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SMOOTHED PARTICLE MAGNETOHYDRODYNAMICS

The state of the art

Daniel Price (Monash), James Wurster (Monash/Exeter), Terrence Tricco (Monash/Exeter/CITA), Matthew Bate (Exeter), Ben Ayliffe (Monash/Exeter) Numerical techniques in MHD simulations, Cologne, Germany, August 16-18th, 2017

SMOOTHED PARTICLE HYDRODYNAMICS



What is the density?

Weighted sum over

neighbours

$$\rho(\mathbf{r}) = \sum_{j=1}^{N} m_j W(|\mathbf{r} - \mathbf{r}_j|, h)$$

e.g.
$$W = \frac{\sigma}{h^3} e^{-r^2/h^2}$$

RESOLUTION FOLLOWS MASS



 $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}$

5

FROM DENSITY TO HYDRODYNAMICS

$$L_{sph} = \sum_{j} m_{j} \left[\frac{1}{2} v_{j}^{2} - u_{j}(\rho_{j}, s_{j}) \right] \qquad \text{Lagrangian}$$

$$du \stackrel{+}{=} \frac{P}{\rho^{2}} d\rho \qquad \text{1st law of thermodynamics}$$

$$+ \nabla \rho_{i} = \sum_{j} m_{j} \nabla W_{ij}(h) \qquad \text{density sum}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \mathbf{v}} \right) - \frac{\partial L}{\partial \mathbf{r}} = 0 \qquad \text{Euler-Lagrange equations}$$

$$= \sum_{j} m_{j} \left(\frac{P_{i}}{\rho_{i}^{2}} + \frac{P_{j}}{\rho_{j}^{2}} \right) \nabla_{i} W_{ij}(h) \qquad \left(\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} \right)$$

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WHAT THIS GIVES US: ADVANTAGES OF SPH

- Exact solution to the mass continuity equation
- Resolution follows mass
- Zero numerical dissipation
- Advection done perfectly
- Exact and simultaneous conservation of mass, momentum, angular momentum, energy and entropy
- ► A guaranteed minimum energy state

THE "GRID" IN SPH



Monaghan (2005) Price (2012) J. Comp. Phys. 231, 759

- Existence of minimum
 energy state guarantees local
 ordering of particle
 distribution
- BUT: requires positive pressure

SMOOTHED PARTICLE MAGNETOHYDRODYNAMICS see review by Price (2012) J. Comp. Phys. 231, 759

$$L = \int \left(\frac{1}{2}\rho v^2 - \rho u - \frac{1}{2\mu_0}B^2\right) dV$$
$$L = \sum_a m_a \left(\frac{1}{2}v_a^2 - u_a - \frac{B_a^2}{2\mu_0\rho_a}\right)$$

Price & Monaghan (2004a,b, 2005)

Euler-Lagrange equations give discrete form of:

Include div B source term for stability

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho(\nabla \cdot \mathbf{v})$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla \cdot \left[\left(P + \frac{1}{2}\frac{B^2}{\mu_0}\right)\mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{\mu_0}\right] - \frac{\mathbf{B}(\nabla \cdot \mathbf{B})}{\mu_0\rho}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{P}{\rho}(\nabla \cdot \mathbf{v})$$
Method is dissipationless
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho} \cdot \nabla\right)\mathbf{v}$$
Need to separately handle div $\mathbf{B} = 0$

These equations are equivalent to the 8-wave formulation of Powell et al. 1994

$$\delta L = m_a v_a \cdot \delta v_a - \sum_b m_b \left[\frac{\partial u_b}{\partial \rho_b} \Big|_s \delta \rho_b + \frac{1}{2\mu_0} \left(\frac{B_b}{\rho_b} \right)^2 \delta \rho_b - \frac{1}{\mu_0} B_b \cdot \delta \left(\frac{B_b}{\rho_b} \right) \right]$$

$$\delta \rho_b = \sum_c m_c \left(\delta \mathbf{r}_b - \delta \mathbf{r}_c \right) \cdot \nabla_b W_{bc} \qquad \frac{du}{d\rho} \Big|_s = \frac{P}{\rho^2}.$$

$$\delta \left(\frac{B_b}{\rho_b} \right) = \sum_c m_c \left(\delta \mathbf{r}_b - \delta \mathbf{r}_c \right) \frac{B_b}{\rho_b^2} \cdot \nabla_b W_{bc} \qquad \text{Div B source term present in induction equation}$$

$$\frac{dv_a^i}{dt} = \sum_b m_b \left[\left(\frac{S^{ij}}{\rho^2} \right)_a + \left(\frac{S^{ij}}{\rho^2} \right)_b \right] \nabla_a^j W_{ab}, \qquad \text{Obtain CONSERVATIVE momentum equation}$$

