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Smoothed Particle Hydrodynamics

A review

or how I learnt to stop worrying and love Lagrangians

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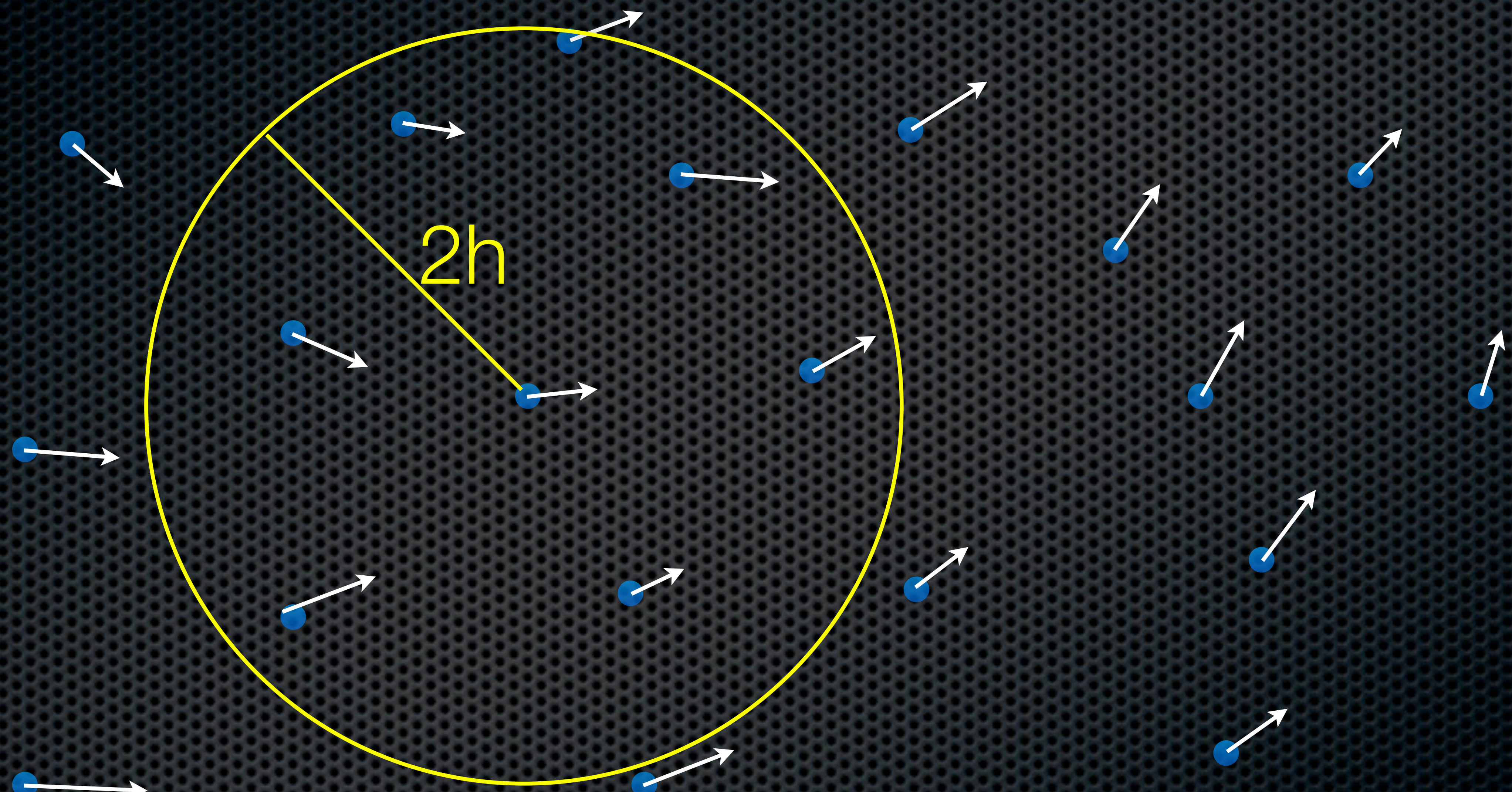
PART I - improvements in the basic physics (hydro, gravity)

Smoothed Particle Hydrodynamics

Lucy (1977), Gingold & Monaghan (1977), Monaghan (1992), Price (2004), Monaghan (2005)



$$\rho(\mathbf{r}) = \sum_{j=1}^{N} m_j W(|\mathbf{r} - \mathbf{r}_j|, h)$$

$$L_{sph} = \sum_j m_j \left[\frac{1}{2} v_j^2 - u_j(\rho_j, s_j) \right]$$ ← Lagrangian

$$+$$

$$du = \frac{P}{\rho^2} d\rho$$ ← 1st law of thermodynamics

$$+$$

$$\nabla \rho_i = \sum_j m_j \nabla W_{ij}(h)$$ ← density sum

$$+$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0$$ ← Euler-Lagrange equations

$$=$$

$$\frac{d\mathbf{v}_i}{dt} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla_i W_{ij}(h)$$ ← equations of motion!

$$\left(\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} \right)$$

SPH gradients 101

$$A_a = \sum_b \frac{m_b}{\rho_b} A_b W_{ab}$$

$$\nabla A_a = \sum_b \frac{m_b}{\rho_b} A_b \nabla W_{ab}$$ BAD

$$\nabla A_a = \sum_b \frac{m_b}{\rho_b} (A_b - A_a) \nabla W_{ab}$$ exact const

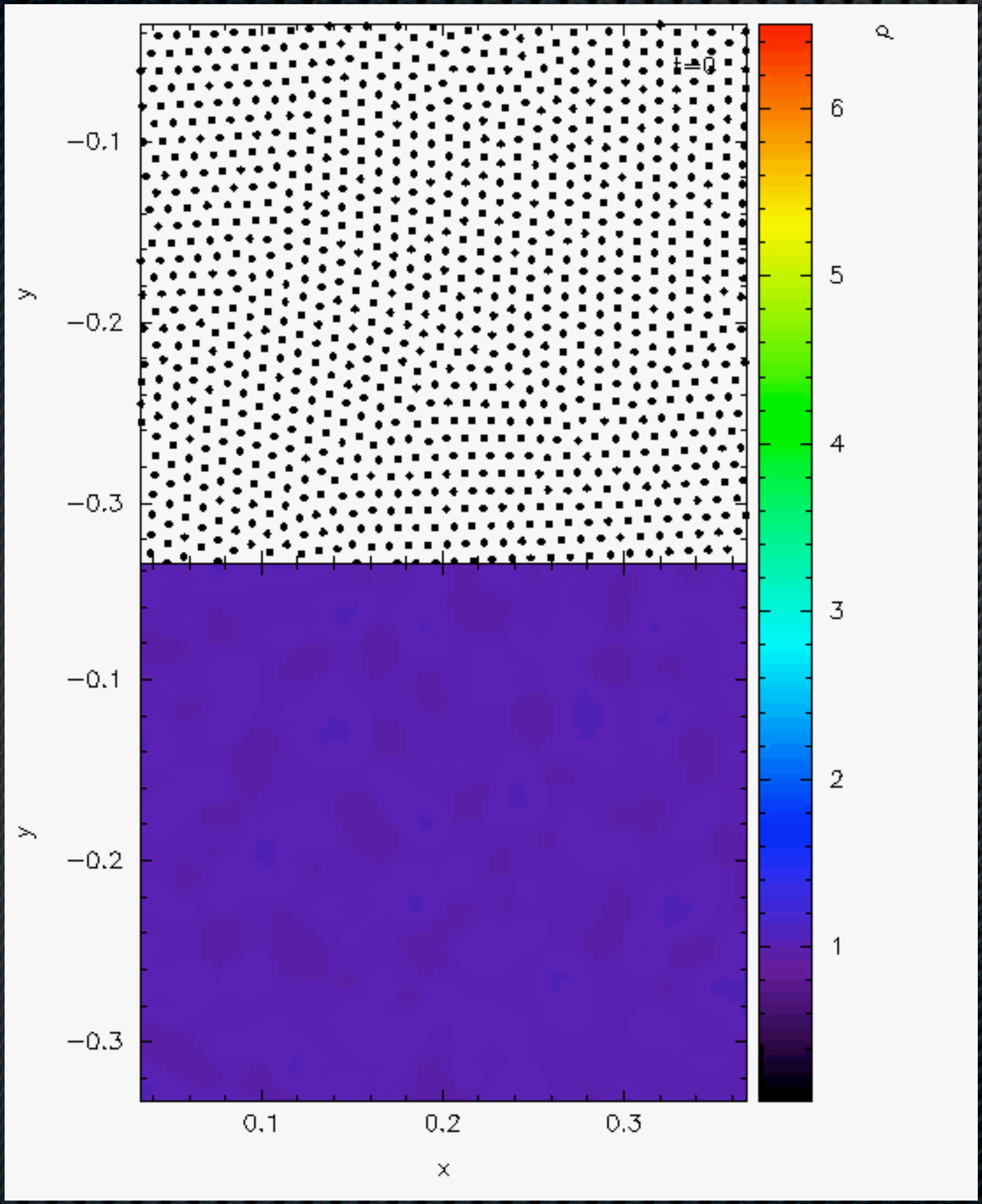
$$\chi_{\mu\nu} \nabla^\mu A_a = \sum_b m_b (A_b - A_a) \nabla_a^\nu W_{ab}$$ exact linear

$$\chi_{\mu\nu} = \sum_b m_b (r_a^\mu - r_b^\mu) \nabla^\nu W_{ab}$$

so what about

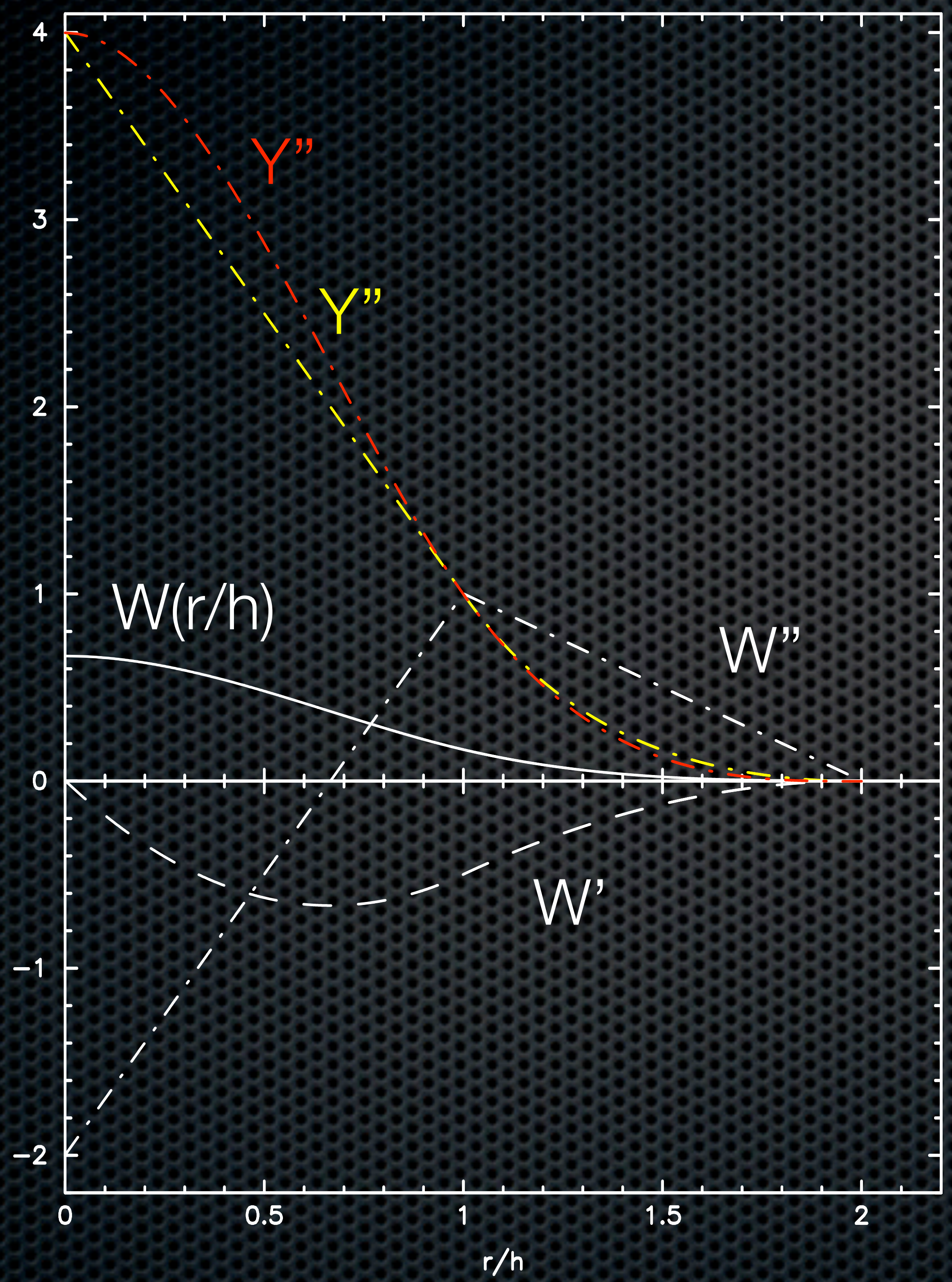
$$\frac{\nabla A_a}{\rho_a} = -\sum_b m_b \left(\frac{A_a}{\rho_a^2} + \frac{A_b}{\rho_b^2} \right) \nabla W_{ab}$$ BAD?

Why SPH works



particles are constrained to remain semi-regular. Do NOT become randomised!

