Smoothed Particle Hydrodynamics: Turbulence and MHD

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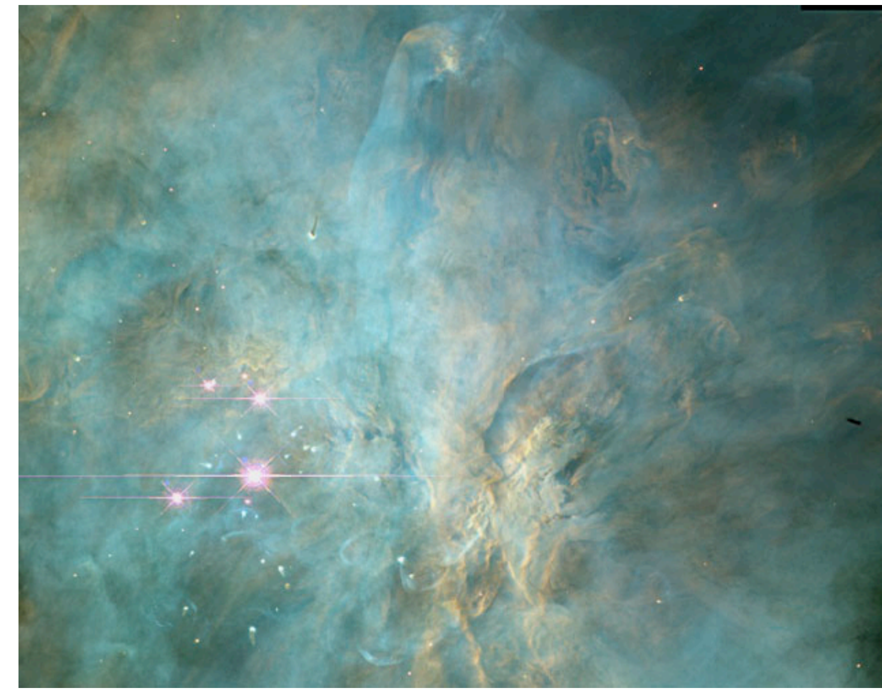
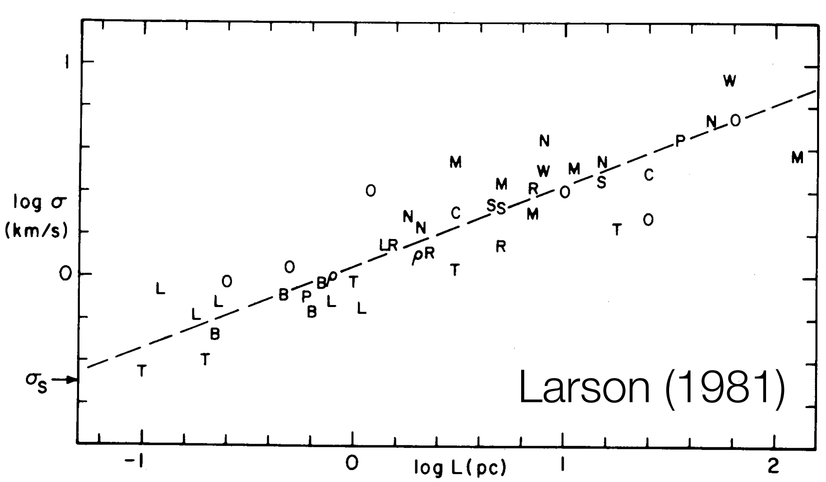
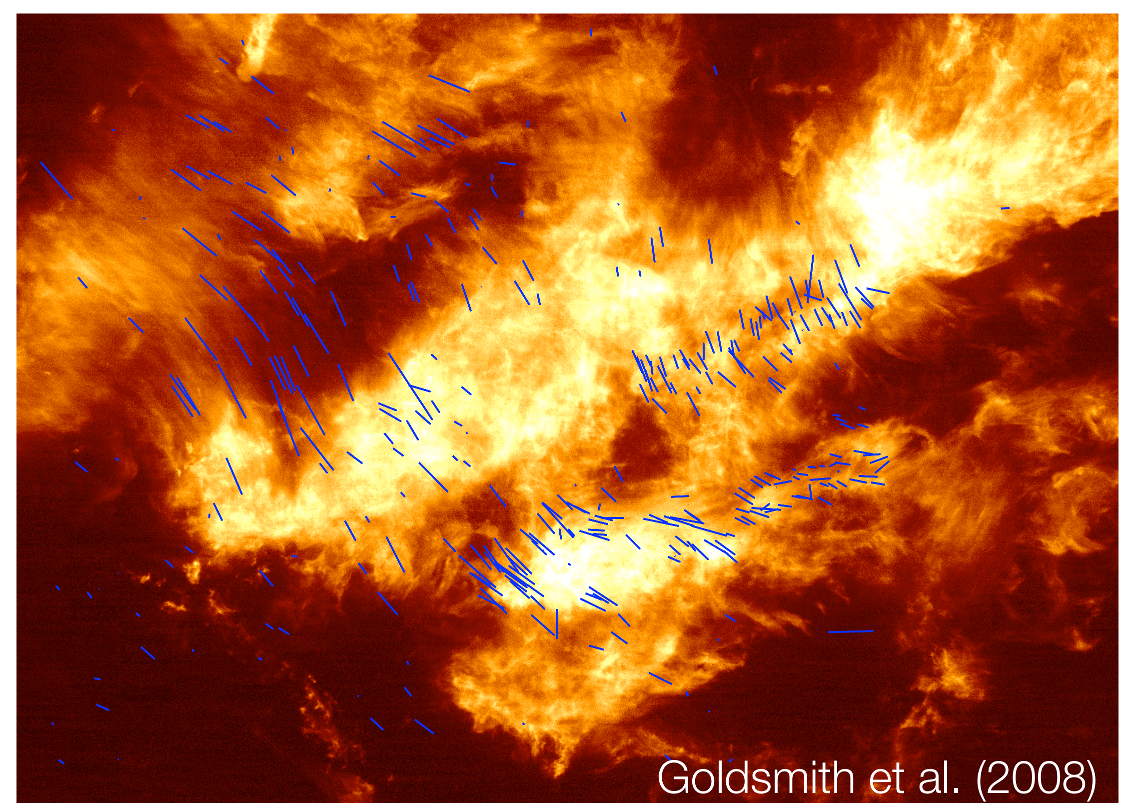
Turbulence in the Interstellar Medium

- highly supersonic, Mach numbers ~ 5-20
- isothermal to good approximation
- unknown driving mechanism, but "large scale"
- super-Alfvenic - magnetic fields mildly important
- statistics of turbulence may determine distribution of stellar masses (IMF) (Padoan & Nordlund 2002)

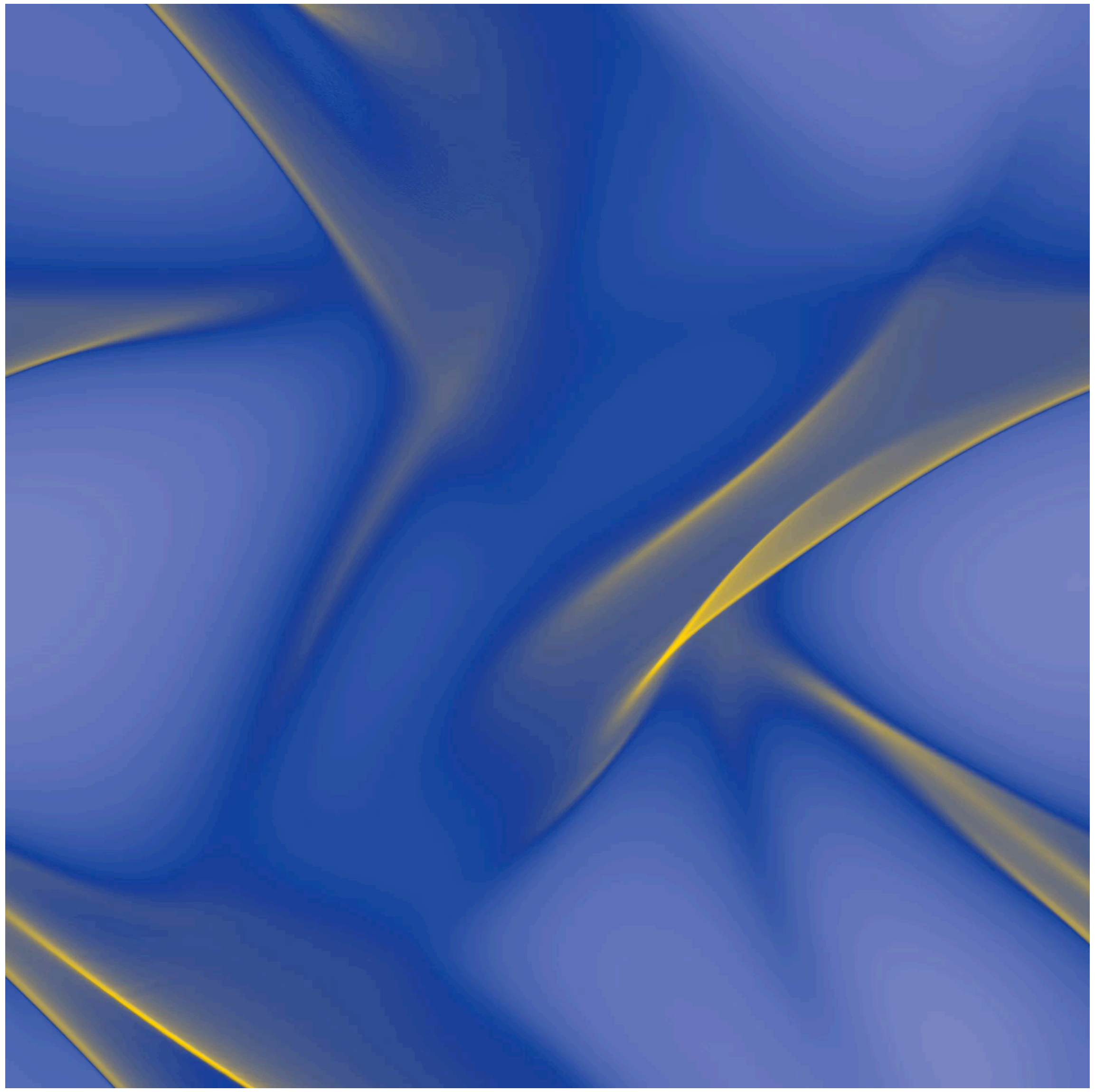




Larson (1981)


Goldsmith et al. (2008)

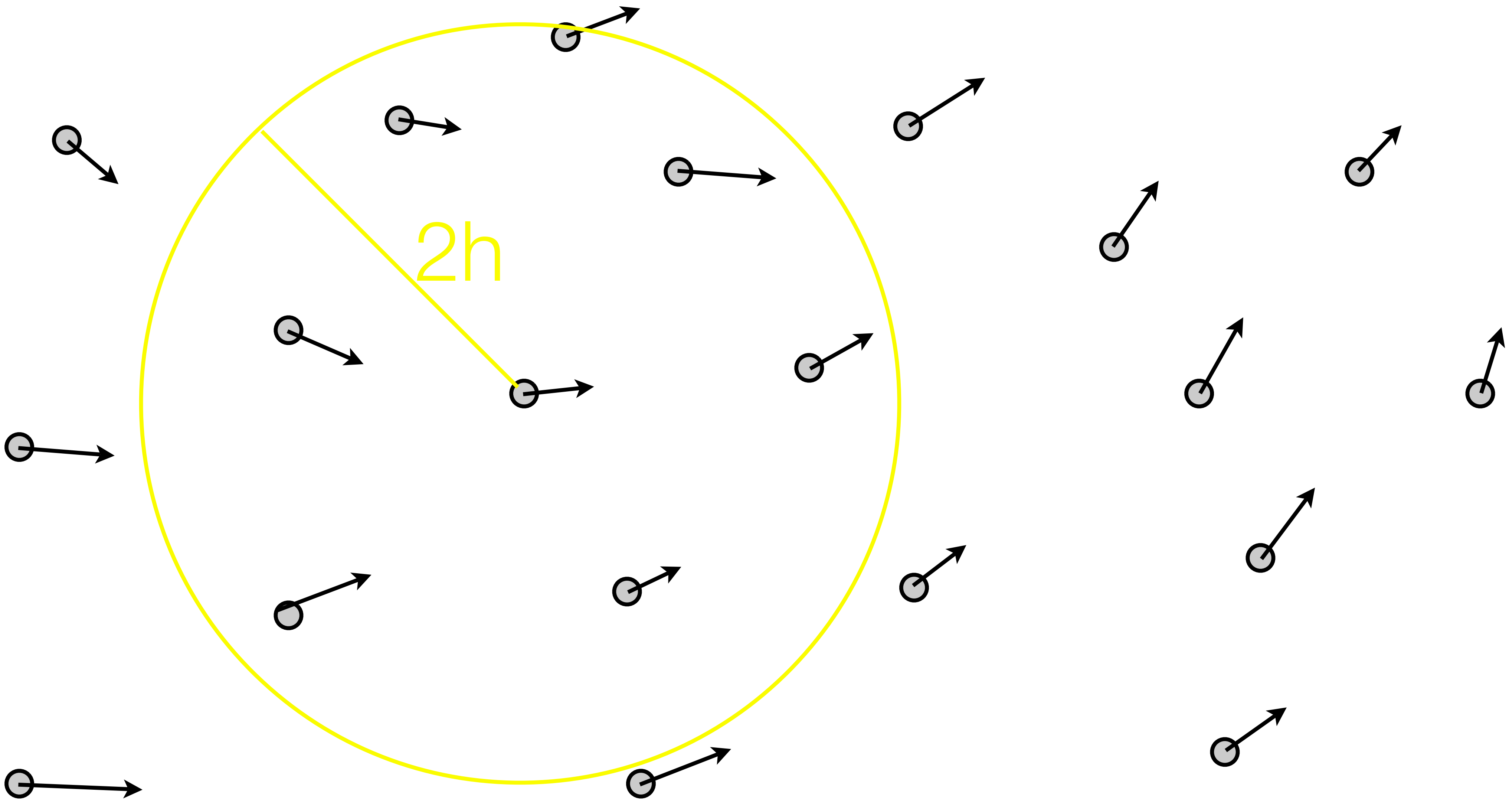
A simple approach is to study isothermal turbulence in periodic box, driven artificially in fourier space at "large scales"