Consistent with Janhunen formulation of div B terms

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PHILLIPS & MONAGHAN (1985): SPH WITH MHD IN CONSERVATIVE FORM IS UNSTABLE WHEN BETA < 1



WITH SOURCE TERM IN MOMENTUM EQUATION

Morris (1996), Børve, Omang & Trulsen (2001, 2004)



ZERO DISSIPATION – EXAMPLE I



Circularly polarised Alfvén wave

ZERO DISSIPATION – II. ADVECTION OF A CURRENT LOOP



first 25 crossings





1000 crossings (Rosswog & Price 2007)

Fig. 3. Gray-scale images of the magnetic pressure $(B_x^2 + B_y^2)$ at t = 2 for an advected field loop $(v_0 = \sqrt{5})$ using the \mathscr{E}_z^{α} (top left), (top right) and \mathscr{E}_z^{c} (bottom) CT algorithm.





Fig. 8. Magnetic field lines at t = 0 (left) and t = 2 (right) using the CTU + CT integration algorithm.

2 crossings (Gardiner & Stone 2005)

SPH

Grid

ZERO DISSIPATION III



$$\frac{\mathrm{d}\boldsymbol{U}_a}{\mathrm{d}t} = \sum_b \frac{m_b}{\bar{\rho}_{ab}} \alpha v_{\mathrm{sig}} (\boldsymbol{U}_a - \boldsymbol{U}_b) \hat{r}_{ab} \cdot \nabla W_{ab}$$

► Formulate dissipative terms similar to approximate Riemann solvers

$$\begin{pmatrix} \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} \end{pmatrix}_{\mathrm{diss}} = \sum_j m_j \frac{\alpha v_{\mathrm{sig}}(\mathbf{v}_i - \mathbf{v}_j) \cdot \hat{\mathbf{r}}_{ij}}{\bar{\rho}_{ij}} \overline{\nabla_i W_{ij}}, \\ \left(\frac{\mathrm{d}e_i}{\mathrm{d}t}\right)_{\mathrm{diss}} = \sum_j m_j \frac{(e_i^* - e_j^*)}{\bar{\rho}_{ij}} \hat{\mathbf{r}}_{ij} \cdot \overline{\nabla_i W_{ij}}, \qquad e = \frac{1}{2}v^2 + u$$

Enforce positive definite contribution to entropy

$$\begin{split} \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)_{\mathrm{diss}} &= -\sum_{j} \frac{m_{j}}{\bar{\rho}_{ij}} \left\{\frac{1}{2} \alpha v_{\mathrm{sig}} (\mathbf{v}_{ij} \cdot \hat{\mathbf{r}}_{ij})^{2} + \alpha_{u} v_{\mathrm{sig}}^{u} (u_{i} - u_{j})\right\} \hat{\mathbf{r}}_{ij} \cdot \nabla_{i} W_{ij} \\ \\ & \text{Viscous heating} \qquad \text{Thermal conduction} \end{split}$$

Gives artificial dissipation terms equivalent to artificial viscosity, conductivity and (in MHD) resistivity

> Viscosity terms = Navier Stokes equations with $\nu = \frac{\alpha}{10} v_{sig} h$ BUT dissipation terms are first order

ARTIFICIAL VISCOSITY



ARTIFICIAL CONDUCTIVITY

Chow & Monaghan (1997) Price (2008), JCP



APPROACHES TO DIV $\mathbf{B} = \mathbf{0}$

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1. Ignore

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- 2. Prevent
- 3. Clean

POWELL'S 8 WAVE METHOD = IGNORE BUT PRESERVE

Powell et al. (1999), Janhunen (2000), Dellar (2001), Tóth (2000)

consistent ∇·B terms



volume conservative form



Preserve



Divergence advection test from Dedner et al. (2002)

Smear	
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Application to SPH: Price & Monaghan (2005)

USE OF POWELL-ONLY DIV B CONTROL IN SPMHD





Orszag-Tang vortex problem in SPMHD (Price & Monaghan 2005, Rosswog & Price 2007)