Second derivatives in SPH



naive way:

$$\nabla^2 A_a = \sum_b \frac{m_b}{\rho_b} A_b \nabla^2 W_{ab}$$

Brookshaw (1985):

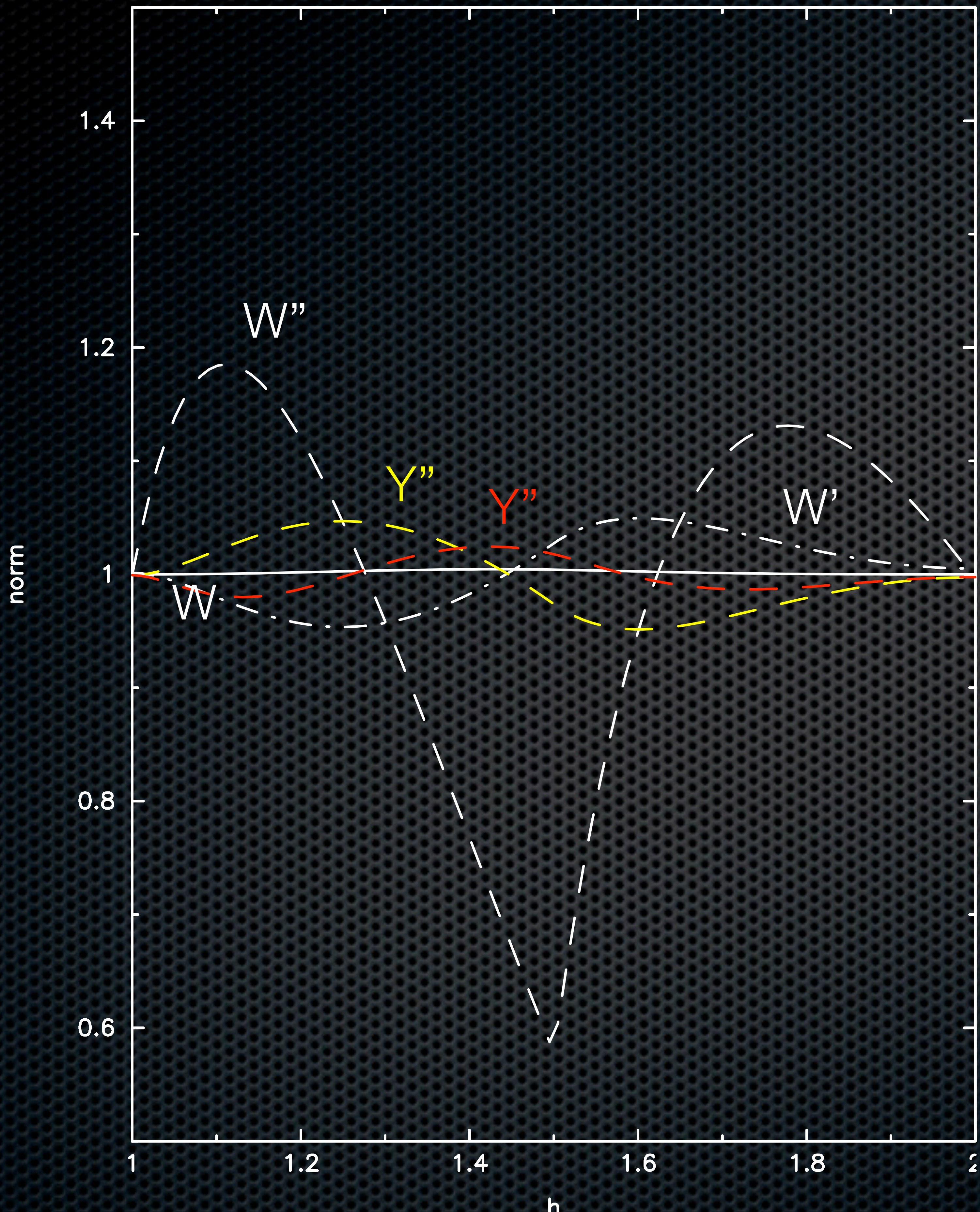
$$\nabla^2 A_a = -2 \sum_b \frac{m_b}{\rho_b} (A_a - A_b) \frac{\hat{\mathbf{r}} \cdot \nabla W_{ab}}{|r_{ab}|}$$

equivalent to:

$$\nabla^2 A_a = \sum_b \frac{m_b}{\rho_b} (A_a - A_b) \nabla^2 Y_{ab}$$

$$Y''_{ab} \equiv -2 \frac{\hat{\mathbf{r}} \cdot \nabla W_{ab}}{|r_{ab}|} = -2W'(q)/q$$

could just use Y" = W



good density estimate:

$$\sum_b \frac{m_b}{\rho_b} W_{ab} \approx 1$$

good gradients:

$$\sum_b \frac{m_b}{\rho_b}(x_a - x_b)\frac{\partial W_{ab}}{\partial x} \approx 1$$

good second derivatives:

$$\frac{1}{2}\sum_b \frac{m_b}{\rho_b}(x_a - x_b)^2\frac{\partial^2 W_{ab}}{\partial x^2} \approx 1$$

Artificial viscosity interpreted

- second derivatives for vector quantities: (Espanol & Revenga 2003)

$$\nabla^2 \mathbf{A}_a \approx -2 \sum_b \frac{m_b}{\rho_b} (\mathbf{A}_a - \mathbf{A}_b) \frac{F_{ab}}{|r_{ab}|}$$

$$\nabla W_{ab} = \hat{\mathbf{r}}_{ab} F_{ab}$$

$$\nabla(\nabla \cdot \mathbf{A})_a \approx -\sum_b \frac{m_b}{\rho_b} [(5\mathbf{A}_{ab} \cdot \hat{\mathbf{r}}_{ab})\hat{\mathbf{r}}_{ab} - \mathbf{A}_{ab}] \frac{F_{ab}}{|r_{ab}|}$$

- artificial viscosity (Monaghan 1997):

$$\left(\frac{d\mathbf{v}}{dt}\right)_{diss} = -\sum_b \frac{m_b}{\bar{\rho}_{ab}} \alpha v_{sig} (\mathbf{v}_a - \mathbf{v}_b) \cdot \hat{\mathbf{r}}_{ab} \nabla_a W_{ab}$$

$$\approx \frac{\alpha v_{sig} h}{10} \left[\nabla^2 \mathbf{v} + 2\nabla(\nabla \cdot \mathbf{v})\right]$$

(e.g. Murray 1996)

- contains both bulk and shear viscosity
- easy to remove shear component but would no longer conserve angular momentum

$$L_{sph} = \sum_j m_j \left[\frac{1}{2} v_j^2 - u_j(\rho_j, s_j) \right]$$

Lagrangian

+PHYSICS

$$+$$

$$du = \frac{P}{\rho^2} d\rho$$

1st law of thermodynamics

CHANGE SOMETHING HERE

$$+$$

$$\nabla \rho_i = \sum_j m_j \nabla W_{ij}(h)$$

density sum

$$+$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0$$

Euler-Lagrange equations

$$=$$

$$\frac{d\mathbf{v}_i}{dt} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla_i W_{ij}(h)$$

equations of motion!

$$\left(\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} \right)$$

Variable h

(Springel & Hernquist 2002, Monaghan 2002, Price & Monaghan 2004b)

$$\rho_a = \sum_b m_b W(\mathbf{r}_a - \mathbf{r_b}, \mathbf{h_a}).$$

$$h_a = \eta \left(\frac{m_a}{\rho_a}\right)^{(1/\nu)}$$

Nonlinear equation for rho(x)
(different to “number of neighbours” approach - can solve to arbitrary precision, i.e. fractions of neighbours)

$$\frac{d\mathbf{v}_a}{dt} = -\sum_b m_b \left[\frac{P_a}{\Omega_a \rho_a^2} \nabla_a W_{ab}(h_a) + \frac{P_b}{\Omega_b \rho_b^2} \nabla_a W_{ab}(h_b)\right]$$

$$\Omega_a = \left[1 - \frac{dh_a}{d\rho_a} \sum_c m_c \frac{\partial W_{ab}(h_a)}{\partial h_a}\right]$$

Adaptive gravitational force softening

Price & Monaghan (2007), MNRAS, 374, 1347

$$\nabla^2\Phi = 4\pi G\rho(\mathbf{r})$$

NOT

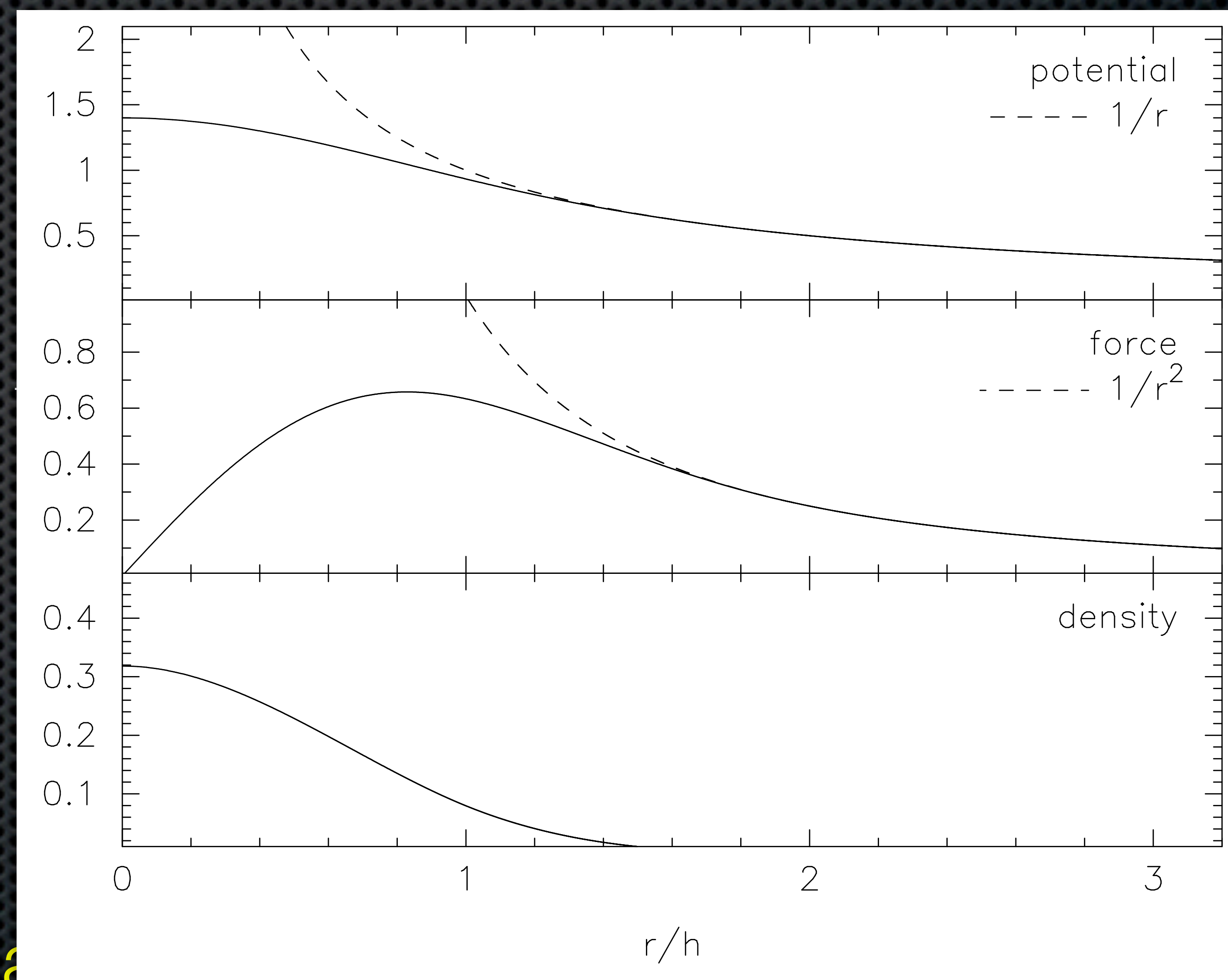
$$\mathbf{F} = -G\sum_j \frac{m_i m_j \mathbf{r}_{ij}}{|r_{ij}|^3}$$

$$\rho(\mathbf{r}) = \sum_{j=1}^{N} m_j W(|\mathbf{r}-\mathbf{r}_j|, h)$$

$$\phi' = \frac{4\pi}{r^2}\int_0^r W r'^2 dr',$$

$$\hat{\mathbf{F}}(\mathbf{r}) = -G\sum_{b=1}^{N} m_b \phi'(|\mathbf{r}-\mathbf{r}_b|, h)\frac{\mathbf{r}-\mathbf{r}_b}{|r-r_b|},$$

(e.g. Dehnen 2001, Athanassoula et al. 2000)



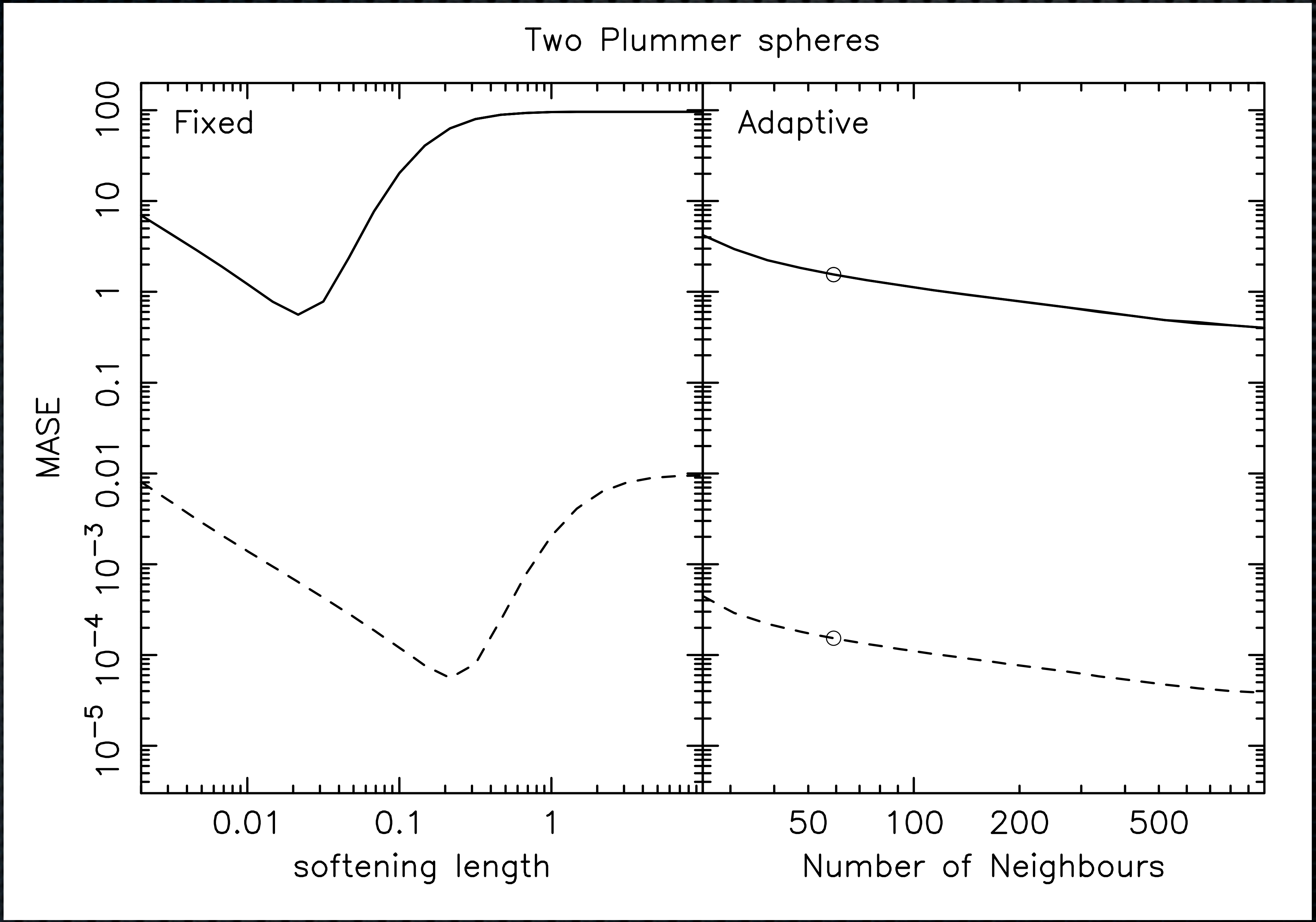
now use:

$$h_a = \eta\left(\frac{m_a}{\rho_a}\right)^{(1/\nu)}$$

ada

momentum, angular momentum and energy (and phase space)!