- previous disagreement between SPH and grid codes (Padoan et al., 2007; Ballesteros-Peredes et al., 2006)
- but based on very low resolution SPH simulations (~58^3 particles)

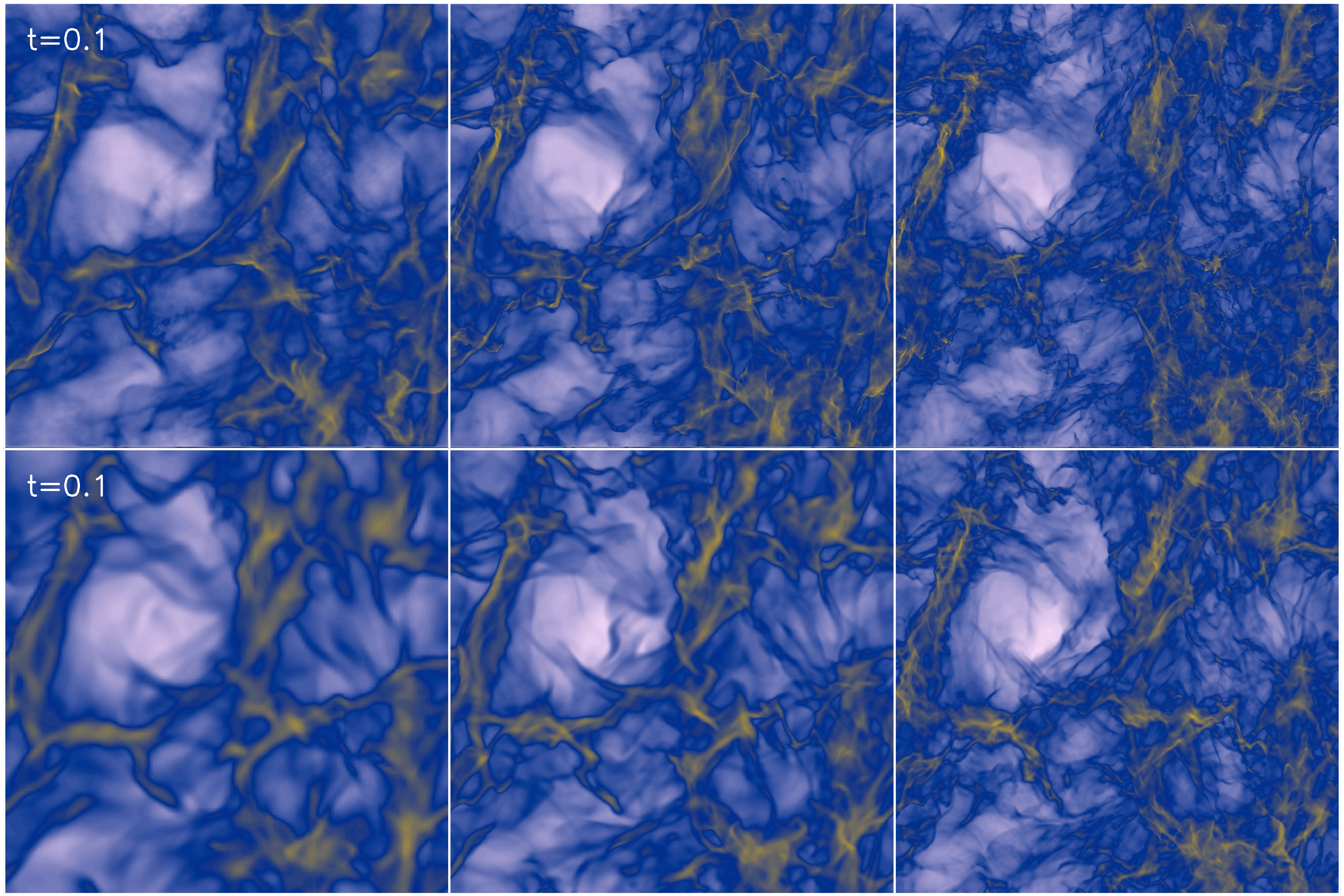
Smoothed Particle Hydrodynamics

Lucy (1977), Gingold & Monaghan (1977), Monaghan (1992), Price (2004), Monaghan (2005)

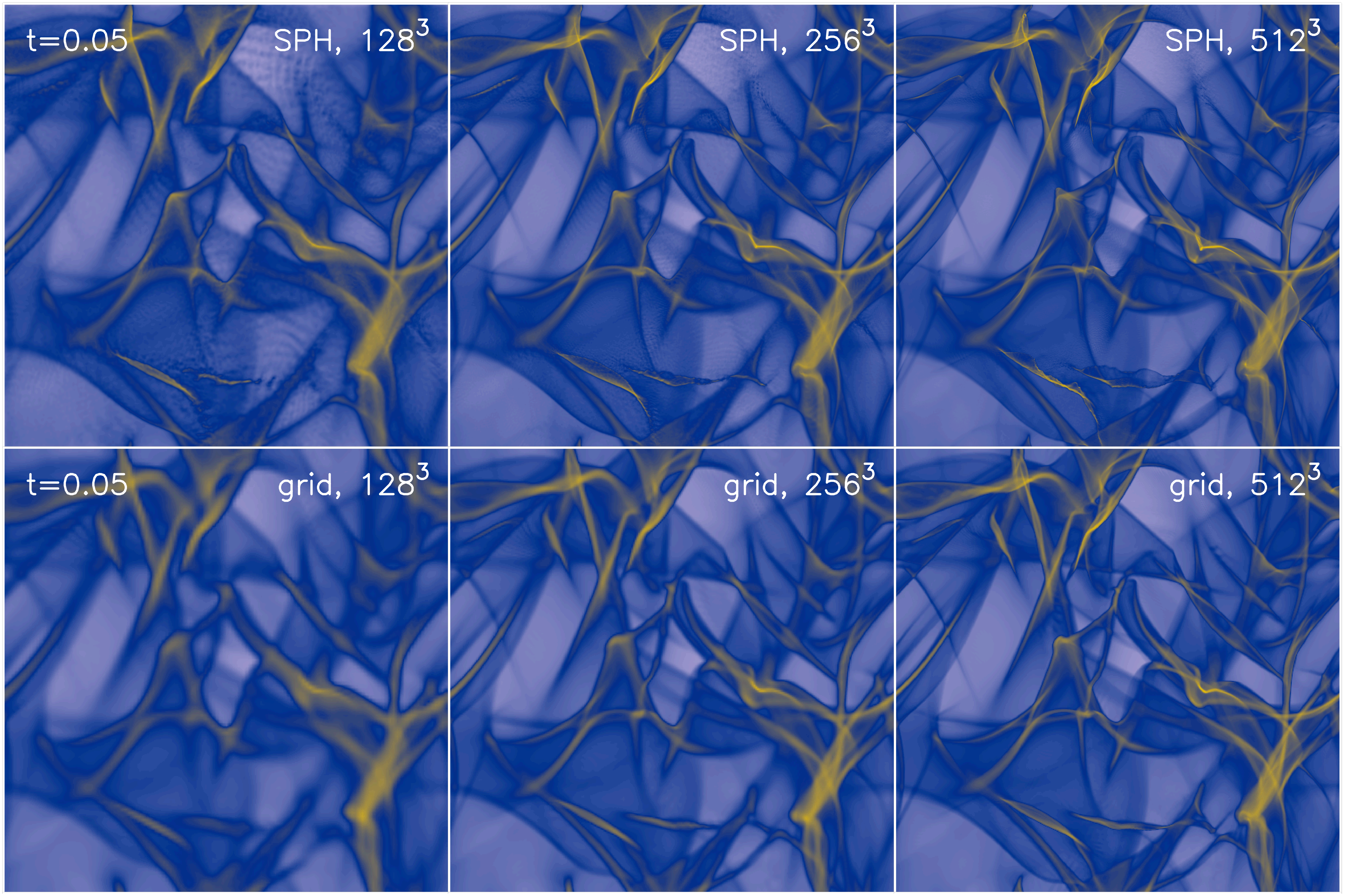


$$\rho(\mathbf{r}) = \sum_{j=1}^{N} m_j W(|\mathbf{r} - \mathbf{r}_j|, h)$$

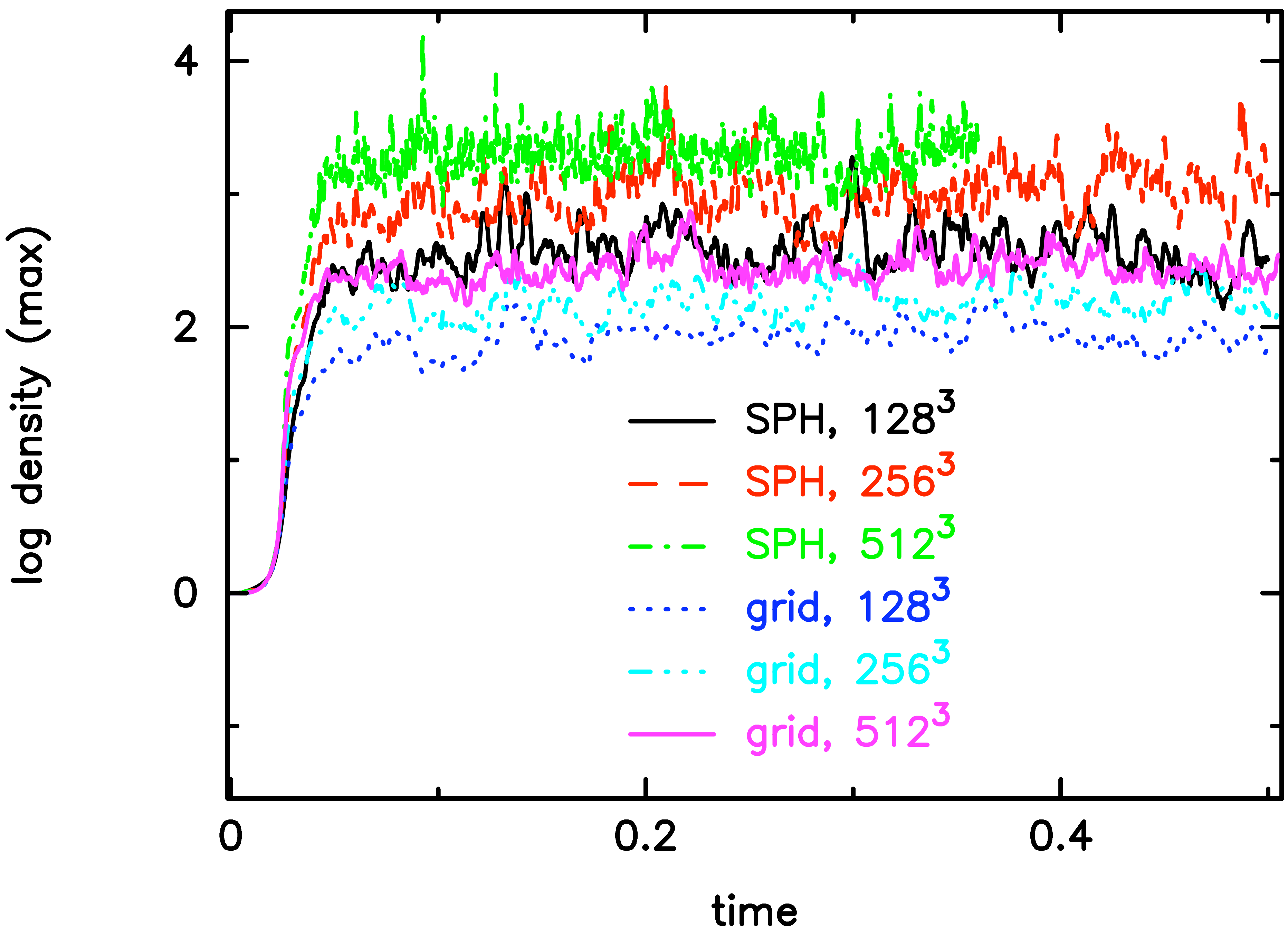
SPH (PHANTOM) vs. Grid (FLASH)



SPH vs. Grid


t=0.05
SPH, 128^3
SPH, 256^3
SPH, 512^3
t=0.05
grid, 128^3
grid, 256^3
grid, 512^3

