Appendix A
Discretization scheme for non-relativistic equations

The discretization scheme used in Chapter 2 for the non-relativistic fluid equations is summarised in Figure 2.1. Fluxes are calculated on the half grid points while the other terms are calculated on the integer points. We solve (2.1)-(2.5) in the following manner: The numerical equations are solved first for velocity on the half grid points (dropping the superscript r for convenience),

$$\begin{aligned} \mathrm{v}_{i+1/2}^{n+1} &= \mathrm{v}_{i+1/2}^{n+1} - \Delta t \left[\mathrm{v}_{i+1/2}^{n} \left(\frac{\mathrm{v}_{i+3/2}^{n} - \mathrm{v}_{i+1/2}^{n}}{r_{i+3/2} - r_{i+1/2}} \right) - \frac{1}{\rho_{i+1/2}^{n}} \left(\frac{P_{i+1}^{n} - P_{i}^{n}}{r_{i+1} - r_{i}} \right) - \frac{1}{r_{i+1/2}^{2}} \right] \quad (\mathrm{v} < 0) \\ &= \mathrm{v}_{i+1/2}^{n+1} - \Delta t \left[\mathrm{v}_{i+1/2}^{n} \left(\frac{\mathrm{v}_{i+1/2}^{n} - \mathrm{v}_{i-1/2}^{n}}{r_{i+3/2} - r_{i+1/2}} \right) - \frac{1}{\rho_{i+1/2}^{n}} \left(\frac{P_{i+1}^{n} - P_{i}^{n}}{r_{i+1} - r_{i}} \right) - \frac{1}{r_{i+1/2}^{2}} \right] \quad (\mathrm{v} > 0) \end{aligned} \tag{A.1}$$

where the superscript n refers to the nth timestep and the subscript i refers to ith grid point ($\mathrm{v}_{i+1/2}$,$\rho_{i+1/2}$ thus being points on the staggered velocity grid). The quantity $\rho_{i+1/2}$ is calculated using linear interpolation between the grid points, ie. $\rho_{i+1/2} = \frac{1}{2}(\rho_i + \rho_{i+1})$. We then solve for the density and internal energy on the integer grid points using the updated velocity,

$$\begin{aligned} \rho_i^{n+1} &= \rho_i^n - \Delta t \left[\mathrm{v}_i^{n+1} \left(\frac{\rho_{i+1}^n - \rho_i^n}{r_{i+1} - r_i} \right) - \frac{\rho_i^n}{r_i^2} \left(\frac{r_{i+1/2}^2 \mathrm{v}_{i+1/2}^{n+1} - r_{i-1/2}^2 \mathrm{v}_{i-1/2}^{n+1}}{r_{i+1/2} - r_{i-1/2}} \right) \right] \quad (\mathrm{v} < 0) \\ &= \rho_i^n - \Delta t \left[\mathrm{v}_i^{n+1} \left(\frac{\rho_i^n - \rho_{i-1}^n}{r_i - r_{i-1}} \right) - \frac{\rho_i^n}{r_i^2} \left(\frac{r_{i+1/2}^2 \mathrm{v}_{i+1/2}^{n+1} - r_{i-1/2}^2 \mathrm{v}_{i-1/2}^{n+1}}{r_{i+1/2} - r_{i-1/2}} \right) \right] \quad (\mathrm{v} > 0) \end{aligned} \tag{A.2}$$

and similarly,

$$\begin{aligned} \rho u_i^{n+1} &= \rho u_i^n - \Delta t \left[\mathrm{v}_i^{n+1} \left(\frac{\rho u_{i+1}^n - \rho u_i^n}{r_{i+1} - r_i} \right) - \left[\frac{P_i^n + \rho u_i^n}{r_i^2} \right] \left(\frac{r_{i+1/2}^2 \mathrm{v}_{i+1/2}^{n+1} - r_{i-1/2}^2 \mathrm{v}_{i-1/2}^{n+1}}{r_{i+1/2} - r_{i-1/2}} \right) + \rho_i^n \Lambda_i \right] \quad (\mathrm{v} < 0) \\ &= \rho u_i^n - \Delta t \left[\mathrm{v}_i^{n+1} \left(\frac{\rho u_i^n - \rho u_{i-1}^n}{r_i - r_{i-1}} \right) - \left[\frac{P_i^n + \rho u_i^n}{r_i^2} \right] \left(\frac{r_{i+1/2}^2 \mathrm{v}_{i+1/2}^{n+1} - r_{i-1/2}^2 \mathrm{v}_{i-1/2}^{n+1}}{r_{i+1/2} - r_{i-1/2}} \right) + \rho_i^n \Lambda_i \right] \quad (\mathrm{v} > 0) \end{aligned}$$

where $\Delta t = t^{n+1} - t^n$ and the timestep is regulated according to the Courant condition

$$\Delta t < \frac{\min(\Delta r)}{\max(|\mathrm{v}|) + \max(c_s)} \tag{A.3}$$

where c_s is the adiabatic sound speed in the gas given by $c_s^2 = \gamma P/\rho$. We typically set Δt to half of this value.