Why fixed softening lengths are evil

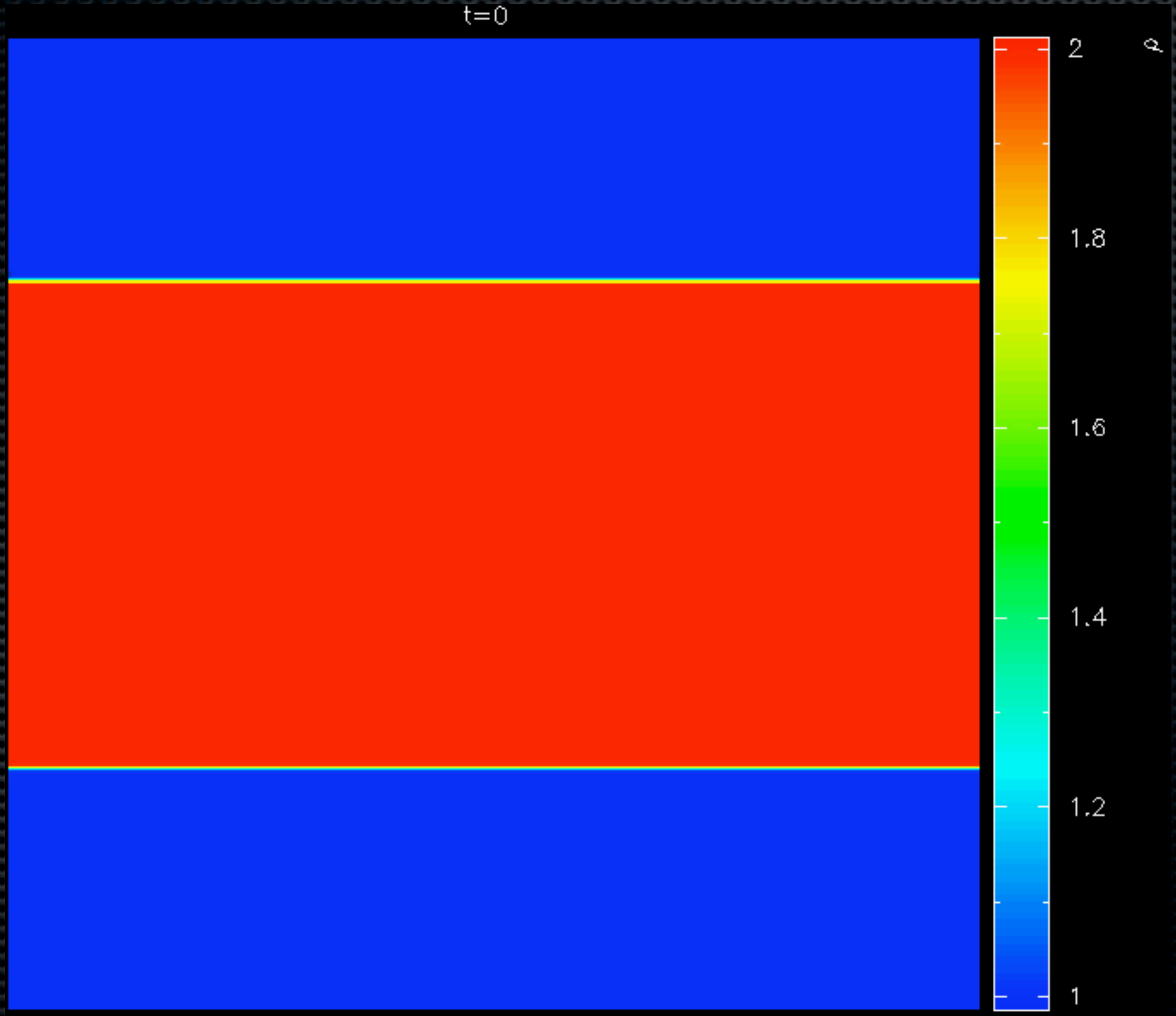


Also, in SPH $h_{grav} \neq h_{gas}$ can result in artificial fragmentation (Bate & Burkert 1997)

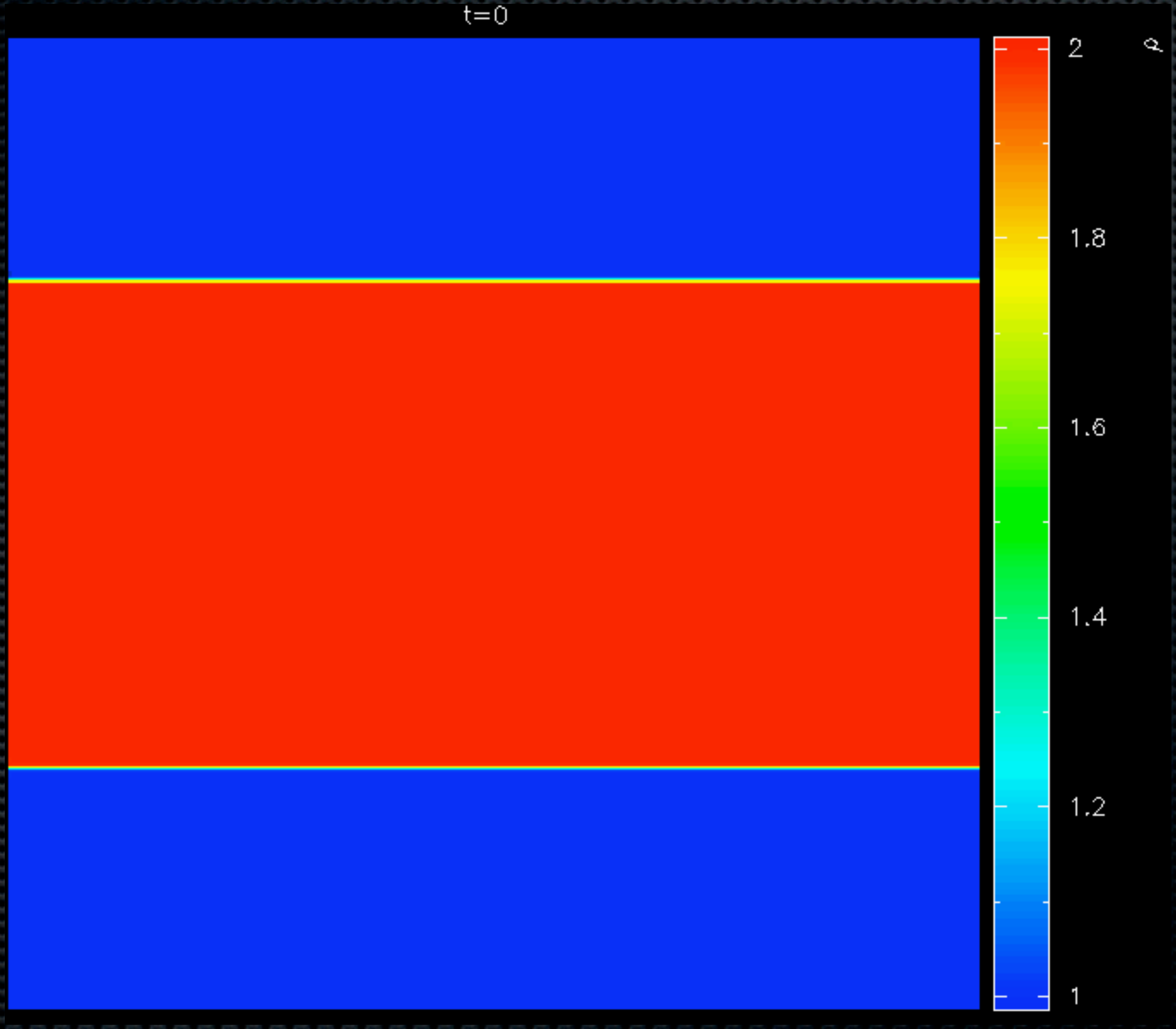
The now-infamous Kelvin-Helmholtz problem

Agertz et al. 2007, Price 2008

KH instability across a 2:1 density jump: no dissipation

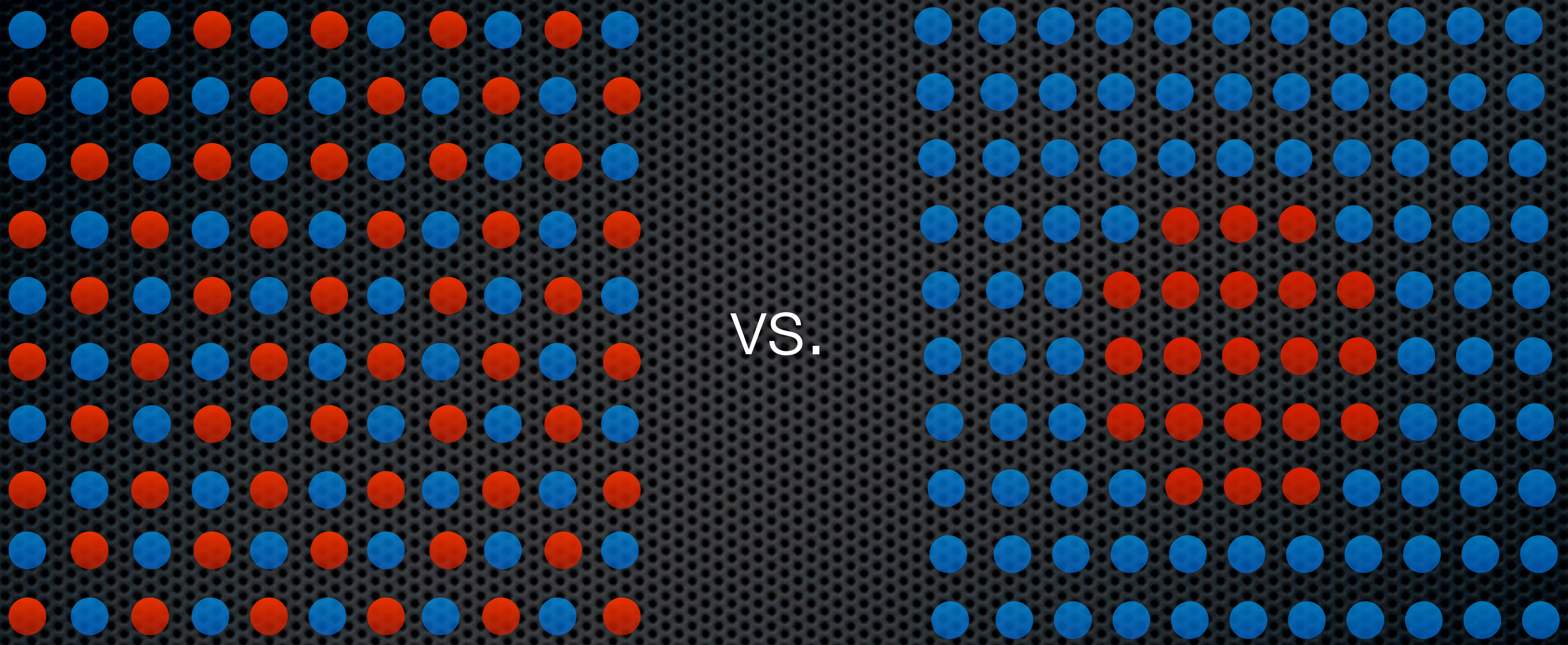


with viscosity



Entropy

- Volker's argument (paraphrased):



- entropy in both configurations is the same
- if energy penalty associated with surfaces, right will be preferred - leads to surface-tension like effect

Integral vs. differential form

$$\frac{d\rho_i}{dt} = \sum_j m_j (\mathbf{v}_i - \mathbf{v}_j) \cdot \nabla W_{ij}(h_i)$$

differential

vs.