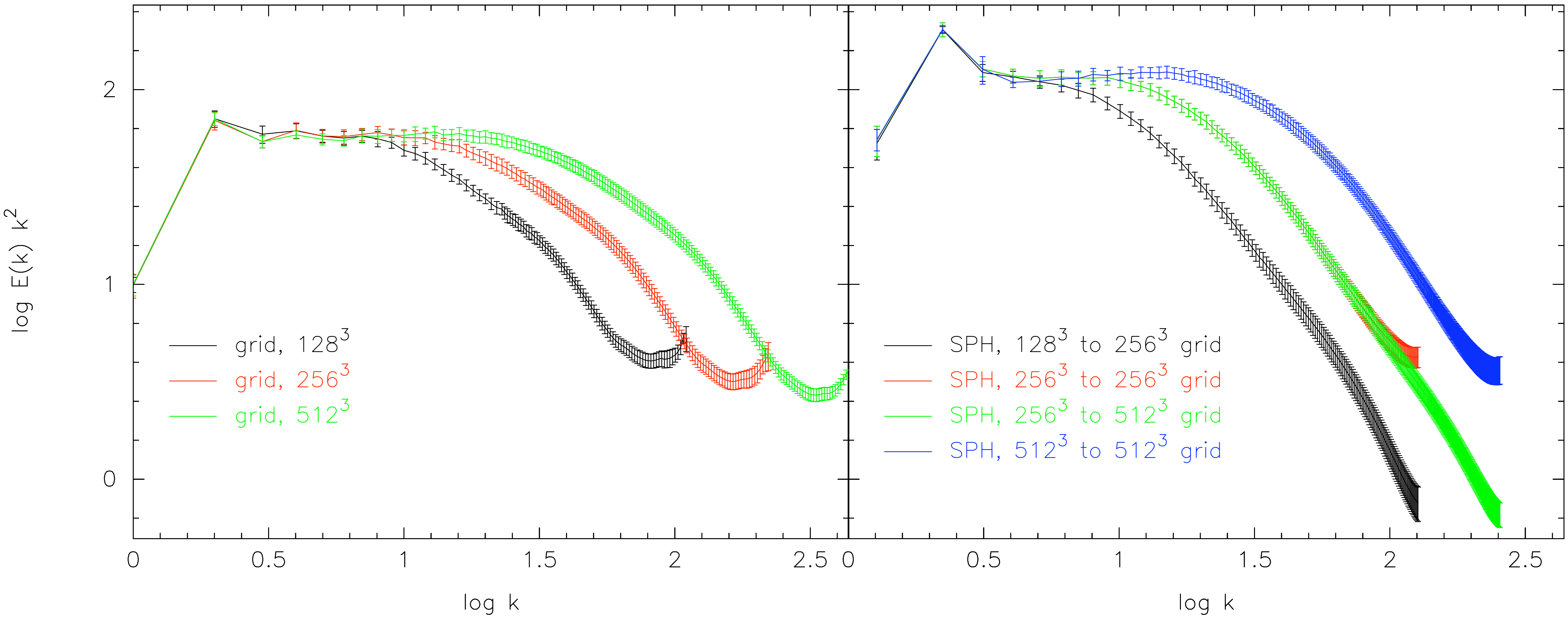
Max density



log density (max)
time
4
2
0
0
0.2
0.4
SPH, 128^3
SPH, 256^3
SPH, 512^3
grid, 128^3
grid, 256^3
grid, 512^3

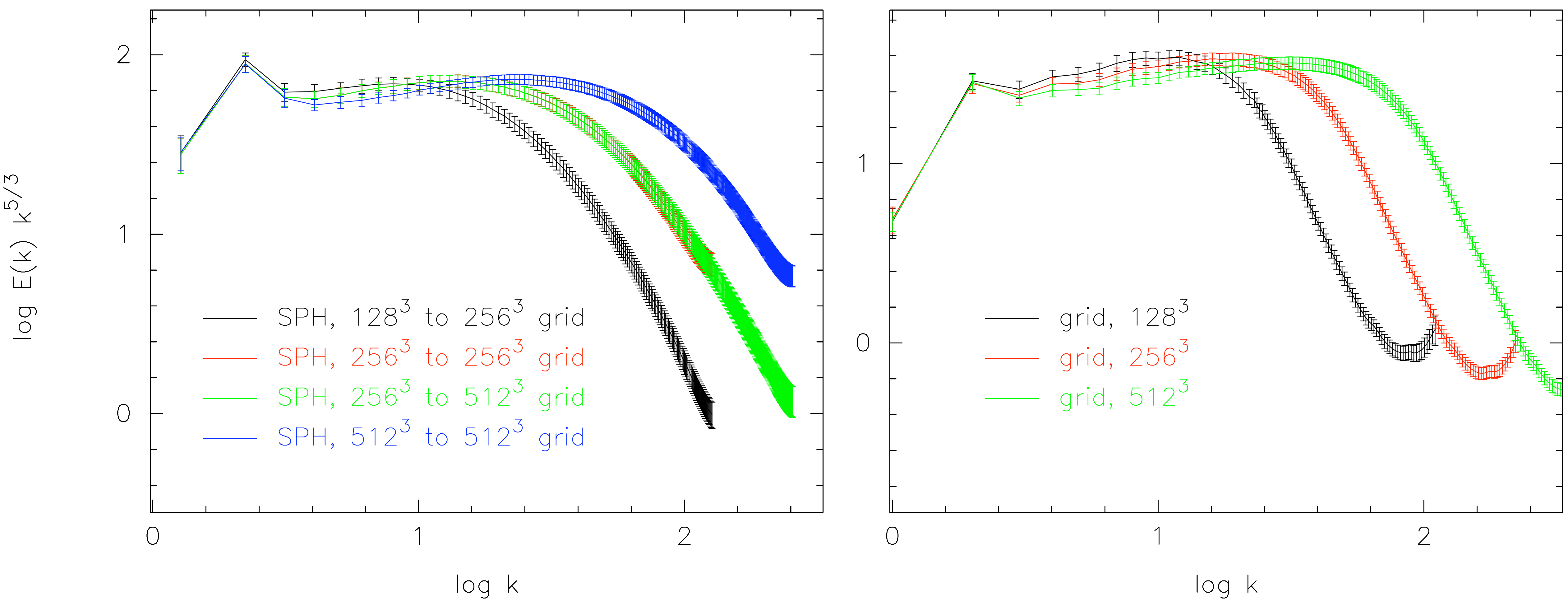
Power spectra

- Kinetic energy goes like k^{-2} -"Burgulence"



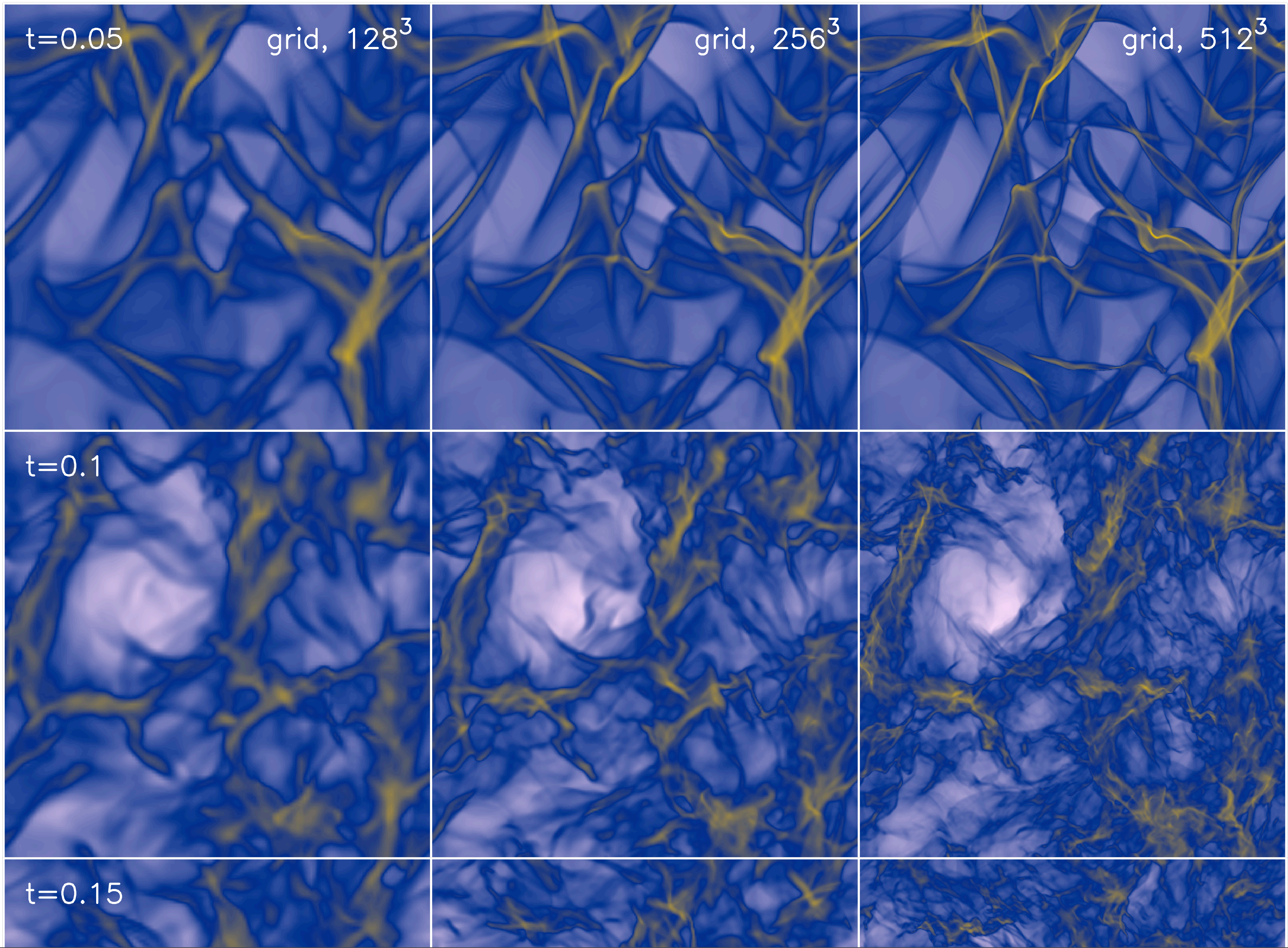
A new universality?

- Kritsuk et al. (2007) suggest $\rho^{1/3}$ v should scale like Kolmogorov ($k^{-5/3}$)