PREVENT: DIV B = 0 by construction in SPH

 $\mathbf{B} = \nabla \alpha \times \nabla \beta$

Euler potentials (e.g. Stern, 1976)



 $\begin{aligned} \frac{d\alpha}{dt} &= 0 & (advection \ of \\ magnetic \ field \ lines \\ \frac{d\beta}{dt} &= 0 & by \ Lagrangian \\ particles) \end{aligned}$

PRICE & BATE (2007): EFFECT OF MAGNETIC FIELDS ON SINGLE AND BINARY STAR FORMATION



...problem forming discs and binaries in the presence of magnetic fields?

see also Allen et al. (2003), Galli et al. (2006), Mellon & Li (2008), Hennebelle & Fromang (2008), Commerçon et al. (2010), Krasnopolsky et al. (2010), Seifried et al. (2012), Santos-Lima et al. (2012), Joos et al. (2013) and many others

LIMITATIONS OF THE EULER POTENTIALS APPROACH

Rosswog & Price (2007), Price & Bate (2008), Brandenburg (2010)

$$\mathbf{B} = \nabla \alpha \times \nabla \beta$$

$$\frac{d\alpha}{dt} = 0$$
$$\frac{d\beta}{dt} = 0$$

- > advection of magnetic fields: no change in topology (A.B = 0)
- does not follow wind-up of magnetic fields
- Ifficult to model resistive effects reconnection processes not treated correctly

HYPERBOLIC/PARABOLIC DIVERGENCE CLEANING

Dedner et al. (2002) Price & Monaghan (2005) Mignone & Tzeferacos (2010)



Hyperbolic term only

WHEN CLEANING ATTACKS



Divergence advection test (Dedner et al. 2002) with 10:1 jump in density

"CONSTRAINED" HYPERBOLIC/PARABOLIC DIVERGENCE CLEANING Tricco & Price (2012); Tricco, Price & Bate (2016)

Define energy associated with cleaning field

$$E = \int \left[\frac{1}{2}\frac{B^2}{\mu_0} + \frac{1}{2}\frac{\psi^2}{\mu_0 c_h^2}\right] dV$$

Enforce energy conservation in hyperbolic terms

Can enforce exact energy conservation in SPH discretisation

CONSTRAINED HYPERBOLIC/PARABOLIC CLEANING



Parabolic term is negative definite!

WHAT IF THE CLEANING SPEED VARIES?

Tricco, Price & Bate (2016) J. Comp. Phys. 322, 326



Hyperbolic terms conserve energy even with variable wave speed!



Non-conservative method Conservative method

APPLICATION TO FINITE VOLUME SCHEMES

Price, this week See also: Derigs, Gassner, Walch & Winters (2017) + Florian Hindenlang talk

Original method (Dedner et al. 2003):



Constrained hyperbolic cleaning with variable wave speed:

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho v \\ E' \\ B \\ \phi \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho v + (P + \frac{1}{2}B^2)\mathcal{I} - BB \\ v(E' + P + \frac{1}{2}B^2) - B(v \cdot B) + \psi B \\ vB - Bv \\ \phi v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{\psi^2}{c_h^2 \tau} \\ v(\nabla \cdot B) \\ -\frac{c_h}{\sqrt{\rho}} (\nabla \cdot B) - \frac{\phi}{\tau} \end{bmatrix}$$

$$E' = E + \frac{1}{2} \frac{\psi^2}{c_h^2} \qquad \phi = \frac{\psi}{c_h \sqrt{\rho}}$$

APPLICATION TO FINITE VOLUME CODES



Tricco, Price & Bate (2016) J. Comp. Phys. 322, 326



Achievable in principle, not currently practical

SHOCK DISSIPATION SWITCHES

Cullen & Dehnen (2010)
 switch for shock viscosity

$$A = \max\left[-\frac{\mathrm{d}}{\mathrm{d}t}(\nabla \cdot \mathbf{v}), 0\right] \quad \alpha_{loc} = \min\left(\frac{10h^2A}{c_{\mathrm{s}}^2 + h^2A}, 1\right)$$



Figure 3. Shocktube test 5A from RJ95 performed in 2D with left state (ρ , P, v_x , v_y , B_y) = (1, 1, 0, 0, 1) and right state (ρ , P, v_x , v_y , B_y) = (0.125, 0.1, 0, 0, -1) with B_x = 0.75 at t = 0.1. Black circles represent the particles and the red line represents the solution obtained with the ATHENA code using 10⁴ grid cells.