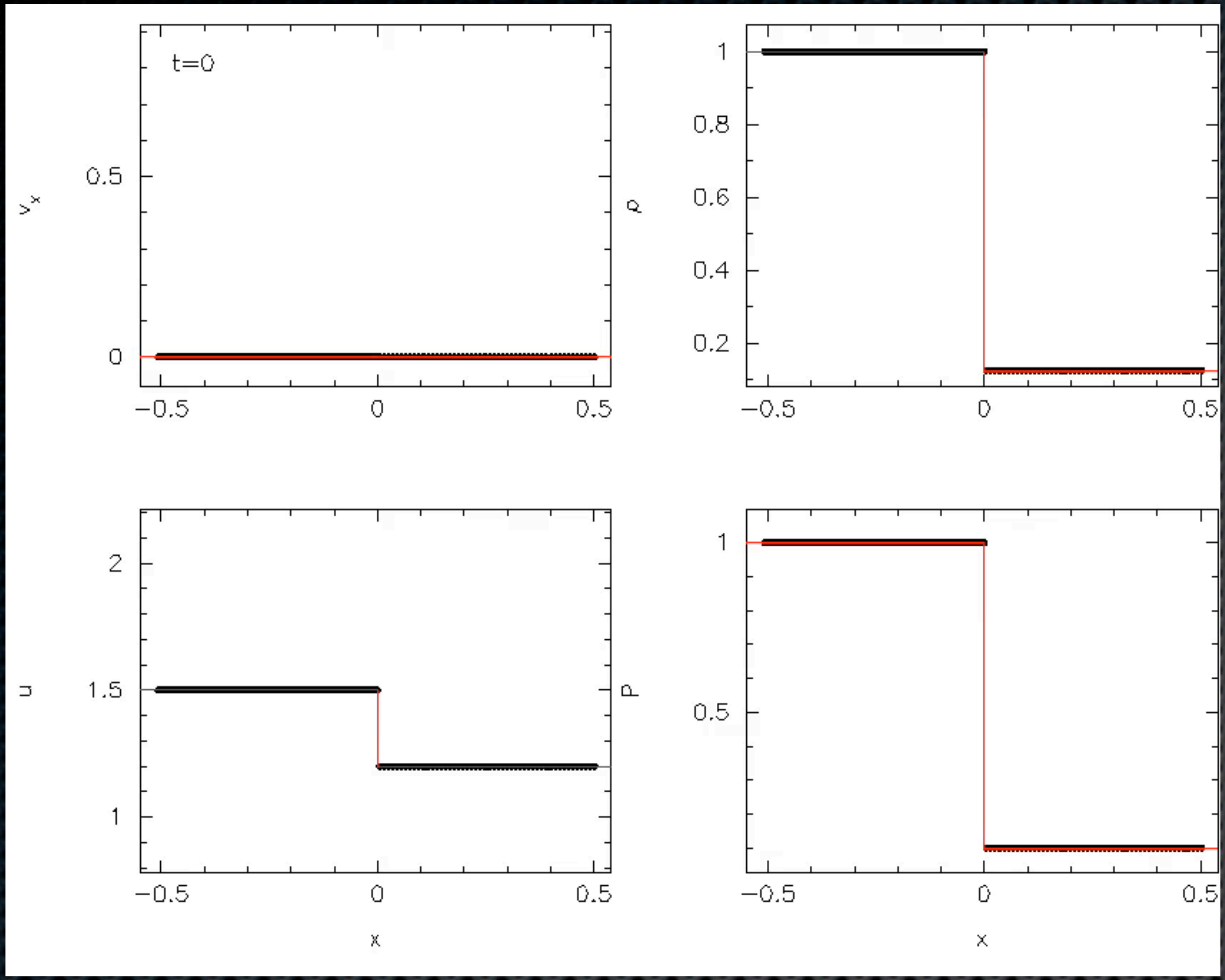
$$\rho_i = \sum_j m_j W_{ij}(h_i)$$

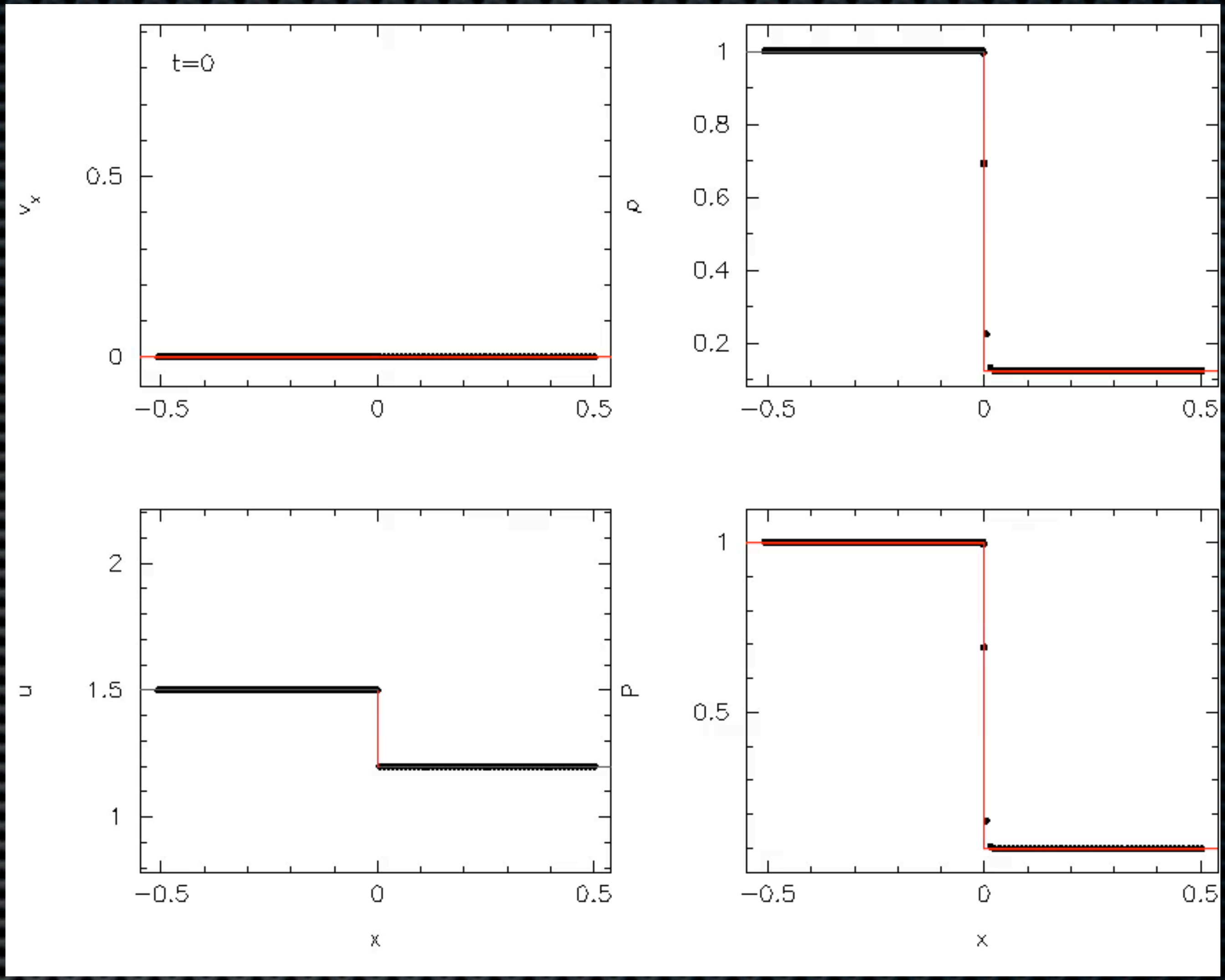
integral

$$\rho(\mathbf{r}) = \int \rho' W(|\mathbf{r} - \mathbf{r}'|, h) dV$$

are they equivalent? (e.g. Monaghan 1997) continuity equation

density sum





$$\int \left[\frac{\partial \rho'}{\partial t} + \nabla' \cdot (\rho' \mathbf{v}') \right] W(|\mathbf{r} - \mathbf{r}'|, h) dV' = 0.$$

$$\frac{d}{dt} \sum_j m_j W_{ij} = \sum_j m_j (\mathbf{v}_i - \mathbf{v}_j) \cdot \nabla W_{ij} - \int [\rho' \mathbf{v}' W] \cdot d\mathbf{S}.$$

ie. they differ at discontinuities...

But what about the thermal energy discontinuity?

1st law of thermodynamics gives: $$\frac{du}{dt} = \frac{P}{\rho^2}\frac{d\rho}{dt}$$

or equivalent:

$$\frac{dS}{dt} = \frac{\gamma - 1}{\rho^{\gamma - 1}}\left(\frac{du}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt}\right) = 0$$

ie. we are forced to use a differential form for the thermal energy / entropy evolution

corollary: discontinuities in u need treatment

$$L_{sph} = \sum_j m_j \left[\frac{1}{2}v_j^2 - u_j(\rho_j, s_j)\right]$$ Lagrangian

$+$

$$du = \frac{P}{\rho^2}d\rho$$ 1st law of thermodynamics

$+$

$$\nabla\rho_i = \sum_j m_j \nabla W_{ij}(h)$$ density sum

$+$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0$$ Euler-Lagrange equations

$=$

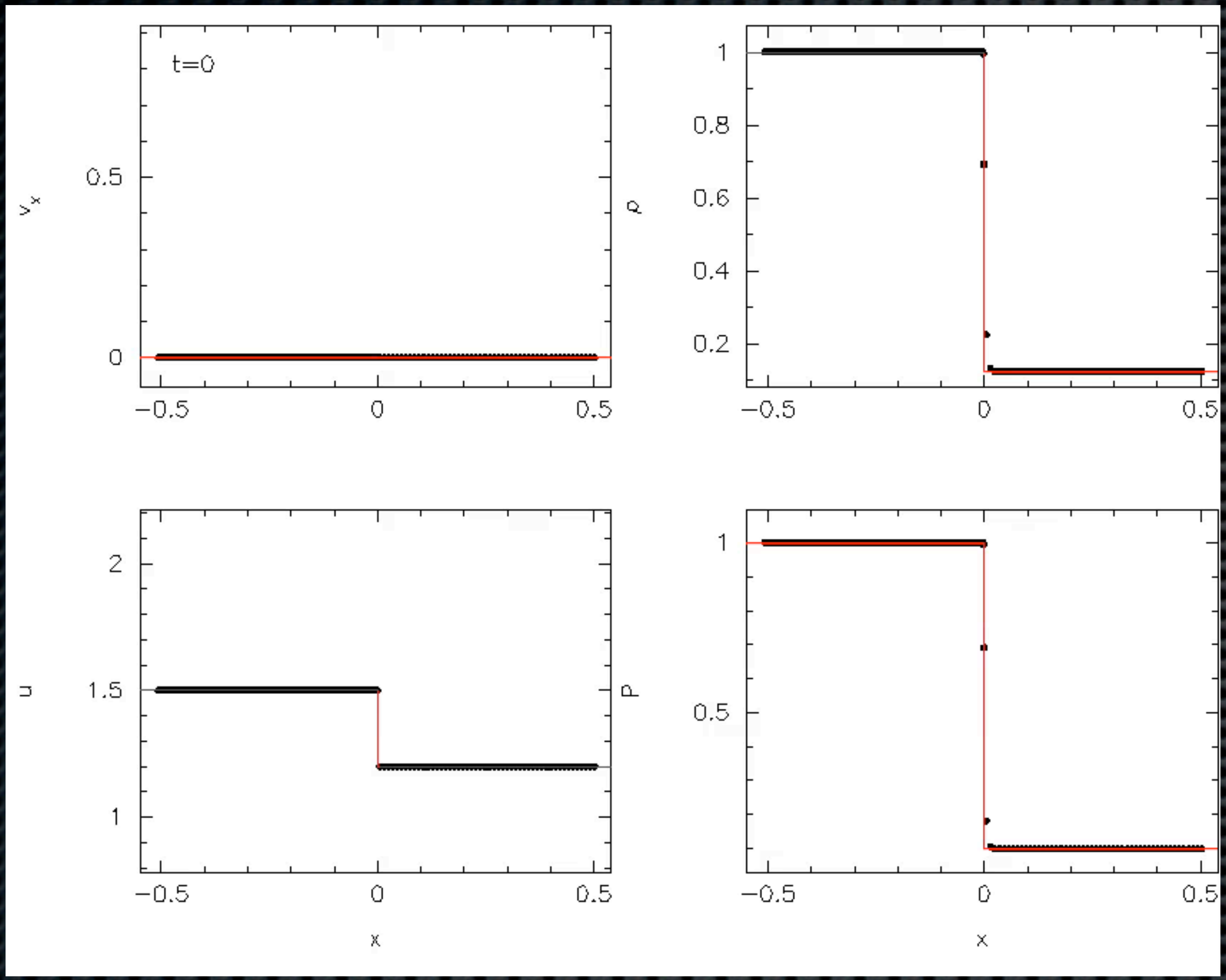
$$\frac{d\mathbf{v}_i}{dt} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right) \nabla_i W_{ij}(h)$$

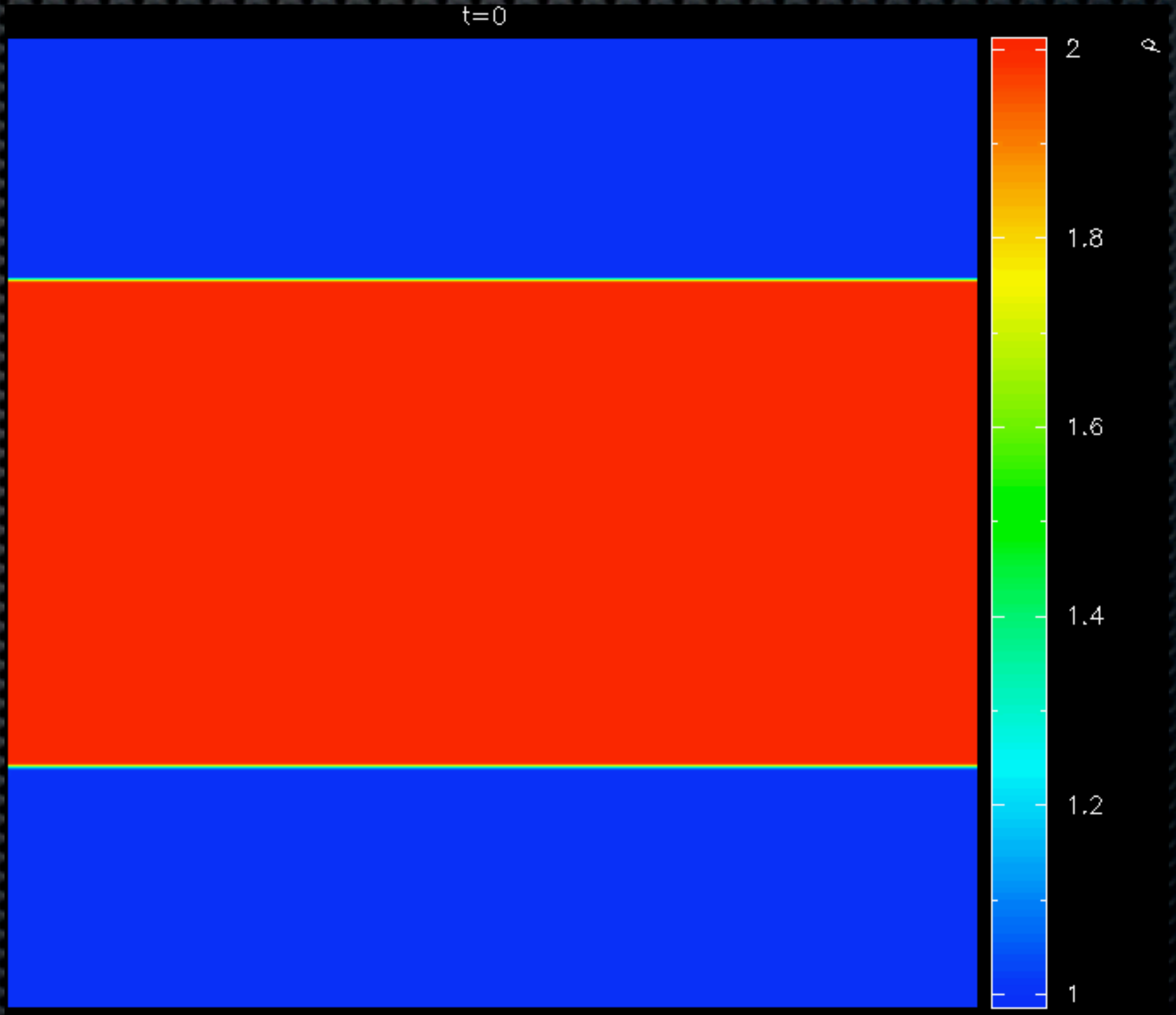
equations of motion!