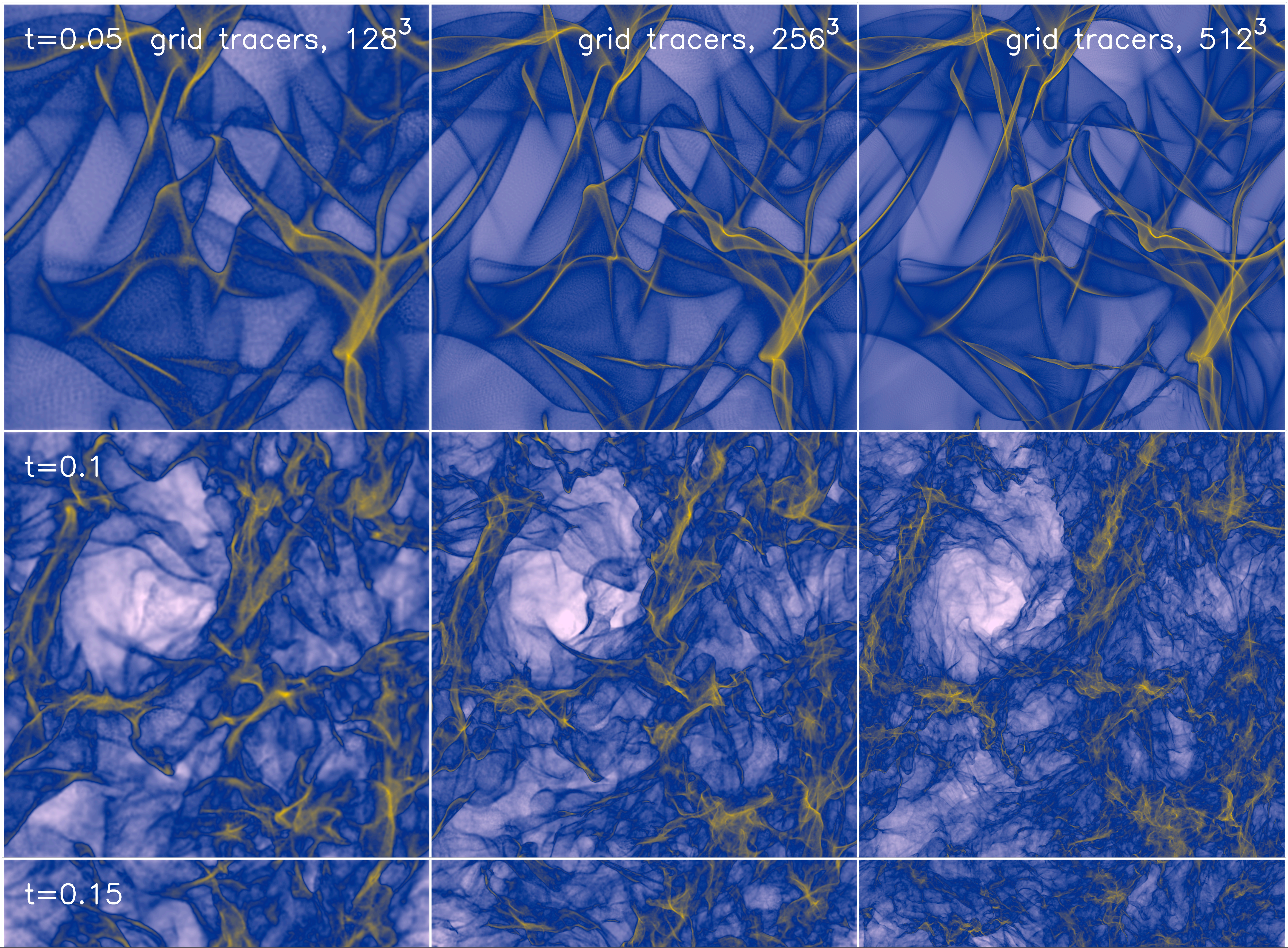


- Some support for this, however not much inertial range even at 512^3

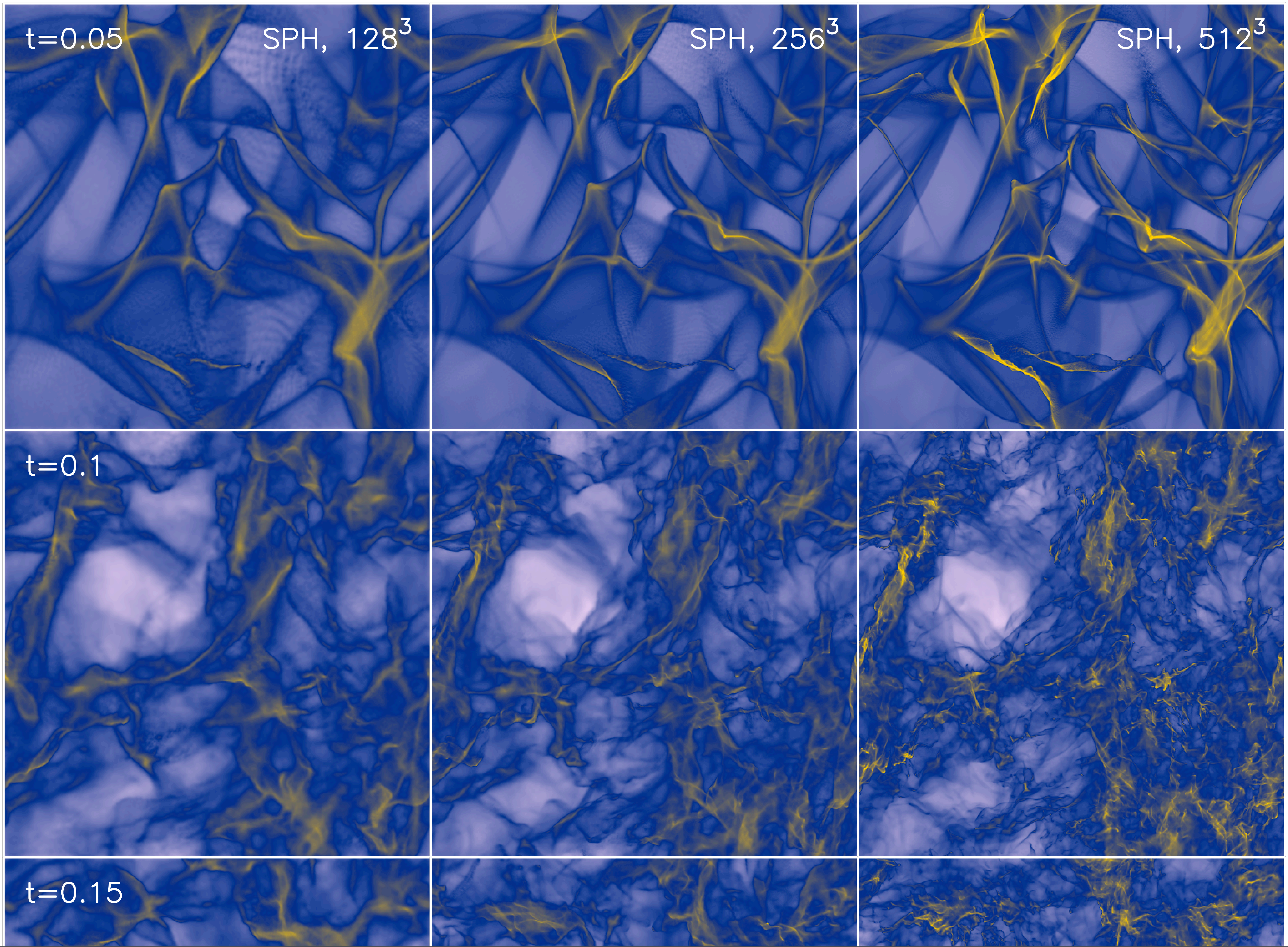
Grid (FLASH)



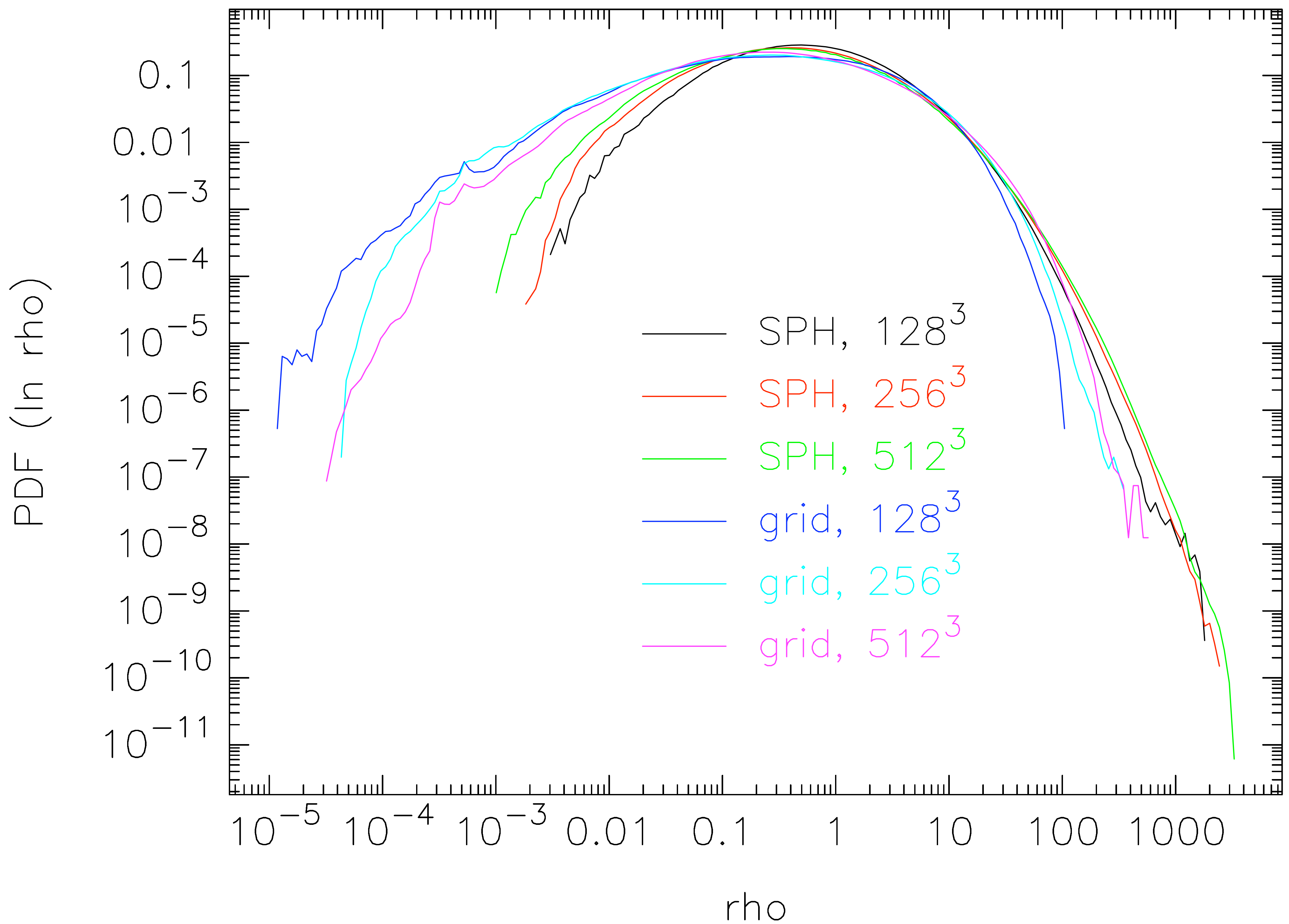
Tracer particles, with SPH density calculation



SPH (PHANTOM)

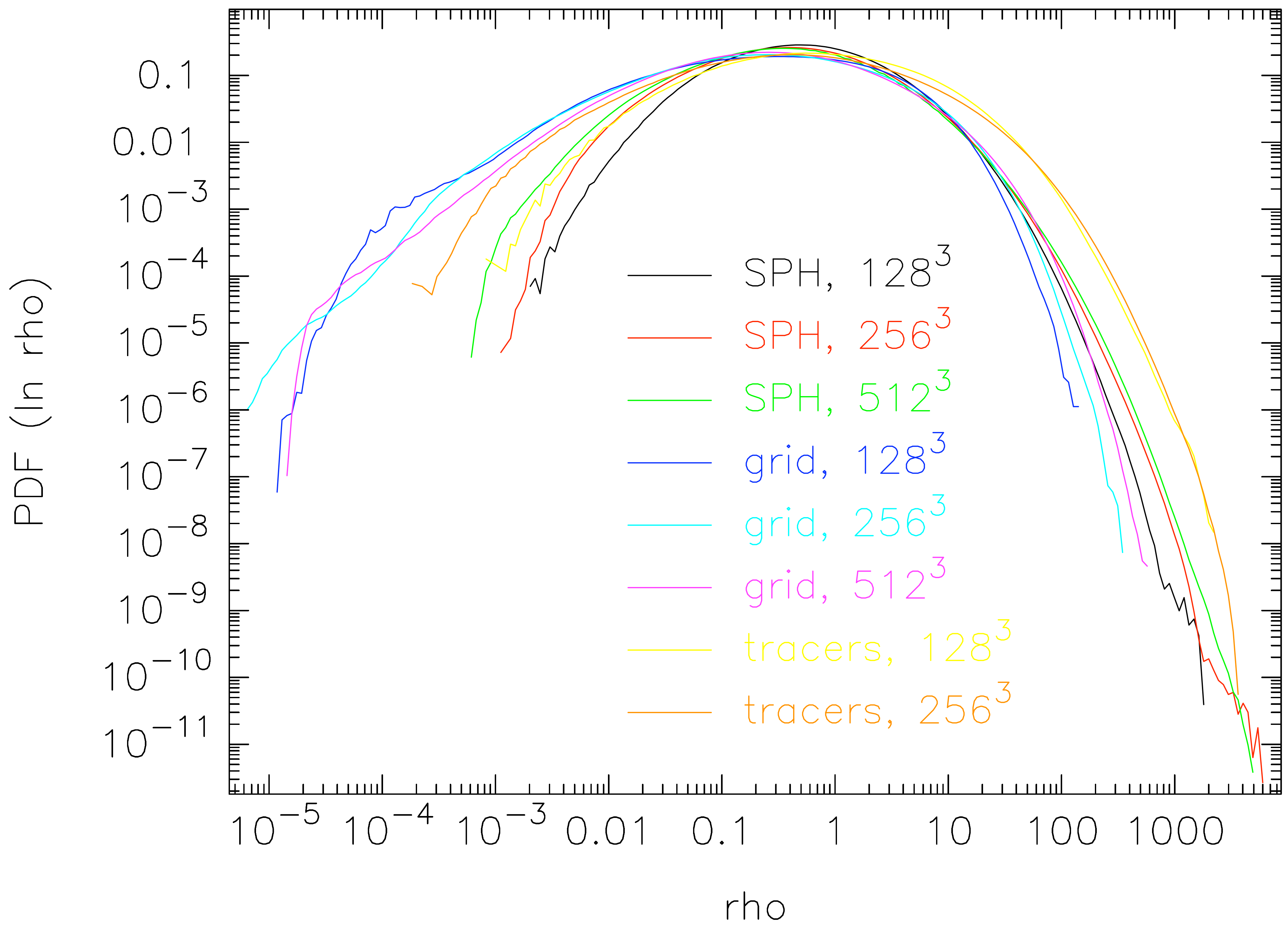


PDFs

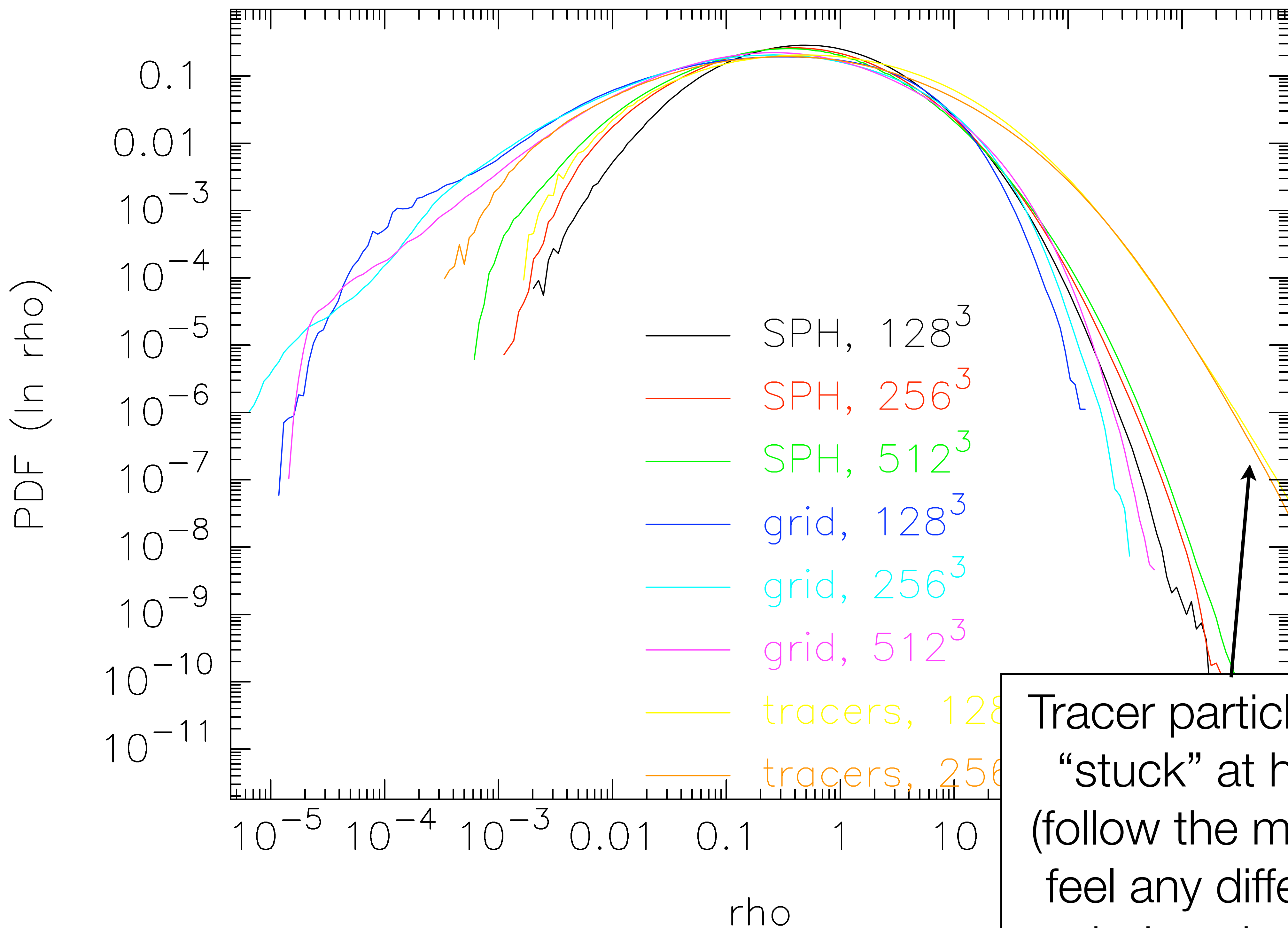


PDF (ln rho)
rho
0.1
0.01
10−3
10−4
10−5
10−6
10−7
10−8
10−9
10−10
10−11
10−5
10−4
10−3
0.01
0.1
1
10
100
1000
SPH, 128³
SPH, 256³
SPH, 512³
grid, 128³
grid, 256³
grid, 512³

