_a}{\rho_{\mathrm{g},a}} \sum_b m_b \left[\frac{P_a}{\Omega_a \rho_a^2} \nabla W_{ab}(h_a) + \frac{P_b}{\Omega_b \rho_b^2} \nabla W_{ab}(h_b) \right], + \frac{1}{\rho_a \Omega_a} \sum_b m_b (\mathbf{v}_a - \mathbf{v}_b) \Delta\mathbf{v}_a . \nabla W_{ab}(h_a) + \frac{1}{2\rho_a \Omega_a} \sum_b m_b \left[(1-2\epsilon_a) \Delta\mathbf{v}_a^2 - (1-2\epsilon_b) \Delta\mathbf{v}_b^2 \right] \nabla W_{ab}(h_a), \tag{29}$$

$$\frac{\mathrm{d}u_a}{\mathrm{d}t} = \frac{P_a}{\Omega_a \rho_a \rho_{\mathrm{g},a}} \sum_b m_b \left(\mathbf{v}_{\mathrm{g},a} - \mathbf{v}_{\mathrm{g},b} \right) \cdot \nabla W_{ab}(h_a) - \frac{\epsilon_a}{\Omega_a \rho_a} \sum_b m_b (u_a - u_b) \Delta\mathbf{v}_a . \nabla W_{ab}(h_a) + \epsilon_a \frac{\Delta\mathbf{v}_a^2}{t_{\mathrm{s,a}}}. \tag{30}$$

ONE set of particles representing the mixture

Tests of one fluid method

Laibe & Price (2014b)



no drag

strong drag

The diffusion approximation for dust

Laibe & Price (2014)
Price & Laibe (2015) MNRAS 451, 5332

“Terminal velocity approximation”

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho(\nabla \cdot \mathbf{v})$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho} + \mathbf{f}$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\frac{1}{\rho}\nabla \cdot (\epsilon t_{\mathrm{s}} \nabla P)$$

Use SPH second derivative

Valid when $t_{\mathrm{stop}} < \Delta t$

Laibe & Price (2014)