$$\left(\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho}\right)$$

KH with conductivity

Price (2008)

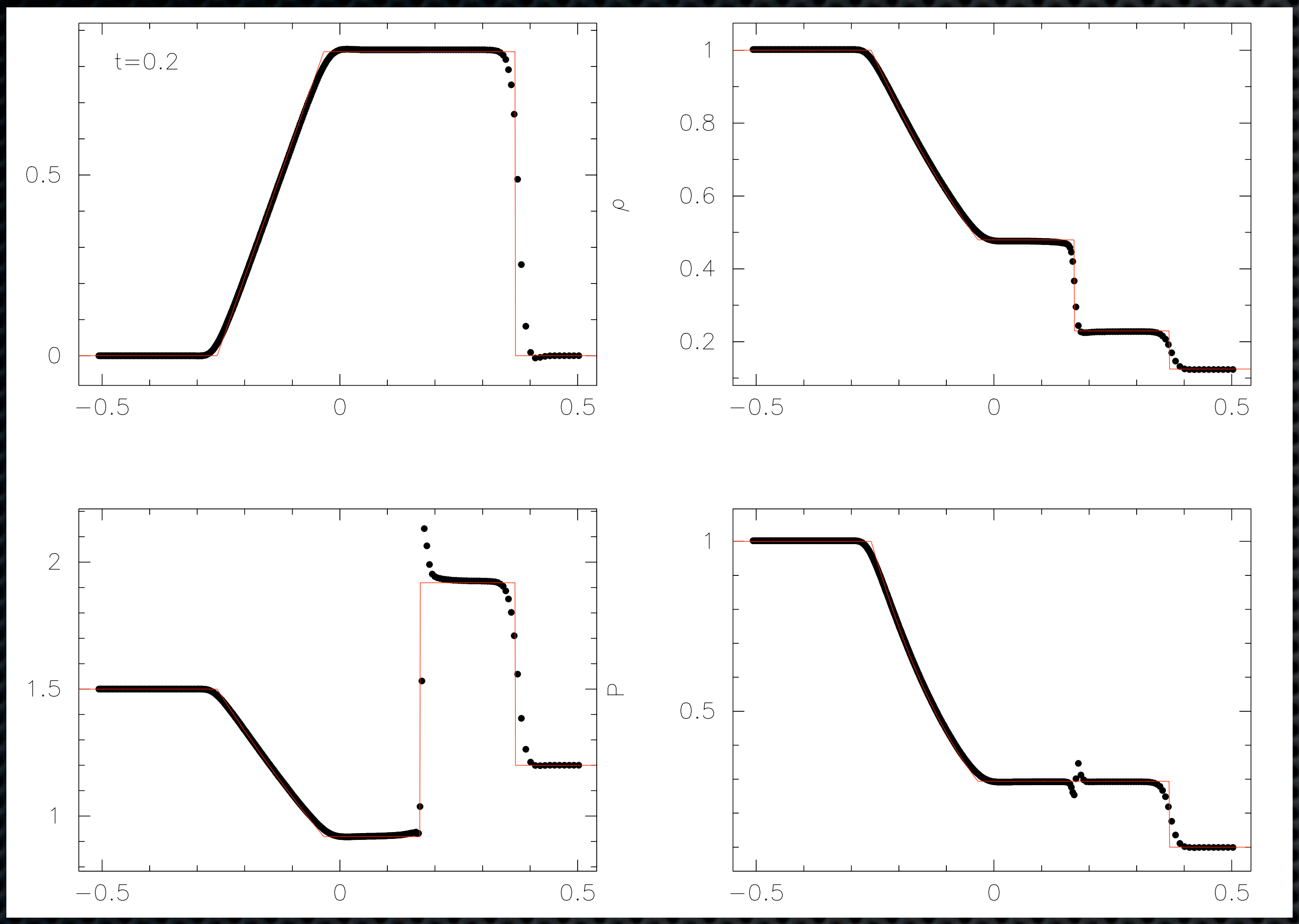


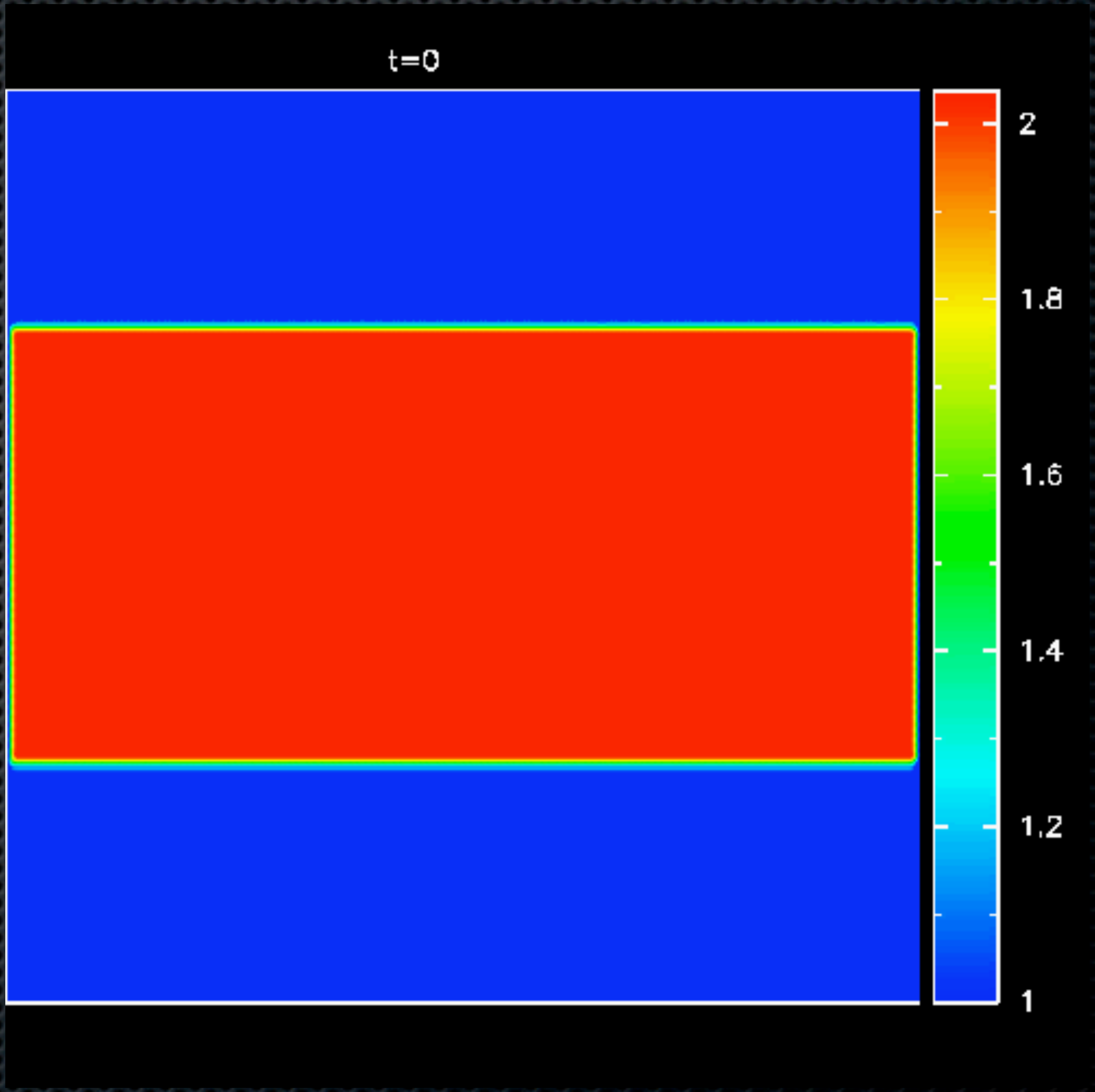


conductivity also proposed by Wadsley et al. (2008) w.r.t. entropy mixing problems in galaxy clusters

Godunov-SPH

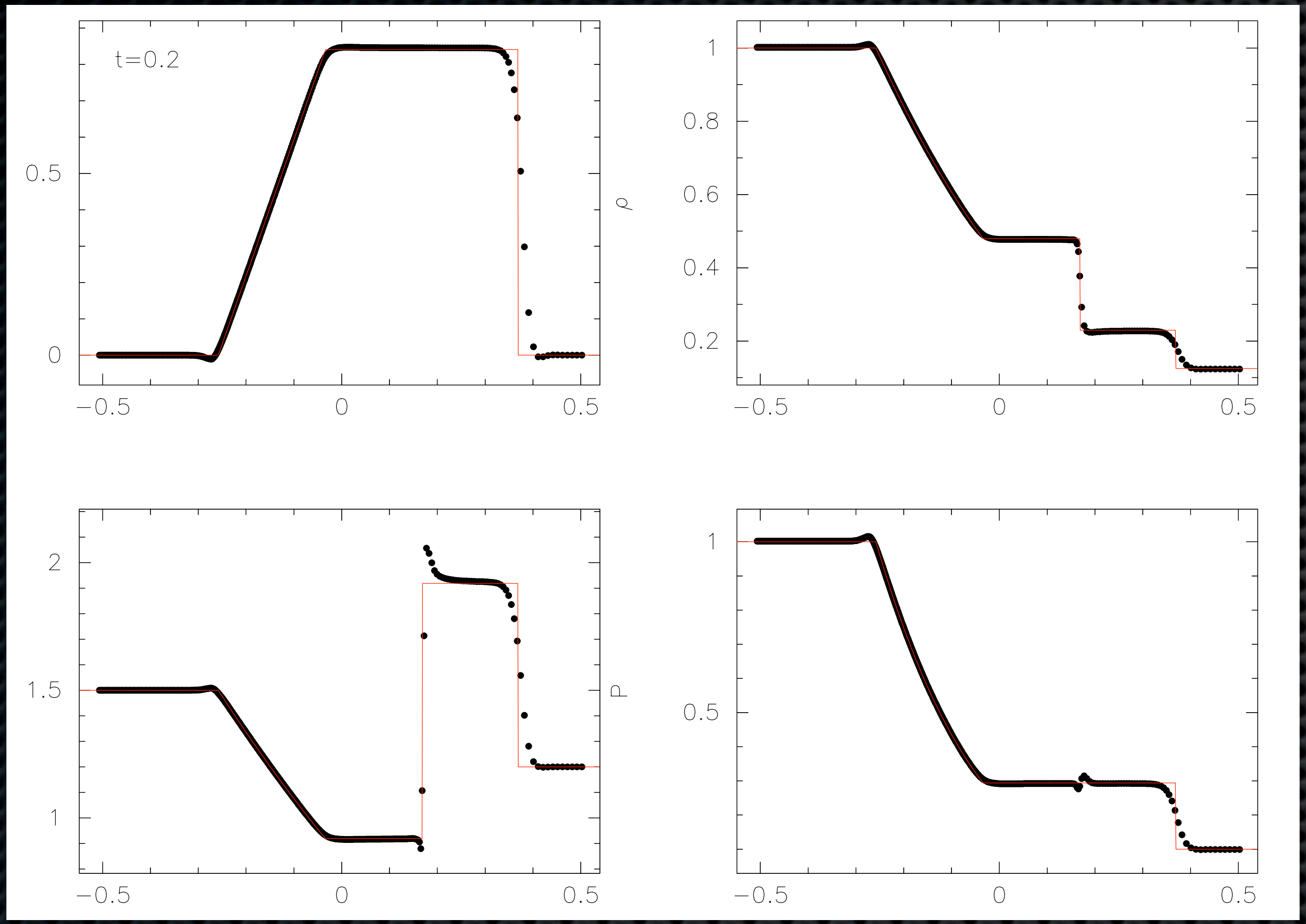
Inutsuka (2002), Cha & Whitworth (2004)