PDFs with tracer particles - I



PDFs with tracer particles - iterated density



Tracer particles tend to get “stuck” at high densities (follow the mass, but don’t feel any differential forces below the grid scale)

MHD

Smoothed Particle Magnetohydrodynamics

Four main issues:

- numerical instability related to B(div B) term in conservative MHD force (particles attract unstoppably) (Phillips & Monaghan 1985)

  Morris (1996), Borve et al. (2001), Price & Monaghan (2004a)

- formulation of dissipative terms associated with MHD shocks

  Price & Monaghan (2004a)

- incorporating variable smoothing length self-consistently

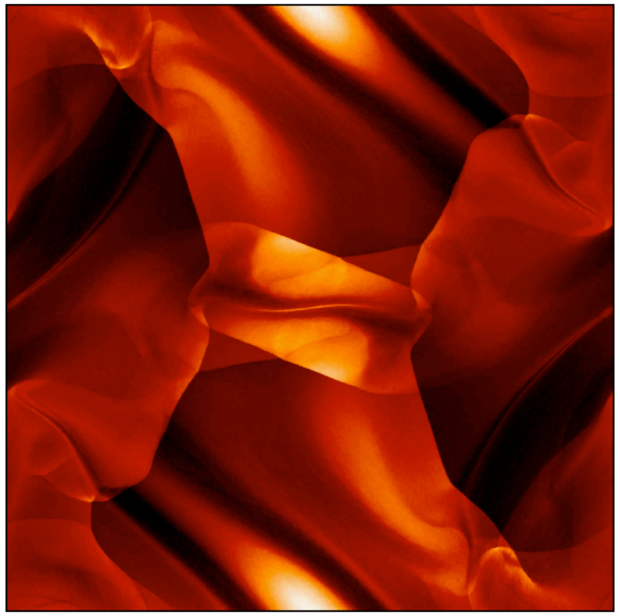
  Price & Monaghan (2004b)

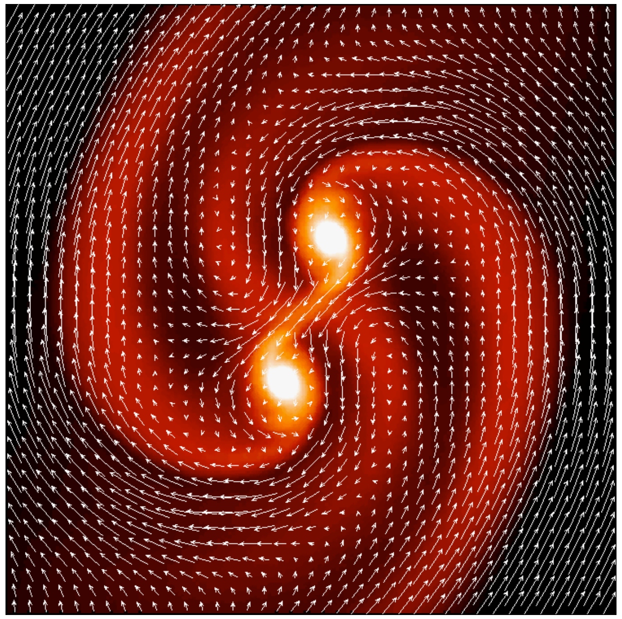
- maintenance of the div B = 0 constraint

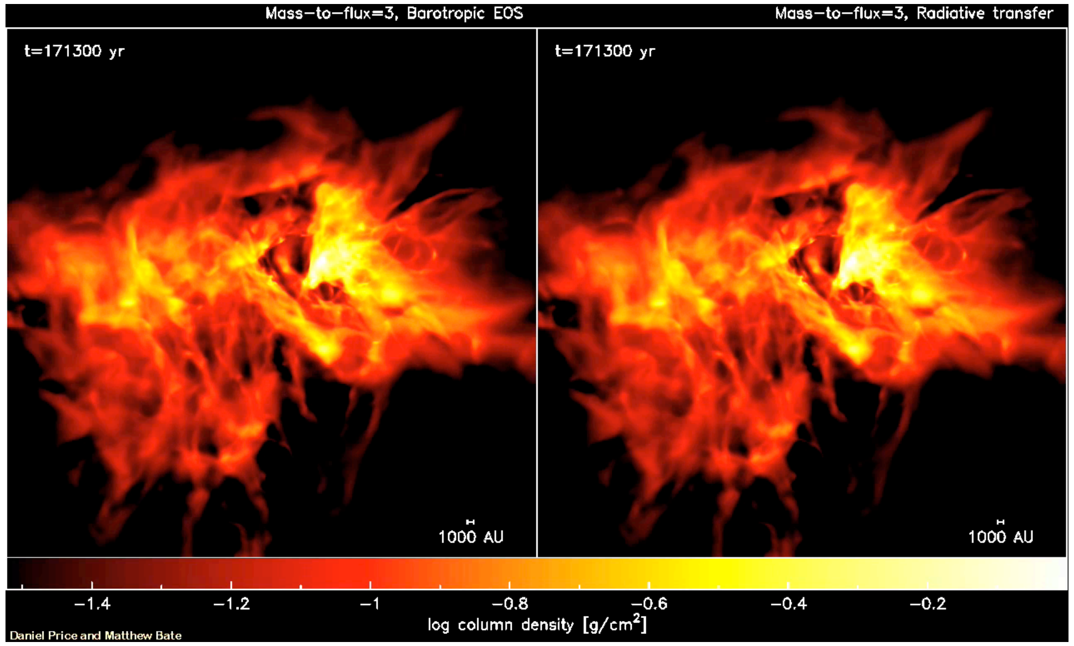
  Price & Monaghan (2005), using divergence cleaning schemes

Euler Potentials / “Clebsch variables”

$$\mathbf{B} = \nabla \alpha \times \nabla \beta$$



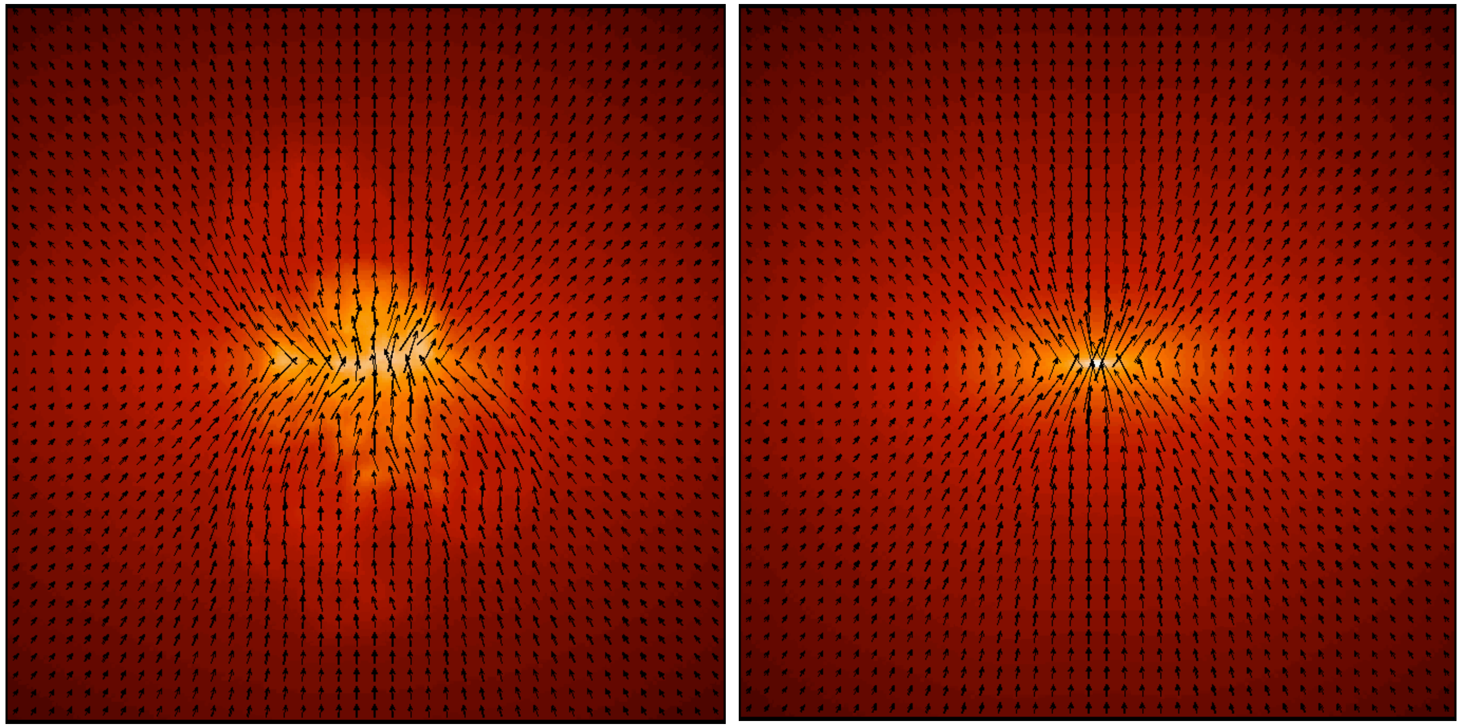




Advantage

$$\frac{d\alpha}{dt} = 0; \;\; \frac{d\beta}{dt} = 0$$

Induction equation



Disadvantage

$$\frac{d\alpha}{dt} = 0; \;\; \frac{d\beta}{dt} = 0$$

- mapping from initial->final particle distribution
- field cannot wind more than once around
- difficult to incorporate non-ideal MHD terms

The Vector Potential $\mathbf{B} = \nabla \times \mathbf{A}$

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \mathbf{B} - \eta \mathbf{J} + \nabla \phi,$$

$$\frac{d\mathbf{A}}{dt} = \mathbf{v} \times \nabla \times \mathbf{A} + (\mathbf{v} \cdot \nabla)\mathbf{A} + \mathbf{v} \times \mathbf{B}_{ext} - \eta \mathbf{J} + \nabla \phi.$$

Use Gauge that gives Galilean invariance: $\phi = \mathbf{v} \cdot \mathbf{A}$

$$\frac{d\mathbf{A}}{dt} = -\mathbf{A} \times (\nabla \times \mathbf{v}) - (\mathbf{A} \cdot \nabla)\mathbf{v} + \mathbf{v} \times \mathbf{B}_{ext} - \eta \mathbf{J}.$$

Also correct low speed ($v << c$) and magnetically dominated ($E < cB$) limit for electromagnetism (de Montigny & Rousseaux 2007, Am. J. Phys 75, 984)

SPMHD with a vector potential

$$L_{sph} = \sum_b m_b \left[\frac{1}{2}\mathbf{v}_b^2 - u_b(\rho_b, s_b) - \frac{1}{2\mu_0}\frac{B_b^2}{\rho_b} \right].$$

take $\delta L = 0$,

Choose:

$$\mathbf{B}_a = (\nabla \times \mathbf{A})_a + \mathbf{B}_{ext} = \frac{1}{\Omega_a \rho_a}\sum_b m_b (\mathbf{A}_a - \mathbf{A}_b) \times \nabla_a W_{ab}(h_a) + \mathbf{B}_{ext},$$

$$\frac{dA_i^a}{dt} = \frac{A_j^a}{\Omega_a \rho_a}\sum_b m_b (v_a^j - v_b^j)\frac{\partial W_{ab}(h_a)}{\partial x_a^i} + \epsilon_{ijk} v_a^j B_{ext,a}^k,$$

Perturbations upon perturbations...

$$\begin{aligned}\delta(\rho_b \mathbf{B}_b) &= \frac{1}{\Omega_b}\sum_c m_c(\mathbf{A}_b - \mathbf{A}_c) \times [(\delta\mathbf{x}_b - \delta\mathbf{x}_c)\cdot\nabla]\nabla_b W_{bc}(h_b) \\ &+ \frac{1}{\Omega_b}\sum_c m_c\left(\delta\mathbf{A}_b - \delta\mathbf{A}_c\right) \times \nabla_b W_{bc}(h_b) + \mathbf{B}_{ext}\delta\rho_b \\ &+ \left[\mathbf{H}_b + \frac{\mathbf{B}_{b,int}}{\Omega_b}\zeta_b\right]\delta\rho_b + \frac{\mathbf{B}_{b,int}\rho_b}{\Omega_b}\frac{\partial h_b}{\partial \rho_b}\sum_c m_c\left[(\delta\mathbf{x}_b - \delta\mathbf{x}_c)\cdot\nabla_b\right]\frac{\partial W_{bc}(h_b)}{\partial h_b},\end{aligned}$$

$$\delta A_k^b = \frac{A_m^b}{\Omega_b \rho_b}\sum_d m_d(\delta x_b^m - \delta x_d^m)\frac{\partial W_{bd}(h_b)}{\partial x_b^k} + \epsilon_{kmn}\delta x_b^m B^n_{ext,b}.$$

several months of your life later...