Diffusion approximation: tests

"Fall of a layer of dust" from Monaghan (1997), JCP

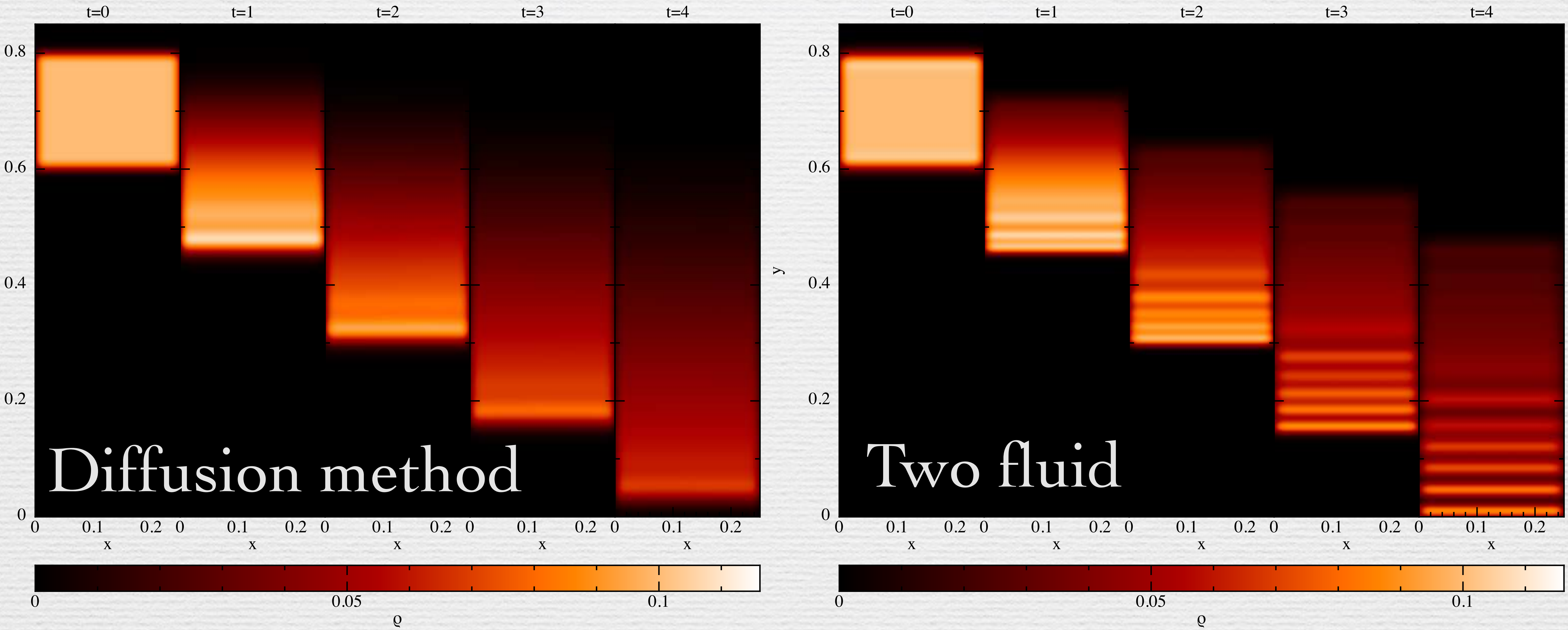
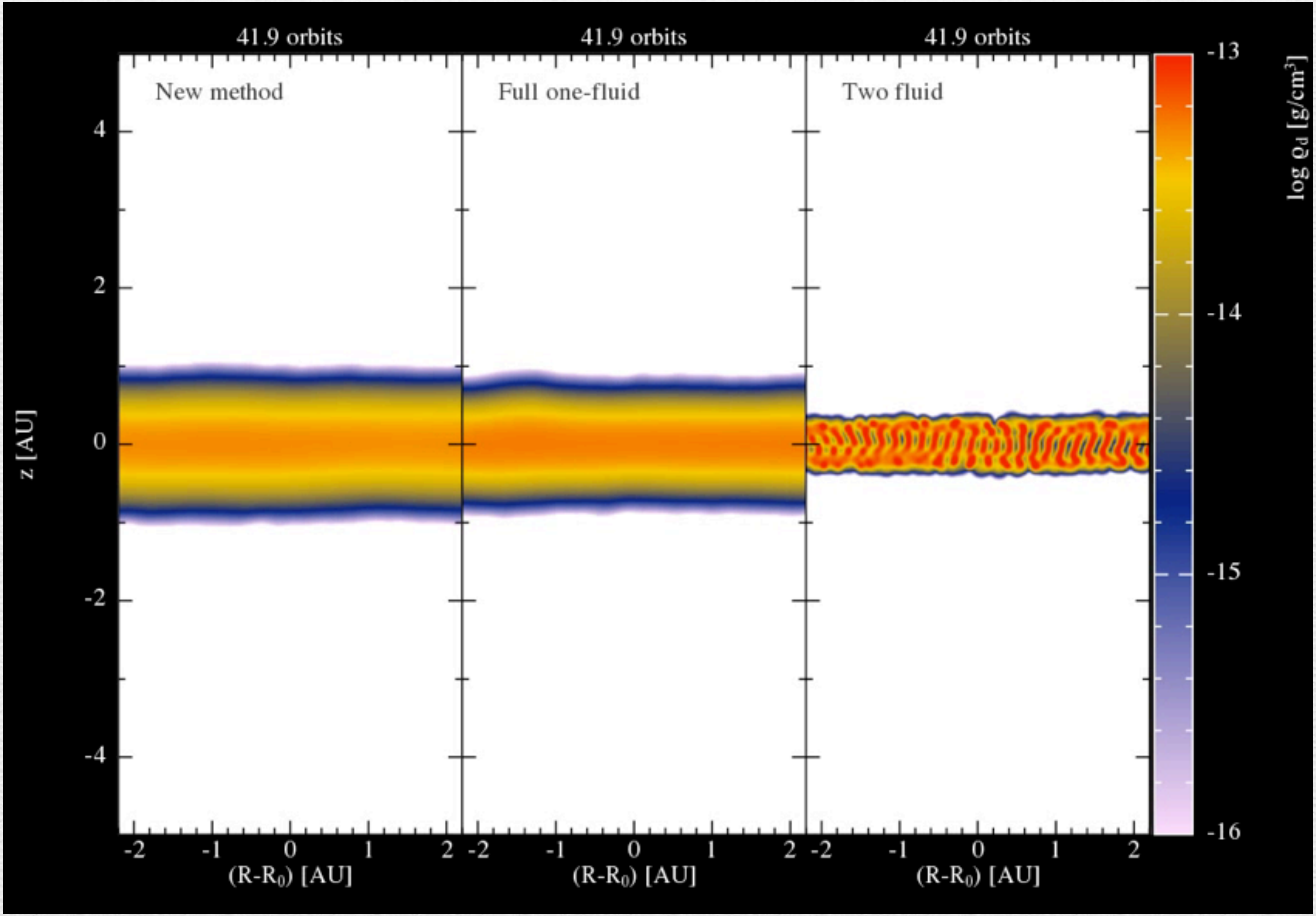