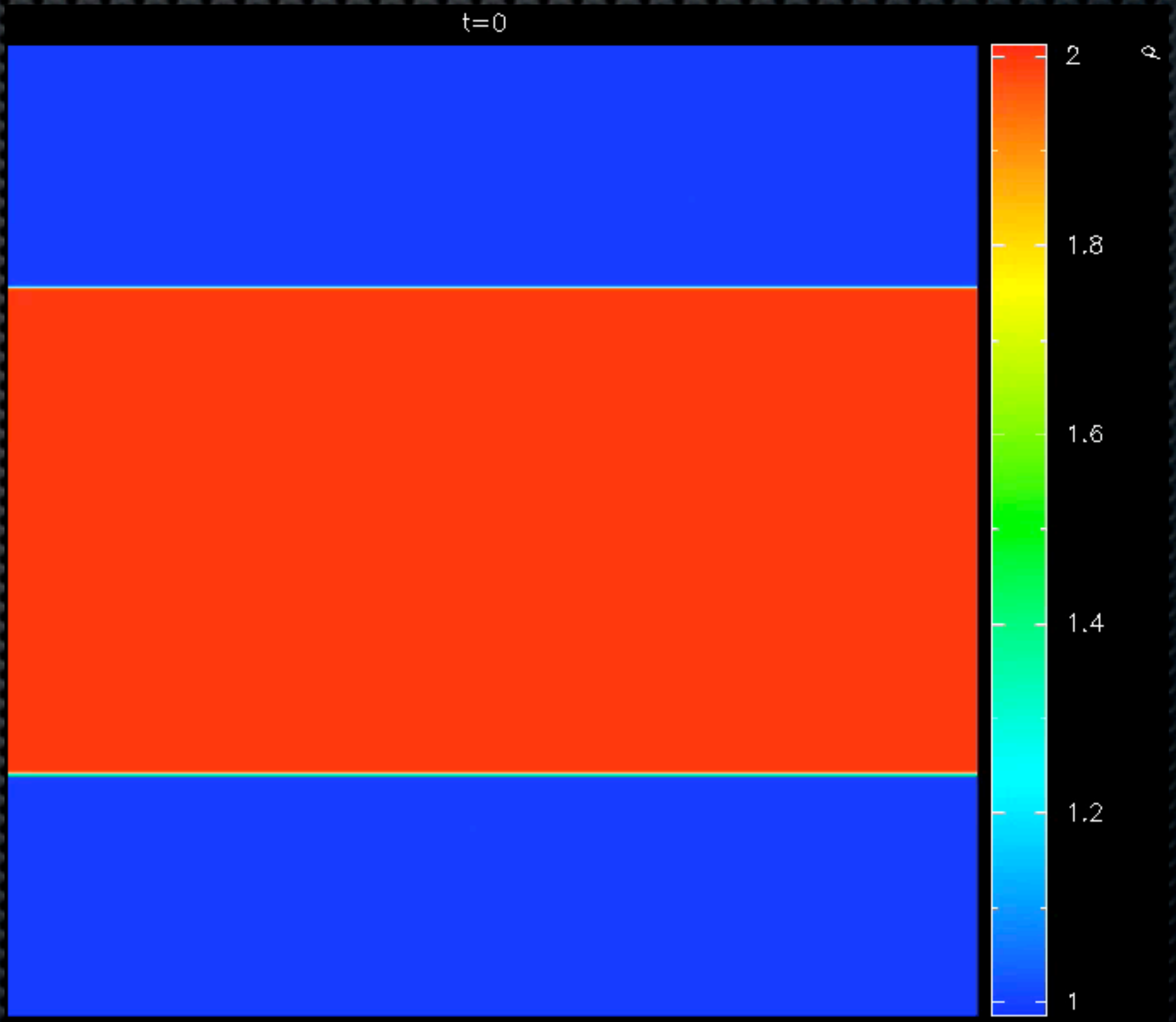




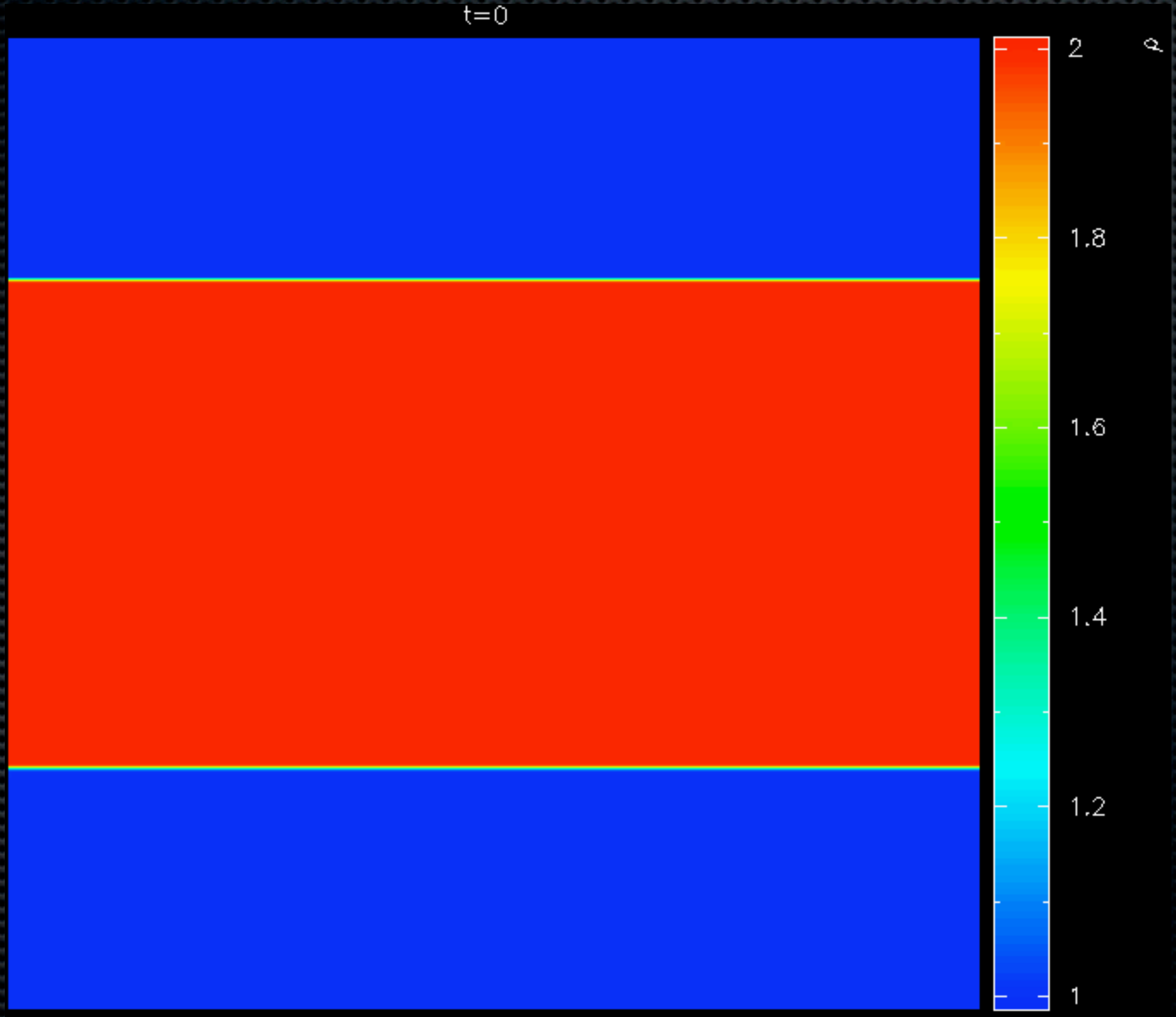
Ritchie & Thomas (2001) method

$$\frac{d\mathbf{v}_i}{dt} = (1-\gamma)\sum_j m_j \left[\frac{u_j}{\langle\rho_i\rangle}\nabla W_{ij}(h_i) + \frac{u_i}{\langle\rho_j\rangle}\nabla W_{ij}(h_j)\right], \qquad \langle\rho_i\rangle = \frac{\langle P_i\rangle}{(\gamma-1)u_i} = \frac{\sum_j m_j u_j W_{ij}(h_i)}{u_i}.$$





conductivity + viscosity using switch



PART II - new physics

Smoothed Particle Magnetohydrodynamics

Price & Monaghan (2004a,b, 2005)

$$L_{sph} = \sum_b m_b \left[\frac{1}{2} v_b^2 - u_b(\rho_b, s_b) - \frac{1}{2\mu_0} \frac{B_b^2}{\rho_b} \right]$$

$$\int \delta L \mathrm{dt} = 0$$

continuity equation

$$\delta \rho_b = \sum_c m_c \left(\delta \mathbf{r}_b - \delta \mathbf{r}_c \right) \cdot \nabla_b W_{bc},$$

mag field evolution

$$\delta \left(\frac{\mathbf{B}_b}{\rho_b} \right) = - \sum_c m_c (\delta \mathbf{r}_b - \delta \mathbf{r}_c) \frac{\mathbf{B}_b}{\rho_b^2} \cdot \nabla_b W_{bc}$$

equations of motion

$$\frac{dv_a^i}{dt} = \sum_b m_b \left[\left(\frac{S^{ij}}{\rho^2} \right)_a + \left(\frac{S^{ij}}{\rho^2} \right)_b \right] \nabla_a^j W_{ab},$$

$$S_a^{ij} = - \left(P_a + \frac{1}{2\mu_0} B_a^2 \right) \delta^{ij} + \frac{1}{\mu_0} \left(B_a^i B_a^j \right),$$

Technical issues

1) Momentum conserving force is unstable

use force which vanishes for constant stress

$$\frac{dv^i}{dt} = -\sum_b m_b \left(\frac{P_a + \frac{1}{2}B_a^2/\mu_0}{\rho_a^2} + \frac{P_b + \frac{1}{2}B_b^2/\mu_0}{\rho_b^2} \right) \frac{\partial W_{ab}}{\partial x^i}$$

$$+ \frac{1}{\mu_0} \sum_b m_b \frac{(B_i B_j)_b - (B_i B_j)_a}{\rho_a \rho_b} \frac{\partial W_{ab}}{\partial x_j}.$$

(Morris 1996)

2) Shocks

formulate artificial dissipation terms (PM04a)

$$\left(\frac{d\mathbf{v}}{dt}\right)_{diss} = -\sum_b m_b \frac{\alpha v_{sig}(\mathbf{v}_a - \mathbf{v}_b)\cdot\hat{r}}{\bar{\rho}_{ab}} \nabla_a W_{ab},$$

$$\left(\frac{d\mathbf{B}}{dt}\right)_{diss} = \rho_a \sum_b m_b \frac{\alpha_B v_{sig}}{\bar{\rho}_{ab}^2} (\mathbf{B}_a - \mathbf{B}_b)\, \hat{r}\cdot\nabla_a W_{ab}$$

$$\left(\frac{de_a}{dt}\right)_{diss} = -\sum_b m_b \frac{v_{sig}(e_a^* - e_b^*)}{\bar{\rho}_{ab}} \hat{r}\cdot\nabla_a W_{ab}$$

3) Variable h

use Lagrangian (Price & Monaghan 2004b)

4) The ∇•B = 0 constraint

- tried lots of things which didn't work (e.g. Dedner et al. cleaning)
- Euler potentials:

Euler (1770), Stern (1976),
Phillips & Monaghan (1985)

use accurate SPH derivatives (Price 2004)

$$\mathbf{B} = \nabla\alpha \times \nabla\beta$$

$$\chi_{\mu\nu}\nabla^{\mu}\alpha_i = -\sum_j m_j(\alpha_i - \alpha_j)\nabla_i^{\nu} W_{ij}(h_i)$$

$$\chi_{\mu\nu} = \sum_j m_j(r_i^{\mu} - r_j^{\mu})\nabla^{\nu} W_{ij}(h_i).$$

add shock dissipation

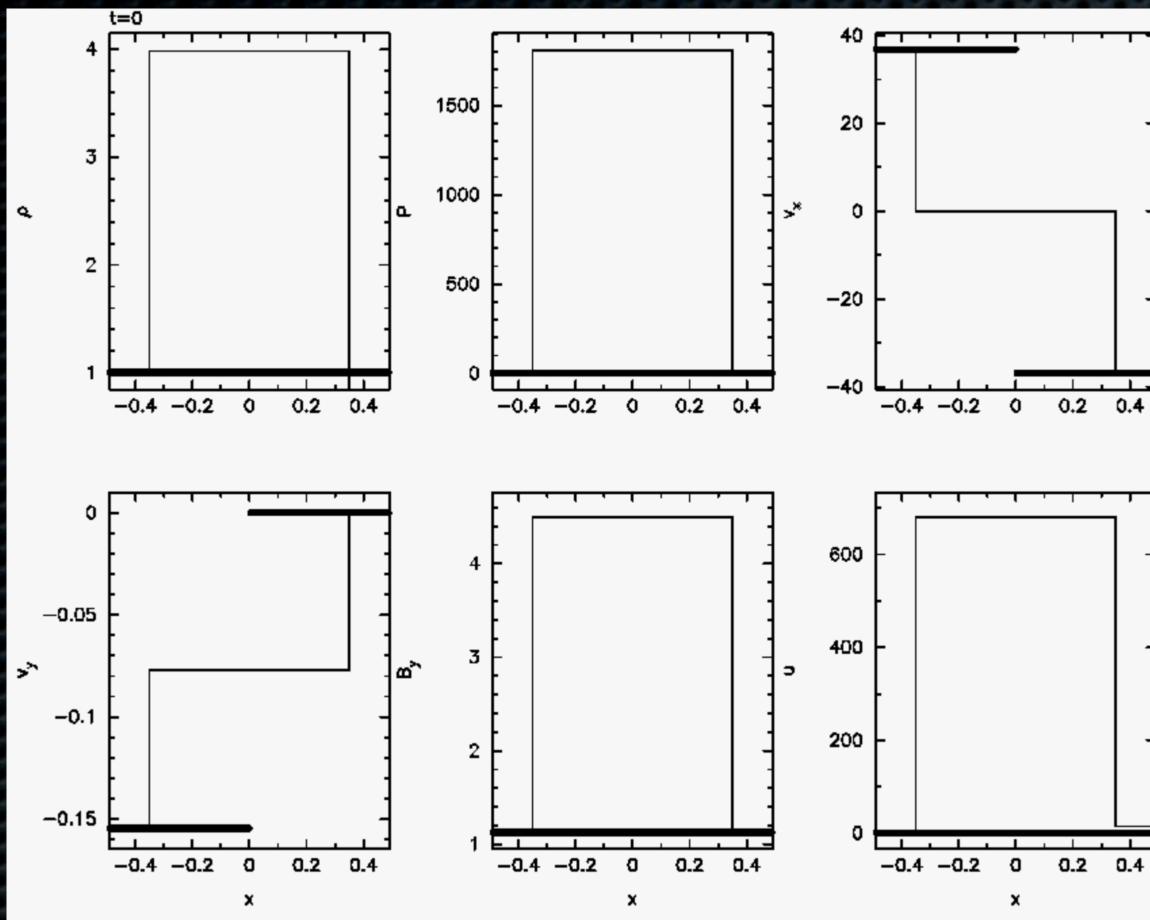
$$\frac{d\alpha}{dt} = 0, \frac{d\beta}{dt} = 0$$

'advection of magnetic field lines'

$$\frac{d\alpha}{dt} = \sum_b m_b \frac{\alpha_B v_{sig}}{\bar{\rho}_{ab}} (\alpha_a - \alpha_b)\,\hat{r} \cdot \nabla_a W_{ab}$$

$$\frac{d\beta}{dt} = \sum_b m_b \frac{\alpha_B v_{sig}}{\bar{\rho}_{ab}} (\beta_a - \beta_b)\,\hat{r} \cdot \nabla_a W_{ab}$$

Test problems

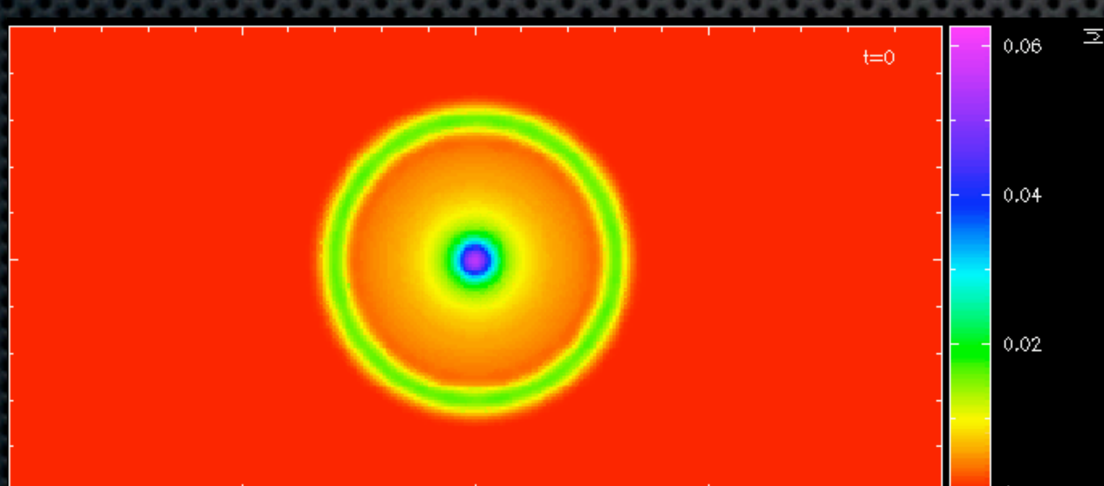


Mach 25 MHD shock (e.g. Balsara 1998)