$$
\begin{aligned}
\int \Bigg\{ -m_a \frac{dv_a^i}{dt} \quad & - \quad \sum_b \frac{m_b}{\Omega_b} \left[\frac{P_b}{\rho_b^2} - \frac{3}{2\mu_0} \left(\frac{B_b}{\rho_b} \right)^2 + \frac{\xi_b}{\rho_b^2} \right] \sum_c m_c \frac{\partial W_{bc}(h_b)}{\partial x_b^i} (\delta_{ba} - \delta_{ca}) \\
& - \quad \frac{\epsilon_{jkl}}{\mu_0} \sum_b \frac{m_b}{\Omega_b} \frac{B_b^j}{\rho_b^2} \sum_c m_c (A_k^b - A_k^c) \frac{\partial^2 W_{bc}(h_b)}{\partial x_b^i \partial x_b^l} (\delta_{ba} - \delta_{ca}) \\
& - \quad \frac{1}{\mu_0} \sum_b \frac{m_b}{\Omega_b} \frac{B_b^j B_{int,b}^j}{\rho_b} \frac{\partial h_b}{\partial \rho_b} \sum_c m_c (\delta_{ba} - \delta_{ca}) \frac{\partial^2 W_{bc}(h_b)}{\partial x_b^i \partial h_b} \\
& - \quad \frac{1}{\mu_0} \sum_b \frac{m_b}{\Omega_b} \frac{B_b^j}{\rho_b^2} \left[2\delta_i^l B_{ext}^j - \delta_i^j B_{ext}^l \right] \sum_c m_c \frac{\partial W_{bc}(h_b)}{\partial x_b^l} (\delta_{ba} - \delta_{ca}) \\
& - \quad \frac{\epsilon_{jkl}}{\mu_0} \sum_b \frac{m_b}{\Omega_b} \frac{B_b^j}{\rho_b^2} \sum_c m_c \frac{A_i^b}{\Omega_b \rho_b} \left[\sum_d m_d \frac{\partial W_{bd}(h_b)}{\partial x_b^k} (\delta_{ba} - \delta_{da}) \right] \frac{\partial W_{bc}(h_b)}{\partial x_b^l} \\
& + \quad \frac{\epsilon_{jkl}}{\mu_0} \sum_b \frac{m_b}{\Omega_b} \frac{B_b^j}{\rho_b^2} \sum_c m_c \frac{A_i^c}{\Omega_c \rho_c} \left[\sum_d m_d \frac{\partial W_{cd}(h_c)}{\partial x_c^k} (\delta_{ca} - \delta_{da}) \right] \frac{\partial W_{bc}(h_b)}{\partial x_b^l} \Bigg\} \delta x_a^i \mathrm{dt} = 0,
\end{aligned}
$$

Equations of motion

$$
\begin{aligned}
\frac{dv_a^i}{dt} &= -\sum_b m_b \left[\frac{P_a - \frac{3}{2\mu_0}B_a^2 + \xi_a}{\rho_a^2 \Omega_a} \frac{\partial W_{ab}(h_a)}{\partial x_a^i} + \frac{P_b - \frac{3}{2\mu_0}B_b^2 + \xi_b}{\rho_b^2 \Omega_b} \frac{\partial W_{ab}(h_b)}{\partial x_a^i} \right] && \Big\} \text{ isotropic term} \\
&- \frac{\epsilon_{jkl}}{\mu_0} \sum_b m_b (A_k^a - A_k^b) \left[\frac{B_a^j}{\Omega_a \rho_a^2} \frac{\partial^2 W_{ab}(h_a)}{\partial x_a^i \partial x_a^l} + \frac{B_b^j}{\Omega_b \rho_b^2} \frac{\partial^2 W_{ab}(h_b)}{\partial x_a^i \partial x_a^l} \right] && \Big\} \text{ 2D term} \\
&- \frac{1}{\mu_0} \sum_b m_b \left[\frac{B_a^j B_{int,a}^j}{\Omega_a \rho_a} \frac{\partial h_a}{\partial \rho_a} \frac{\partial^2 W_{ab}(h_a)}{\partial x_a^i \partial h_a} + \frac{B_b^j B_{int,b}^j}{\Omega_b \rho_b} \frac{\partial h_b}{\partial \rho_b} \frac{\partial^2 W_{ab}(h_b)}{\partial x_a^i \partial h_b} \right] && \Big\} \text{ 2D } \nabla h \text{ term} \\
&- \frac{1}{\mu_0} \left[2\delta_i^l B_{ext}^j - \delta_i^j B_{ext}^l \right] \sum_b m_b \left[\frac{B_a^j}{\Omega_a \rho_a^2} \frac{\partial W_{ab}(h_a)}{\partial x_a^l} + \frac{B_b^j}{\Omega_b \rho_b^2} \frac{\partial W_{ab}(h_b)}{\partial x_a^l} \right] && \Big\} \text{ 2.5D/}\mathbf{B}_{ext} \text{ term} \\
&- \sum_b m_b \left[\frac{A_i^a}{\Omega_a \rho_a^2} J_a^k \frac{\partial W_{ab}(h_a)}{\partial x_a^k} + \frac{A_i^b}{\Omega_b \rho_b^2} J_b^k \frac{\partial W_{ab}(h_b)}{\partial x_a^k} \right], && \Big\} \text{ 3D term}
\end{aligned}
$$

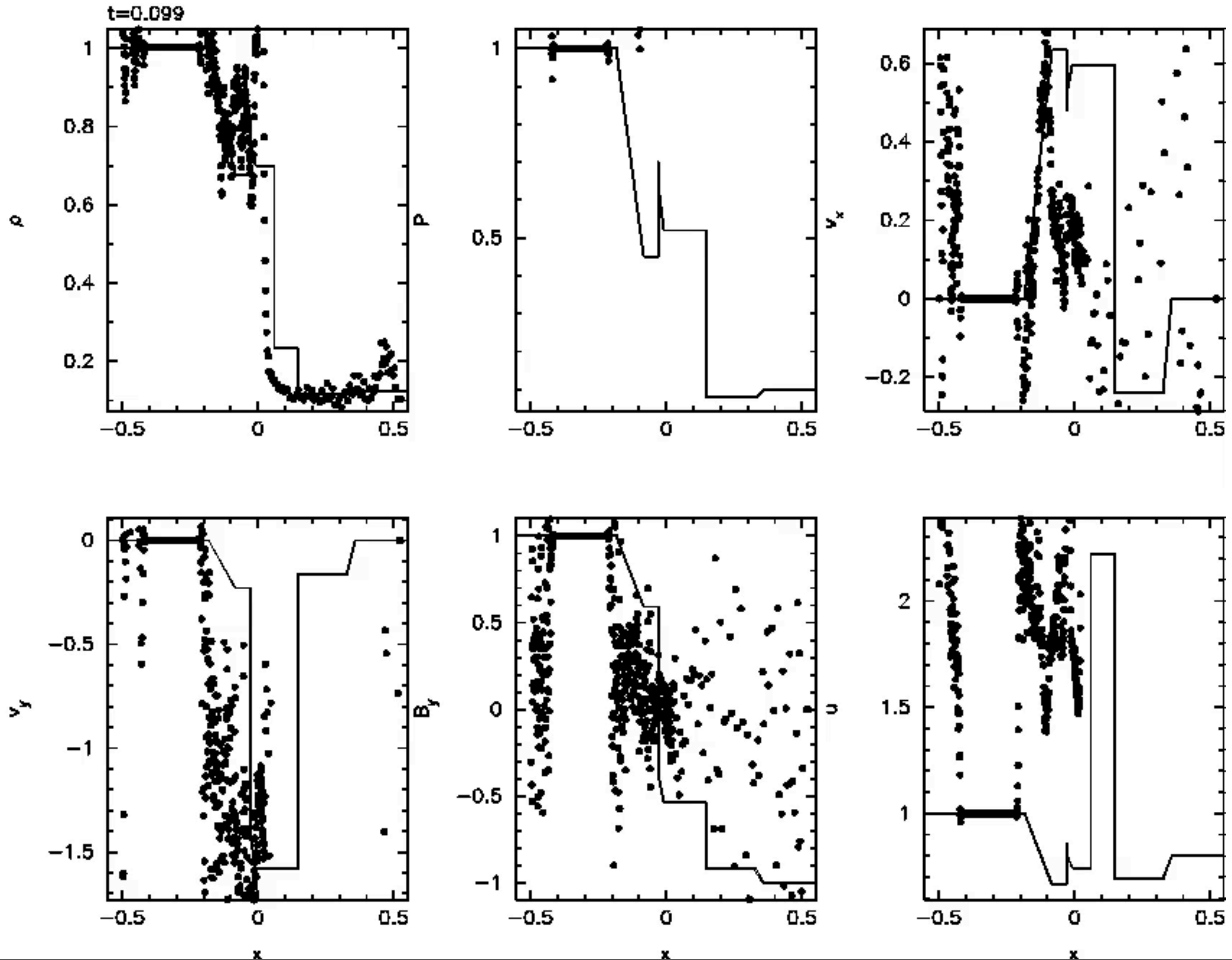
Equations of motion (simplified)

$$\frac{dv_a^i}{dt} = \sum_b m_b \left[\left(\frac{\mathcal{S}_a^{ij}}{\rho_a^2 \Omega_a} + \frac{(A_{ab} \times B_a)^j}{\mu_0 \rho_a^2 \Omega_a} \frac{\partial}{\partial x_a^i} + \psi_a \delta_j^i \frac{\partial}{\partial h_a} \right) \frac{\partial W_{ab}(h_a)}{\partial x_a^j} + \left(\frac{\mathcal{S}_b^{ij}}{\rho_b^2 \Omega_b} + \frac{(A_{ab} \times B_b)^j}{\mu_0 \rho_b^2 \Omega_b} \frac{\partial}{\partial x_a^i} + \psi_b \delta_j^i \frac{\partial}{\partial h_b} \right) \frac{\partial W_{ab}(h_b)}{\partial x_a^j} \right]$$

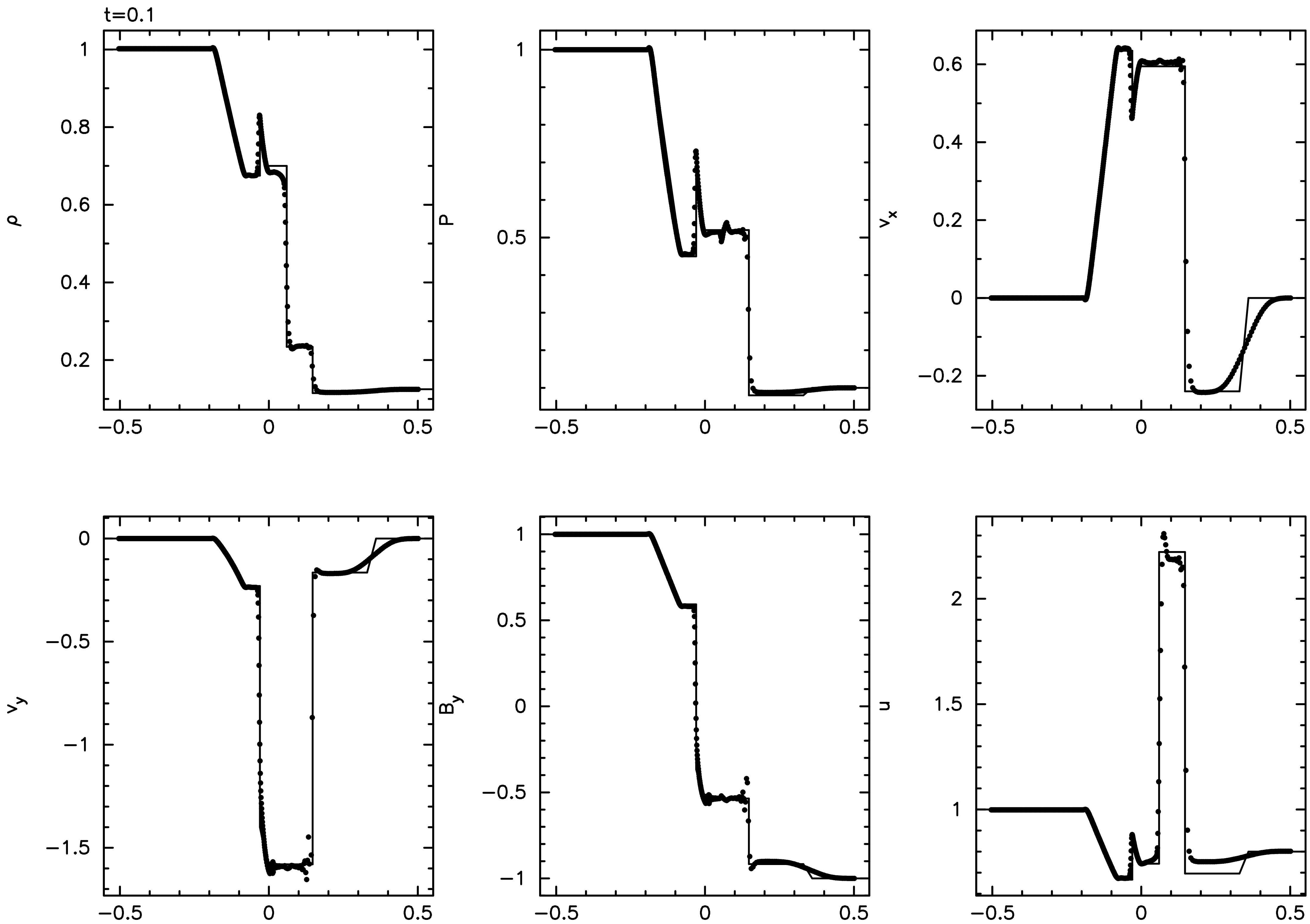
$$\mathcal{S}^{ij} \equiv -P\delta^{ij} + \frac{1}{\mu_0} \left[B^i B^j_{ext} + \delta^{ij} \left(\frac{3}{2} B^2 - 2\mathbf{B} \cdot \mathbf{B}_{ext} - \xi \right) \right] - A^i J^j,$$