- ► Tricco & Price (2013) switch for resistivity $\alpha^{B} = \min\left(\frac{h|\nabla \mathbf{B}|}{|\mathbf{B}|}, 1\right)$
- Revised further in Phantom - 2nd order artificial resistivity, vanishes when v=const

PHANTOM SPMHD CODE

Price et al. (2017), arXiv:1702.03930

0.01

 n_x^{-2}

 10^{2}



Performed with all dissipation, shock capturing and divergence cleaning turned on

JETS FROM THE FIRST CORE

Price, Tricco & Bate (2012); see also Machida et al. (2008)

27140 yrs 1000 AU

PROTOSTELLAR JETS: SECOND COLLAPSE

Bate, Tricco & Price (2014)



Performed with radiation magnetohydrodynamics (gray FLD: Whitehouse & Bate 2004a,b; Whitehouse, Bate & Monaghan 2006)

MAGNETICALLY LAUNCHED OUTFLOWS



First core (100 x 100 au)

Second (protostellar) core (10 x 10 au)

STRONG MAGNETIC FIELDS IMPLANTED IN STARS AT BIRTH



SMALL SCALE DYNAMO: FLASH VS PHANTOM Tricco, Price & Federrath (2016)



Phantom



MAGNETIC FIELDS IN TIDAL DISRUPTION EVENTS Bonne (2017)

Bonnerot, Price, Rossi, Lodato (2017), MNRAS



Artificial dynamo with Powell-terms only

NON-IDEAL SPMHD

Strong coupling approximation: $ho pprox
ho_n; \
ho_i \ll
ho$

 $\begin{aligned} \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} &= -\mathbf{B}(\nabla \cdot \mathbf{v}) + (\mathbf{B} \cdot \nabla)\mathbf{v} \\ &- \nabla \times \left[\eta_O \mathbf{J} + \eta_H \mathbf{J} \times \hat{\mathbf{B}} - \eta_A (\mathbf{J} \times \hat{\mathbf{B}}) \times \hat{\mathbf{B}}\right] \\ & \mathbf{Ohmic} \qquad \text{Hall} \qquad \text{Ambipolar} \end{aligned}$







Tests: Mac-Low et al. (1995) O'Sullivan & Downes (2006) Choi et al. (2009) Falle (2003)



- Spatial discretisation exactly conserves energy
- Guaranteed positive definite contribution to entropy
- RKC super-timestepping for ambipolar/Ohmic terms

 (Alexiades et al. 1996;
 O'Sullivan & Downes 2006)



Figure C2. The analytical (solid line) and numerical (crosses) results for the isothermal standing shock. The initial conditions are given in the text. At any given position, the analytical and numerical solutions agree within 3 per cent.

IDEAL MHD: MAGNETIC BRAKING CATASTROPHE

Wurster, Price & Bate (2016) MNRAS 457, 1037



NON-IDEAL MHD: ALIGNED INITIAL FIELD



NON-IDEAL MHD: ANTI-ALIGNED INITIAL FIELD

Wurster, Price & Bate (2016)



see also Tsukamoto et al. (2015)

OUTFLOWS – IDEAL MHD

Wurster, Price & Bate (2016)



OUTFLOWS: NON-IDEAL MHD / ALIGNED INITIAL FIELD



OUTFLOWS: NON-IDEAL MHD / ANTI-ALIGNED



Outflows are anti-correlated with disc formation!

WHICH NON-IDEAL EFFECTS ARE IMPORTANT?



- Hall effect is dominant during disc formation
 - Produces counterrotating envelope when B and rotation are misaligned
 - Maybe why half of all stars have planets?

ARE FOSSIL FIELDS POSSIBLE IN NON-IDEAL MHD?

Wurster, Price & Bate (2018)



CONCLUSIONS

- Enforcing div B = 0 is main issue in accurate SPMHD simulations
- Current best approach to enforcing div B = 0 in SPMHD is to use "constrained" hyperbolic/parabolic cleaning
- Phantom SPMHD code now public
- Non-ideal MHD, in particular the Hall effect, plays a crucial role in the formation of protostellar discs
- Can seemingly rule out fossil field hypothesis for origin of magnetic fields in stars