(Price & Monaghan 2004a,b, Price 2004)

Orszag-Tang vortex (everyone)

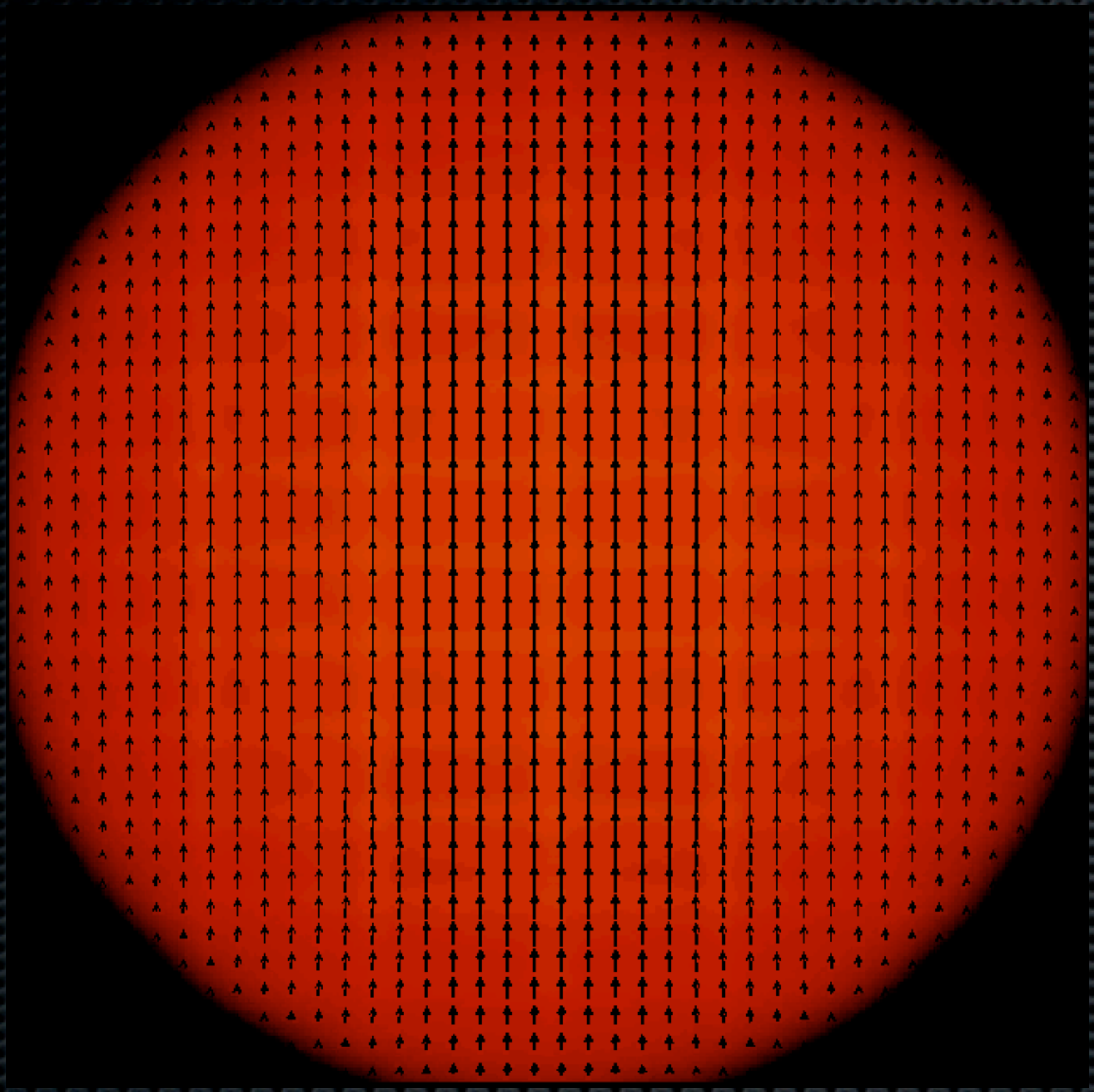
(Price & Monaghan 2005, Rosswog & Price 2007)

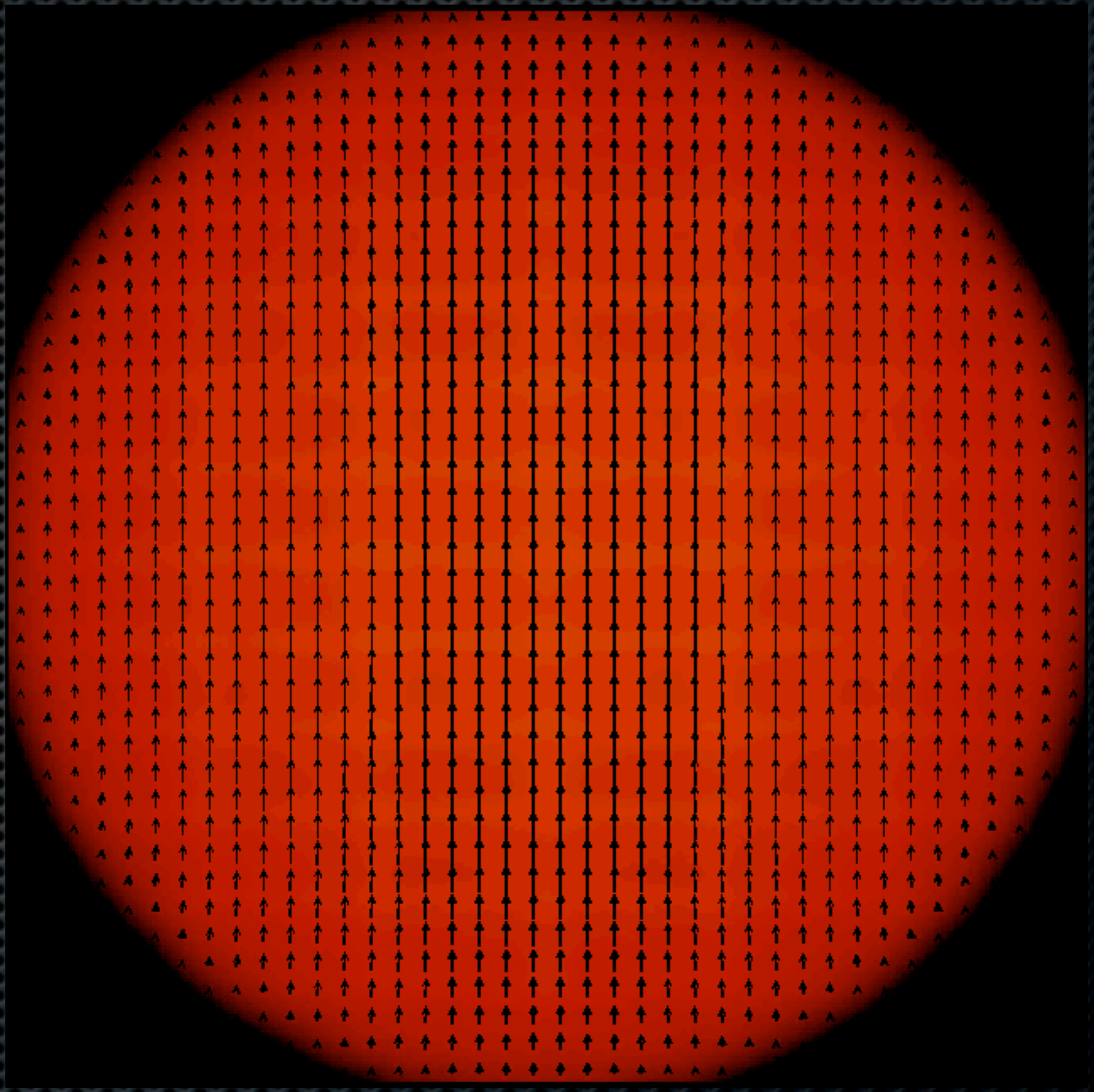


Current loop advection (e.g. Gardiner & Stone 2007)

(Rosswog & Price 2007)

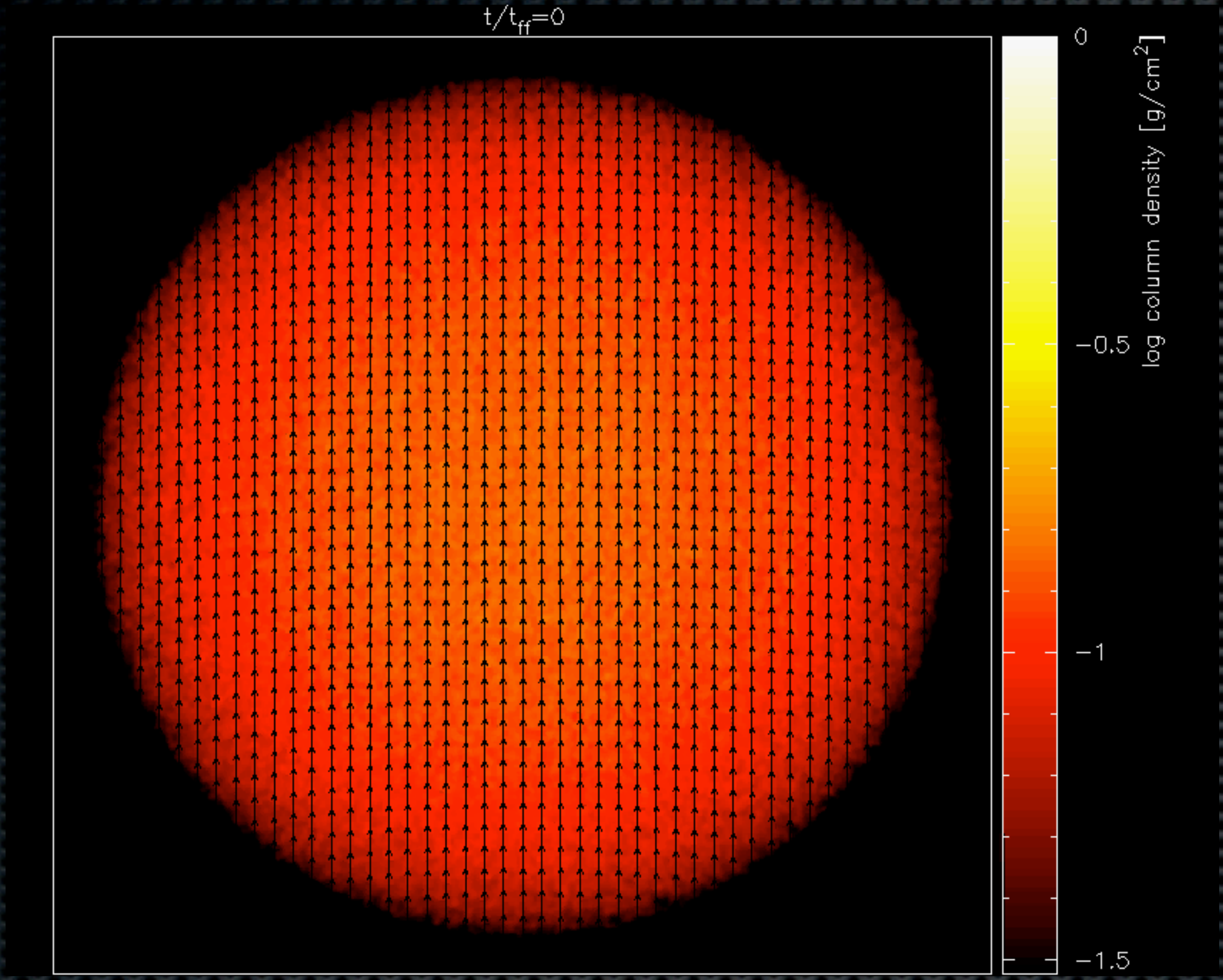
Star formation





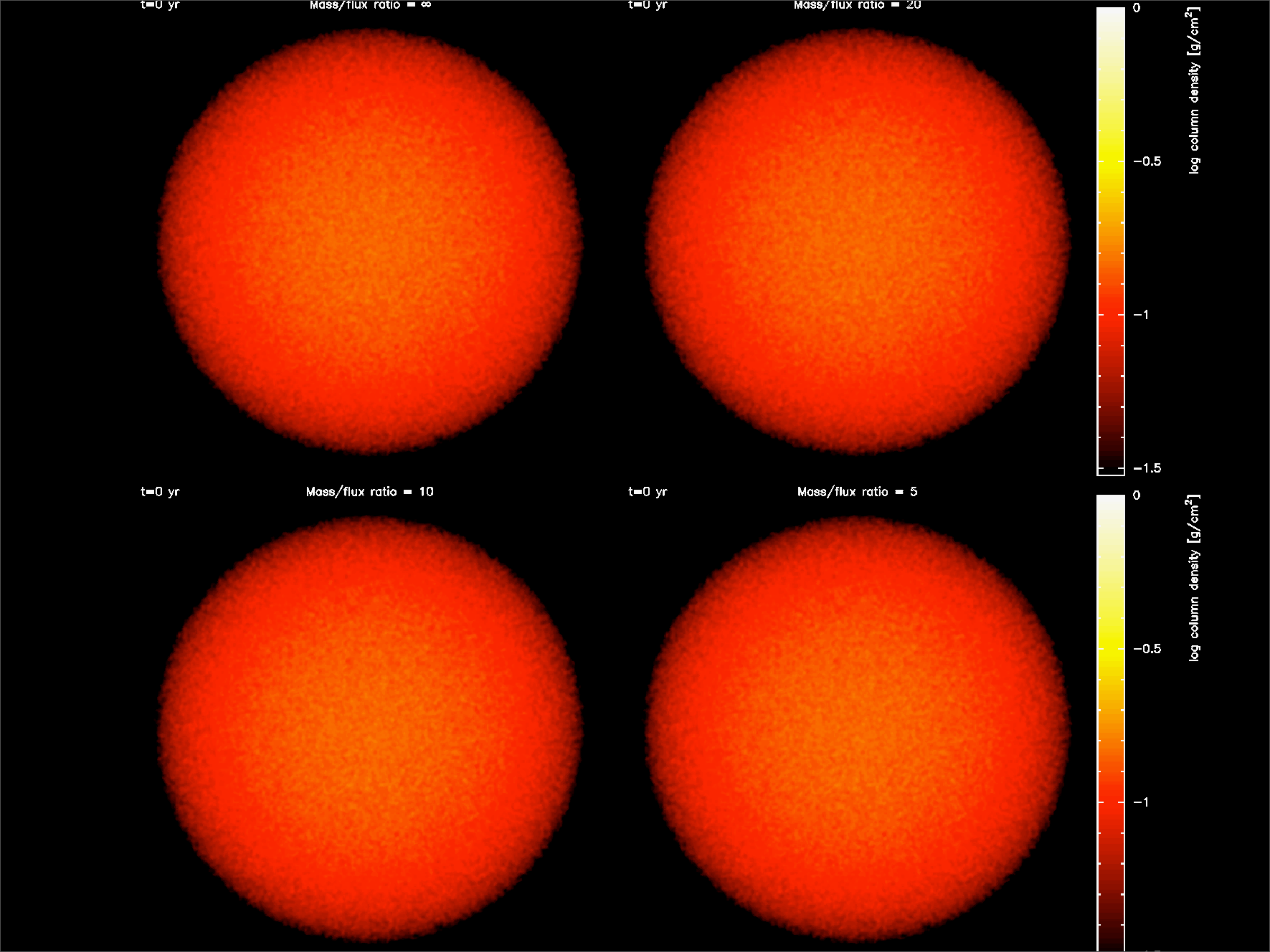
Magnetic fields in star cluster formation