- conserves energy, momentum and entropy exactly and simultaneously

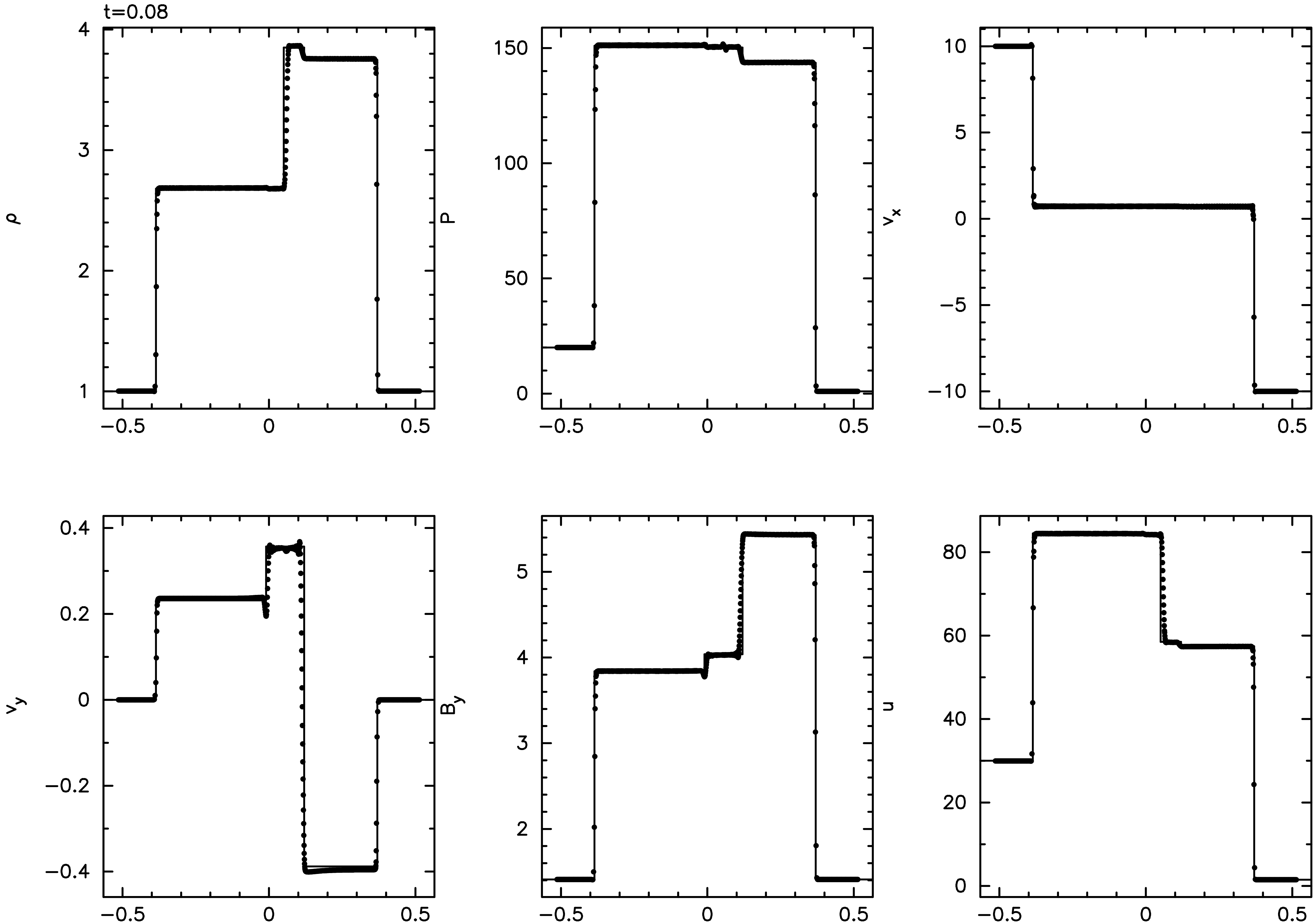
Does it work?



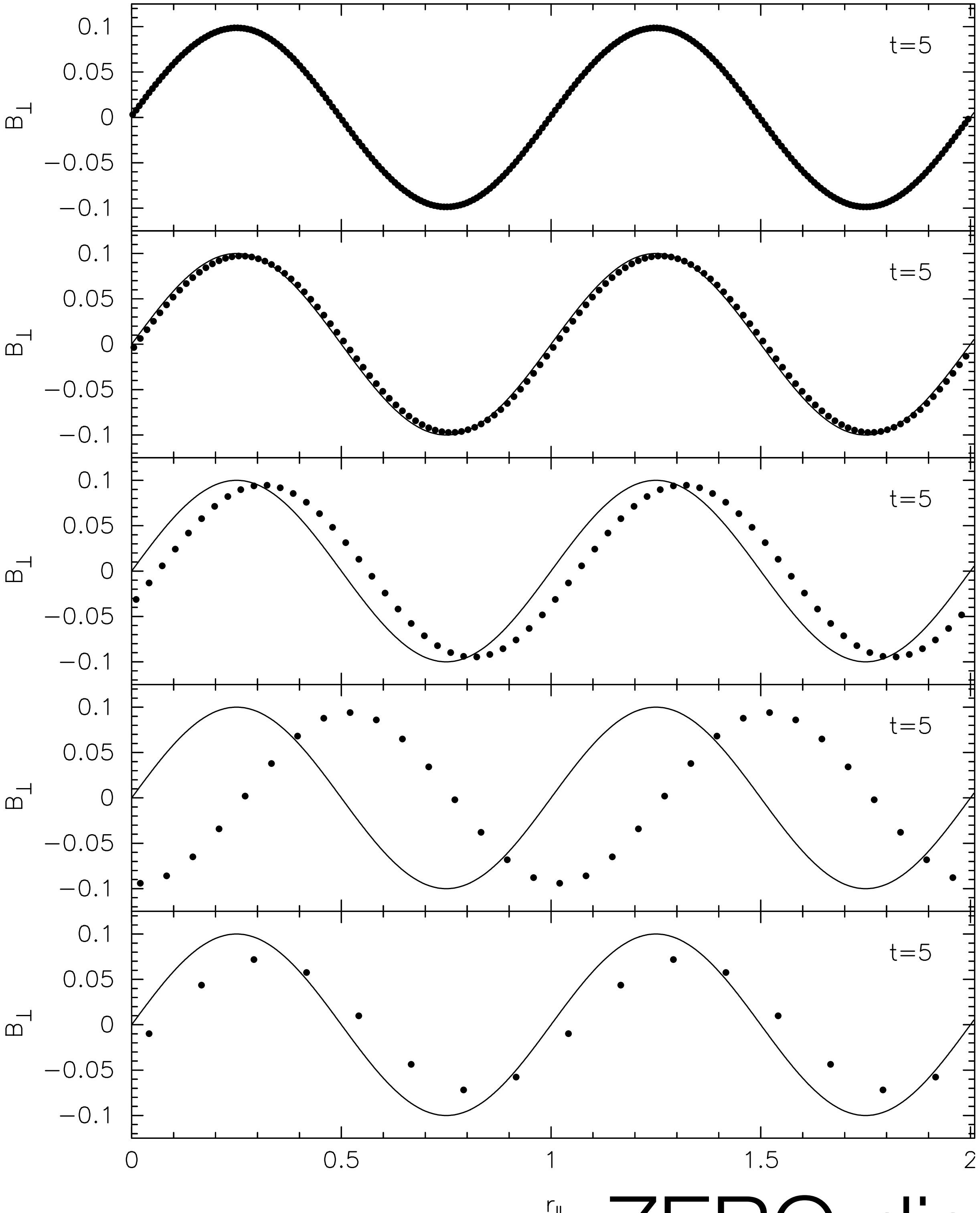
With some hacks...

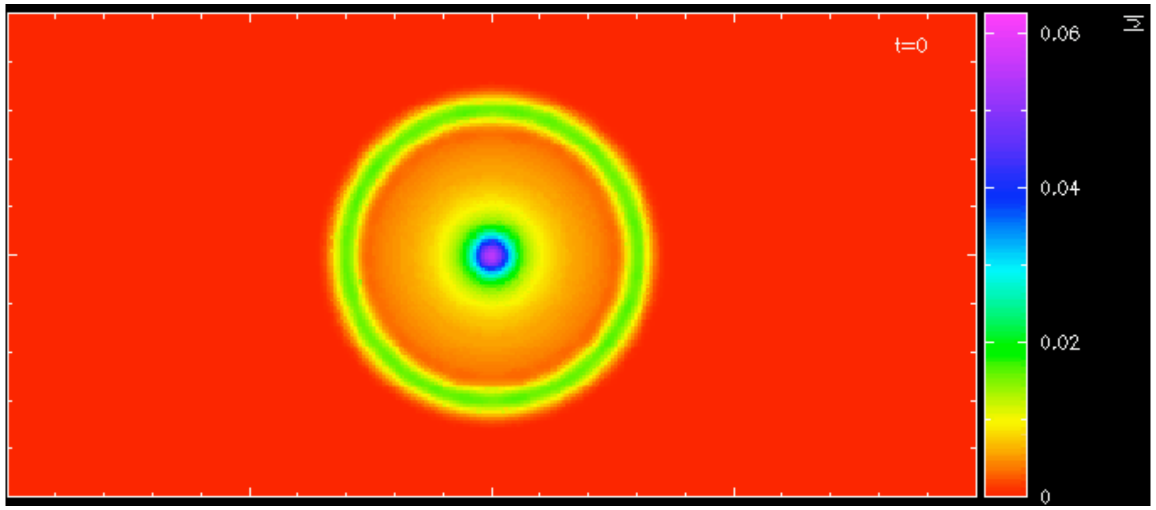


We can do OK

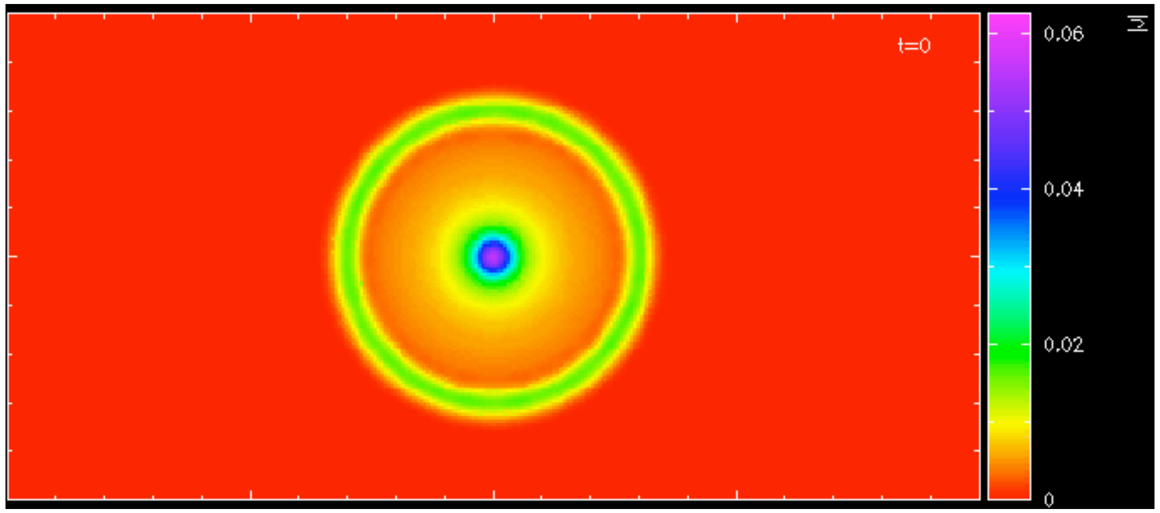


Circularly polarised Alfven wave, field loop advection (2D)