Fig. 8. Fall of a layer of dust in a stratified atmosphere, comparing results with the diffusion method (left) to the two fluid approach (right).

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\sum_b \frac{m_b}{\rho_a \rho_b} \left(\epsilon_a t_{\mathrm{s},a} + \epsilon_b t_{\mathrm{s},b}\right) (P_a - P_b) F_{ab}$$

Explicit timestepping!

Settling of grains in a protoplanetary disc



Some issues

- Epsilon can go negative. Solution: if (eps < 0) eps = 0.
- Found problem with harmonic mean:

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\sum_b \frac{m_b}{\rho_a \rho_b} \left(\frac{4D_a D_b}{D_a + D_b} \right) (P_a - P_b) F_{ab}$$

- Weird discretisation in du/dt - is it correct?

$$\frac{\mathrm{d}u_a}{\mathrm{d}t} = \frac{1}{2(1-\epsilon_a)\rho_a} \sum_b \frac{m_b}{\rho_b} (u_a - u_b)(D_a + D_b)(P_a - P_b) \overline{F}_{ab}$$

APPENDIX A: PROOF THAT EQUATION 46 IS A DISCRETE FORM OF EQUATION 4

Here, we prove that the expression obtained for the second term in Eq. 46 by enforcing the conservation o

$$\frac{1}{2(1-\epsilon_a)\rho_a}\sum_b \frac{m_b}{\rho_b}(u_a-u_b)(D_a+D_b)(P_a-P_b)\frac{F_{ab}}{|r_{ab}|},$$

is indeed a discrete form of the corresponding term in Eq. 4, i.e.

$$-\frac{\epsilon t_{\rm s}}{\rho_{\rm g}}\nabla P\cdot\nabla u.$$

We proceed, following Price (2012), by identifying $-2F_{ab}/|r_{ab}|$ as equivalent to the second derivative of a

$$\nabla^2 Y_{ab}\equiv\frac{-2F_{ab}}{|r_{ab}|}.$$

It may be shown straightforwardly that this new kernel Y_{ab} indeed satisfies the normalisation condition derivative (see Price 2012 for more details). We can then take the Laplacian of the standard SPH summati

$$A_a\simeq\sum_b m_b\frac{A_b}{\rho_b}Y_{ab},$$

to give

$$\nabla^2 A_a\simeq\sum_b m_b\frac{A_b}{\rho_b}\nabla^2 Y_{ab}.$$

By writing (A1) in the form

$$-\frac{1}{4\rho_{\rm g}^a}\sum_b \frac{m_b}{\rho_b}(u_a-u_b)(D_a+D_b)(P_a-P_b)\nabla^2 Y_{ab},$$

we can then use (A5) to translate the various terms. Expanding (A6) we have

Summary

- New general method for dusty gas with SPH
- Small grains/strong drag = usual SPH equations + evolution equation for dust fraction
- Widely applicable

Refs:
Two fluid: Laibe & Price (2012a,b) MNRAS
One fluid: Laibe & Price (2014a,b,c) MNRAS
Diffusion method: Price & Laibe (2015) MNRAS