Price & Bate (2008) MNRAS 385, 1820

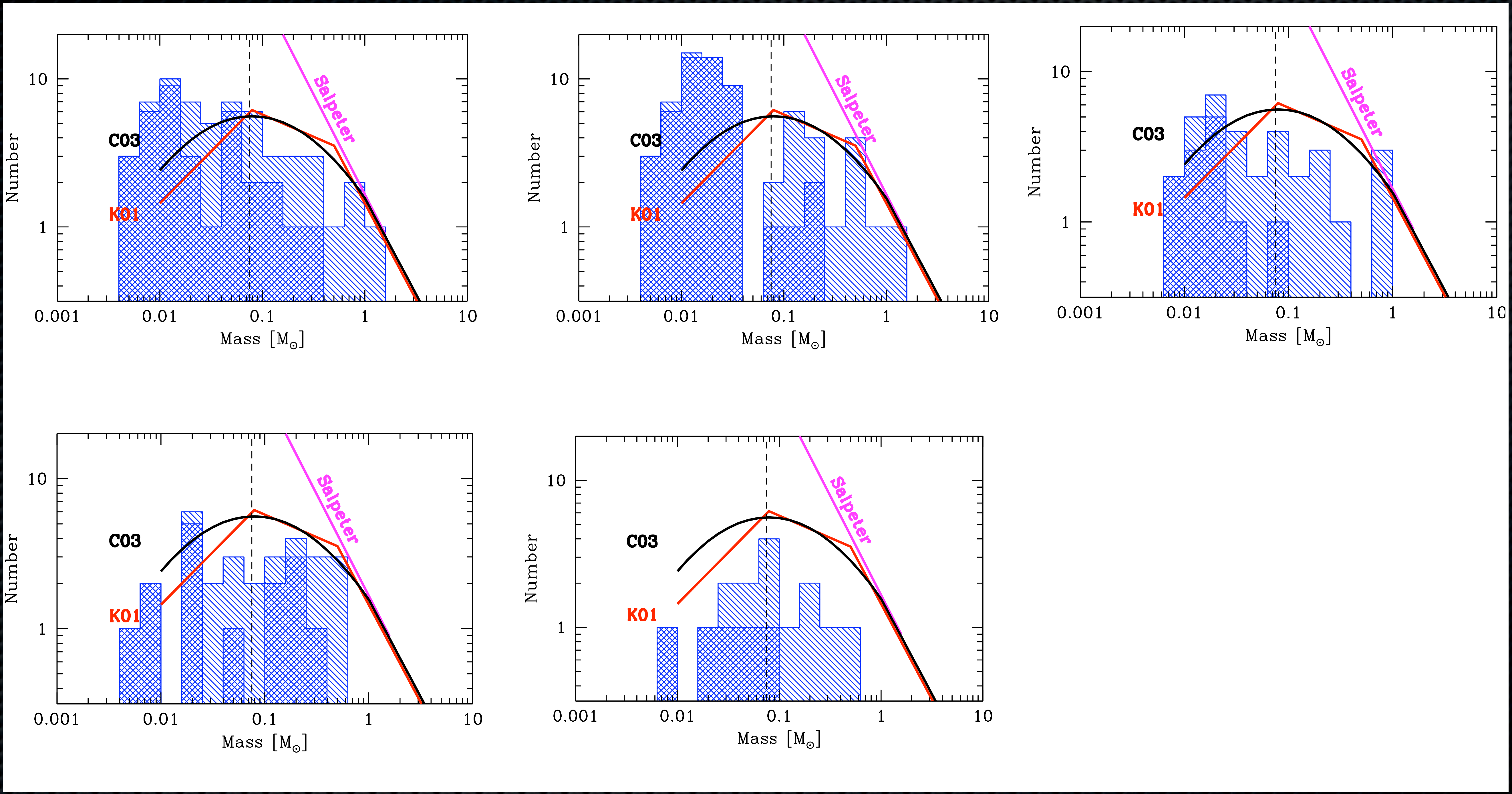


- 50 solar mass cloud
- diameter 0.375 pc, n_{H2} = 3.7 x 10^4
- initial uniform B field
- T ~10K
- turbulent velocity field $P(k) \propto k^{-4}$
- RMS Mach number 6.7
- barytropic equation of state

Bate, Bonnell & Bromm (2003) with magnetic fields...


t=0 yr
Mass/flux ratio = ∞
t=0 yr
Mass/flux ratio = 20
t=0 yr
Mass/flux ratio = 10
t=0 yr
Mass/flux ratio = 5
0
−0.5
−1
−1.5
log column density [g/cm²]
0
−0.5
−1
log column density [g/cm²]

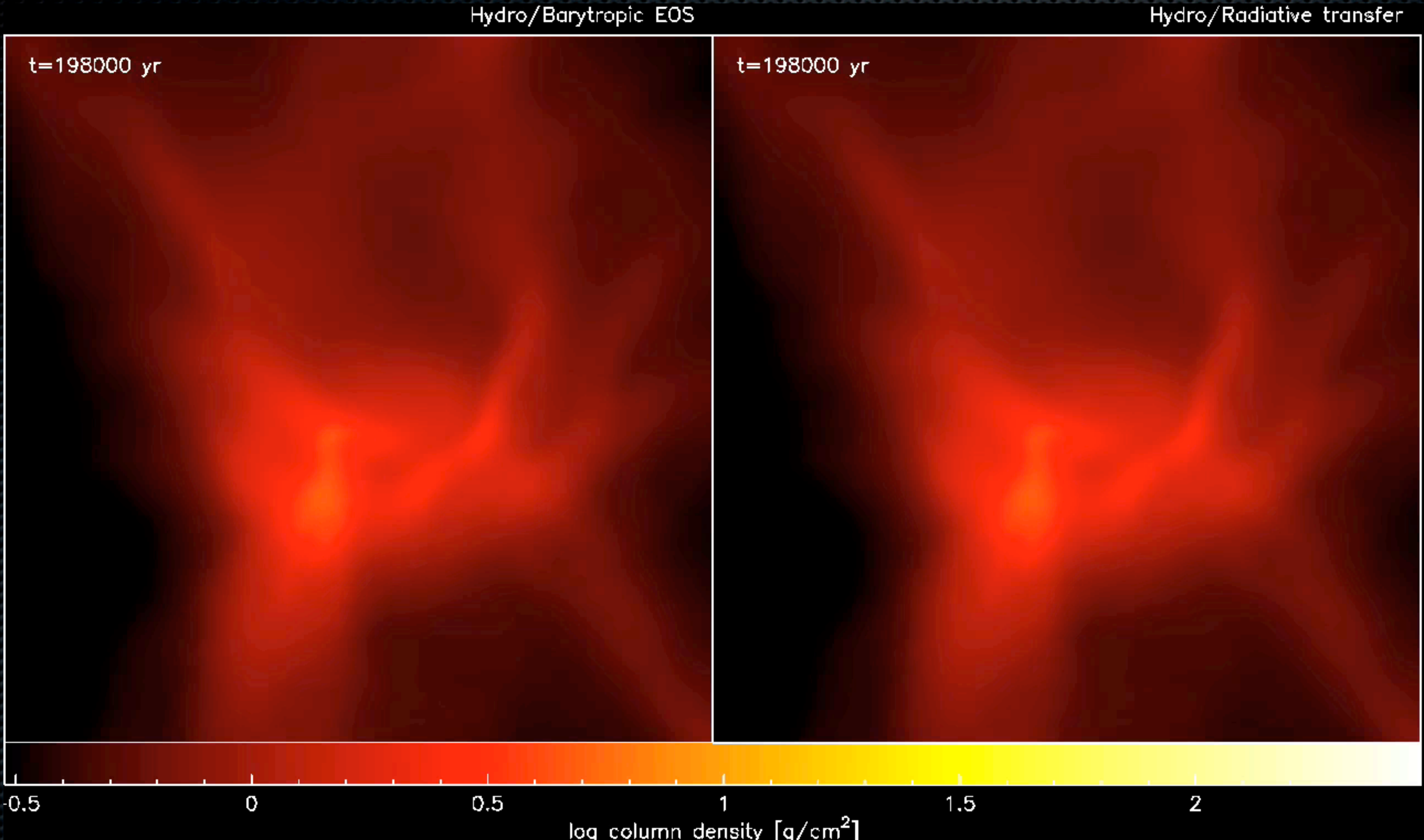
Effect on IMF



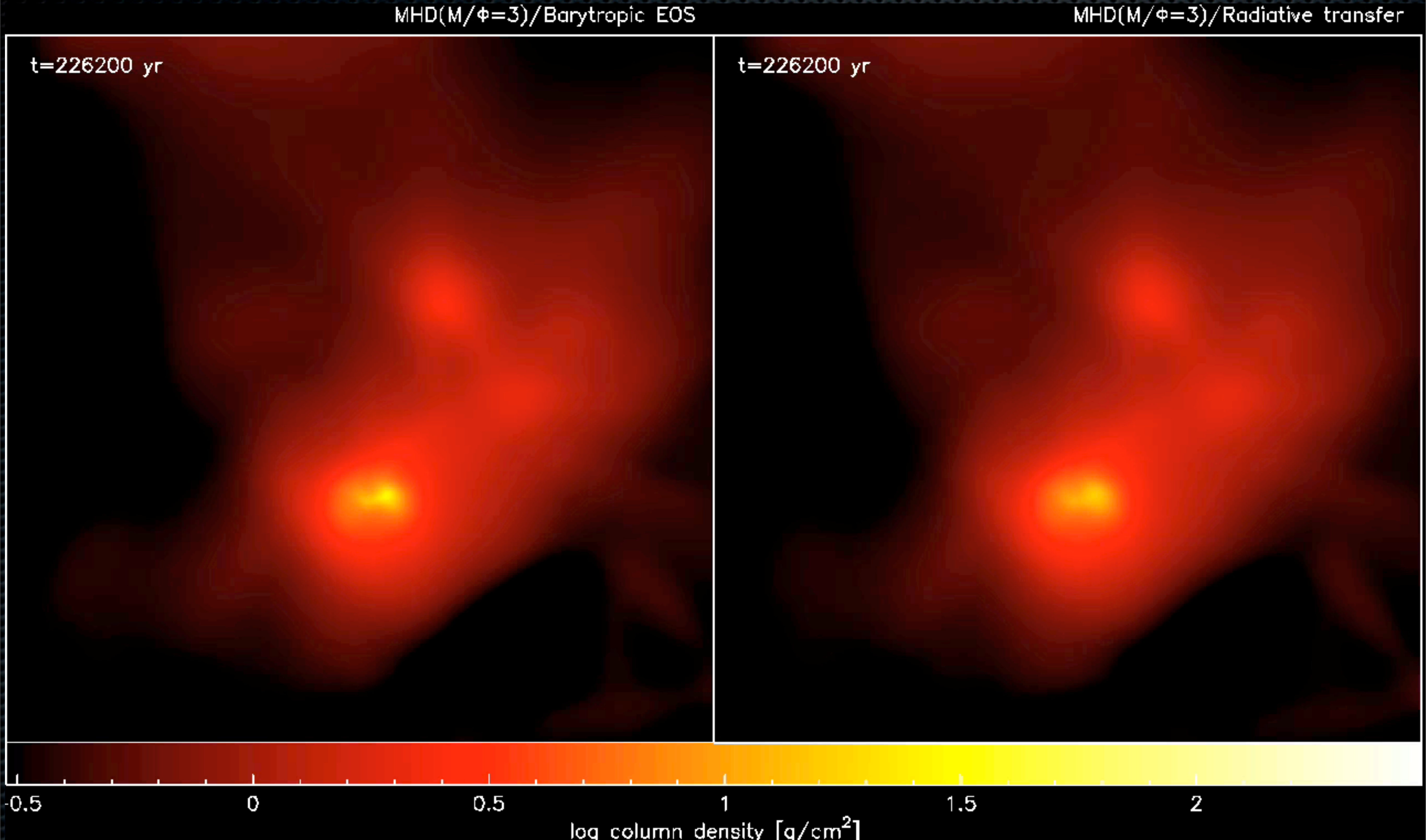
Radiative transfer

- photoionisation/ray-tracing schemes (Dale 2006, SPHRAY, Altay et al. ’08; TRAPHIC, Pawlik)
- SPH+Monte Carlo (e.g. Oxley & Woolfson ’03), single temperature
- approximate methods (Stamatellos et al. ’07)
- single temperature diffusion approximation (implicit) (Bastien et al. ’04, Viau et. al. ’06)
- single temperature flux limited diffusion (Mayer et al. ’06)
- two-temperature flux limited diffusion (implicit) (Whitehouse & Bate ’04, Whitehouse et al. ’05, Whitehouse & Bate ’06)

Radiation hydro



Radiation + MHD



Conclusions