first 25 crossings



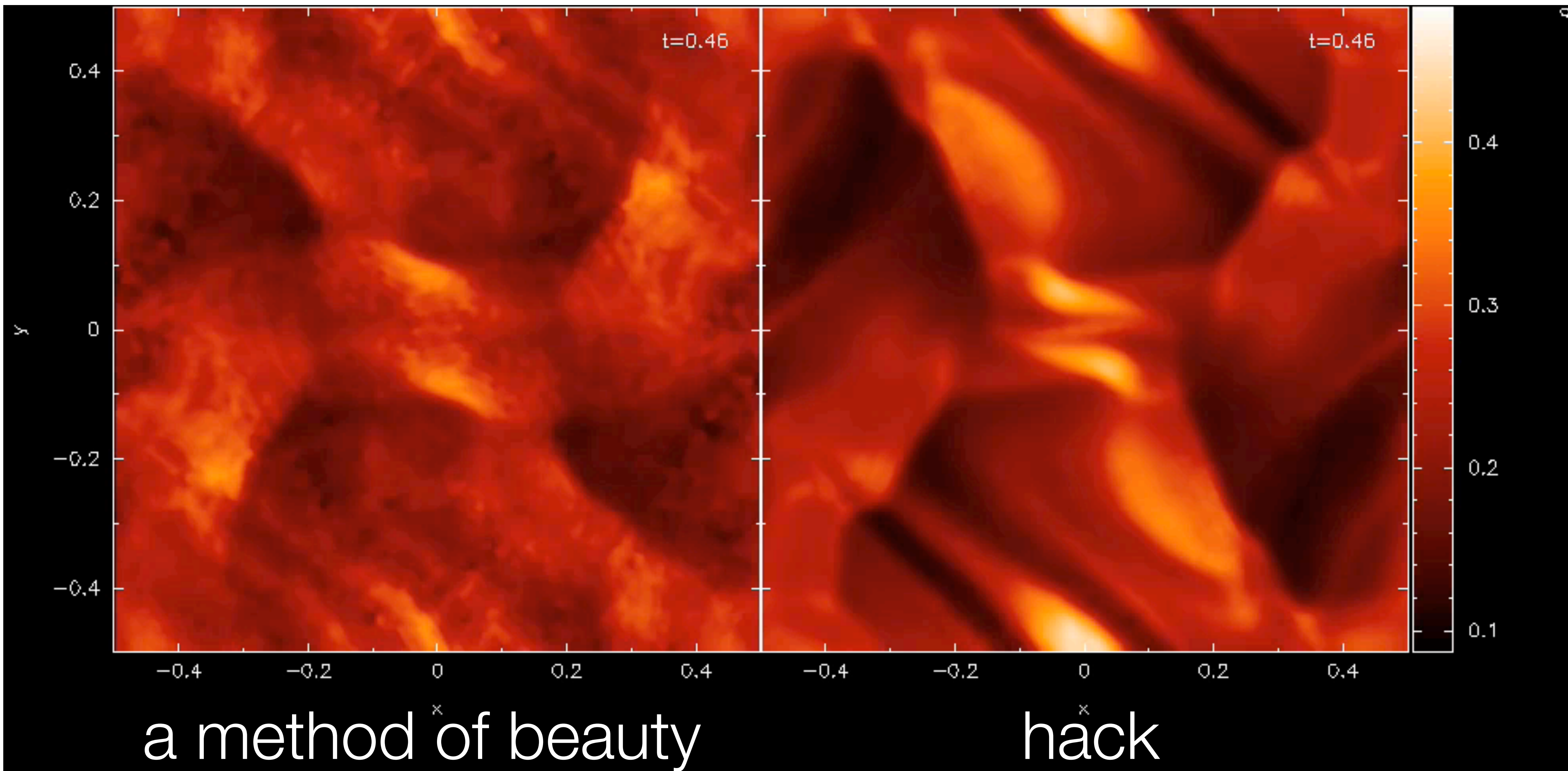
1000 crossings

ZERO dissipation until you add some

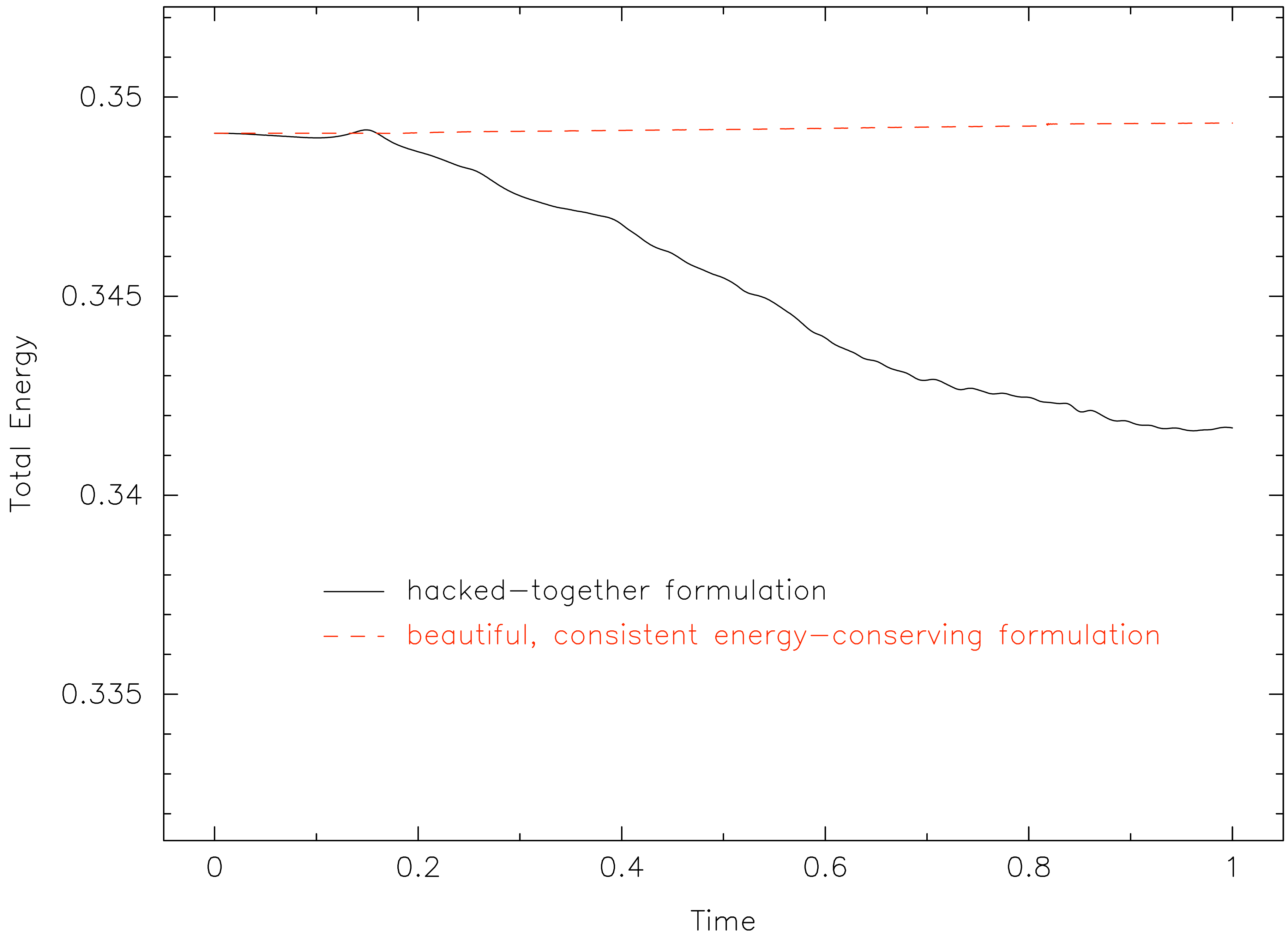
What about all that divergence-free wonder?

- 2D test problem: Orszag-Tang Vortex

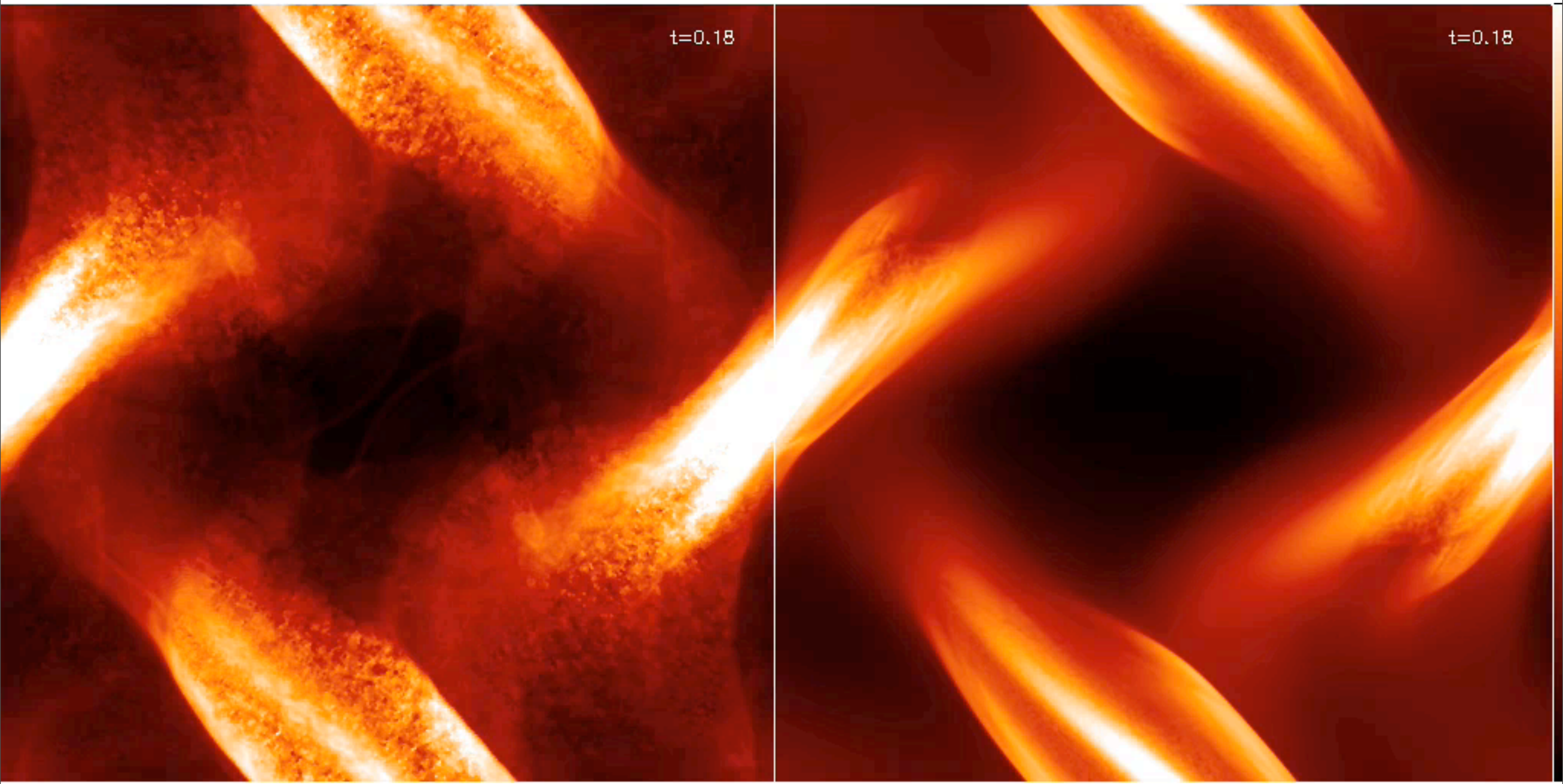
$$[v_x, v_y] = v_0[-\sin(2\pi y), \sin(2\pi x)] \qquad [B_x, B_y] = B_0[-\sin(2\pi y), \sin(4\pi x)]$$



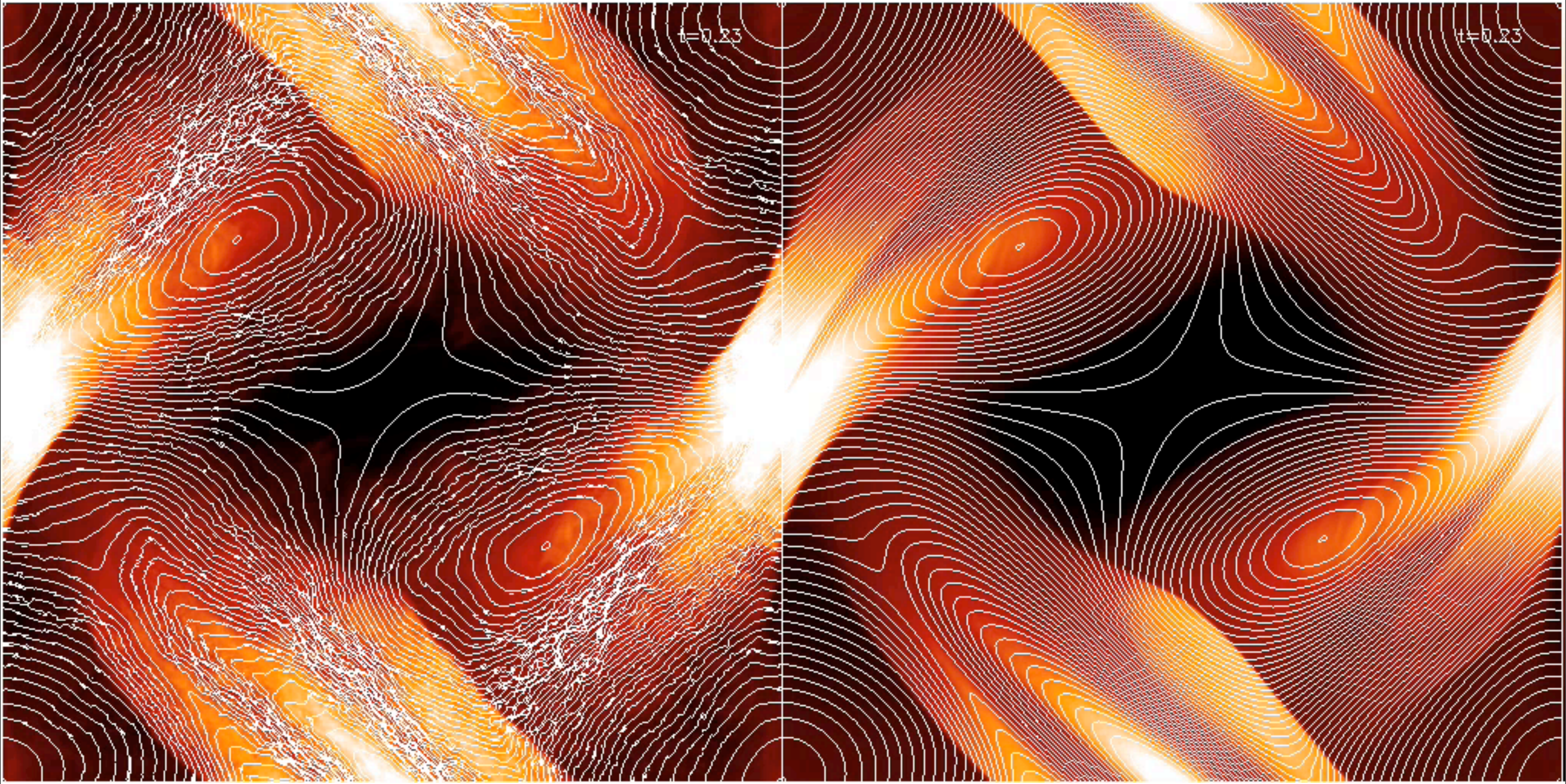
2D Orszag-Tang Vortex: Energy conservation



High(er) resolution version



Field lines



Conclusions

On turbulence:

- We find good agreement between SPH and grid codes on the statistics of supersonic turbulence
- SPH does a good job of simulating highly compressible turbulence by placing resolution in high density regions.
- tracer particles have the possibility of dramatically improving the density resolution in grid-based simulations at little extra cost. A hybrid scheme?

On SPMHD:

- vector potential is not a viable approach for MHD in SPH. Numerical instabilities are MUCH WORSE than in the standard approach.
- better to look at generalised versions of the Euler potentials