Appendix B
SPH stability analysis

In this appendix we perform a stability analysis of the standard SPH formalism derived in §3.3. Since the SPH equations were derived directly from a variational principle, the linearised equations may be derived from a second order perturbation to the Lagrangian (3.46), given by

$$\delta L = \sum_b m_b \left[\frac{1}{2}\mathrm{v}_b^2 - \delta\rho_b \frac{du_b}{d\rho_b} - \frac{(\delta\rho_b)^2}{2}\frac{d^2u_b}{d\rho_b^2}\right] \tag{B.1}$$

where the perturbation to ρ is to second order in the second term and to first order in the third term. The density perturbation is given by a perturbation of the SPH summation (3.42), which to second order is given by[1]

$$\delta\rho_a = \sum_b m_b \delta x_{ab}\frac{\partial W_{ab}}{\partial x_a} + \sum_b m_b \frac{(\delta x_{ab})^2}{2}\frac{\partial^2 W_{ab}}{\partial x_a^2} \tag{B.2}$$

The derivatives of the thermal energy with respect to density follow from the first law of thermodynamics, ie.

$$\frac{du}{d\rho} = \frac{P}{\rho^2}, \qquad \frac{d^2u}{d\rho^2} = \frac{d}{d\rho}\left(\frac{P}{\rho^2}\right) = \frac{c_s^2}{\rho^2} - \frac{2P}{\rho^3}$$

The Lagrangian perturbed to second order is therefore

$$\delta L = \sum_b m_b \left[\frac{1}{2}\mathrm{v}_b^2 - \frac{P_b}{\rho_b^2}\sum_c m_c \frac{(\delta x_{bc})^2}{2}\frac{\partial^2 W_{bc}}{\partial x_a^2} - \frac{(\delta\rho_b)^2}{2\rho_b^2}\left(c_s^2 - \frac{2P_b}{\rho_b}\right)\right] \tag{B.3}$$

The perturbed momentum equation is given by using the perturbed Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathrm{v}_a}\right) - \frac{\partial L}{\partial(\delta x_a)} = 0. \tag{B.4}$$

where

$$\frac{\partial L}{\partial \mathrm{v}_a} = m_a \mathrm{v}_a \tag{B.5}$$

[1] Note that the first order term may be decoded into continuum form to give the usual expression

$$\delta\rho = -\rho_0 \nabla\cdot(\delta\mathbf{r})$$

where ρ_0 refers to the unperturbed quantity.

$$\begin{aligned}\frac{\partial L}{\partial(\delta x_a)} &= -m_a \sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2}\right) \delta x_{ab} \frac{\partial^2 W_{bc}}{\partial x_a^2} \\ &\quad - m_a \sum_b m_b \left[\left(c_s^2 - \frac{2P_b}{\rho_b}\right)\frac{\delta\rho_a}{\rho_a^2} + \left(c_s^2 - \frac{2P_b}{\rho_b}\right)\frac{\delta\rho_b}{\rho_b^2}\right] \frac{\partial W_{ab}}{\partial x_a} \end{aligned} \tag{B.6}$$

giving the SPH form of the linearised momentum equation

$$\begin{aligned}\frac{d^2\delta x_a}{dt^2} &= -\sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2}\right) \delta x_{ab} \frac{\partial^2 W_{bc}}{\partial x_a^2} \\ &\quad - \sum_b m_b \left[\left(c_s^2 - \frac{2P_b}{\rho_b}\right)\frac{\delta\rho_a}{\rho_a^2} + \left(c_s^2 - \frac{2P_b}{\rho_b}\right)\frac{\delta\rho_b}{\rho_b^2}\right] \frac{\partial W_{ab}}{\partial x_a} \end{aligned} \tag{B.7}$$

Equation (B.7) may also be obtained by a direct perturbation of the SPH equations of motion derived in §3.3.2. For linear waves the perturbations are assumed to be of the form

$$x = x_0 + \delta x, \tag{B.8}$$

$$\rho = \rho_0 + \delta\rho, \tag{B.9}$$

$$P = P_0 + \delta P. \tag{B.10}$$

where

$$\delta x_a = X e^{i(kx_a - \omega t)}, \tag{B.11}$$

$$\delta\rho_a = D e^{i(kx_a - \omega t)}, \tag{B.12}$$

$$\delta P_a = c_s^2 \delta\rho_a. \tag{B.13}$$

Assuming equal mass particles, the momentum equation (B.7) becomes

$$-\omega^2 X = -\frac{2mP_0}{\rho_0^2} X \sum_b \left[1 - e^{ik(x_b - x_a)}\right] \frac{\partial^2 W}{\partial x_a^2} - \frac{m}{\rho_0^2}\left(c_s^2 - \frac{2P_b}{\rho_b}\right) D \sum_b \left[1 + e^{ik(x_b - x_a)}\right] \frac{\partial W}{\partial x_a} \tag{B.14}$$

From the continuity equation (3.43) the amplitude D of the density perturbation is given in terms of the particle co-ordinates by

$$D = Xm \sum_b \left[1 - e^{ik(x_b - x_a)}\right] \frac{\partial W}{\partial x_a} \tag{B.15}$$

Finally, plugging this into (B.14) and taking the real component, the SPH dispersion relation (for any equation of state) is given by

$$\begin{aligned}\omega_a^2 &= \frac{2mP_0}{\rho_0^2} \sum_b [1 - \cos k(x_a - x_b)] \frac{\partial^2 W}{\partial x^2}(x_a - x_b, h) \\ &\quad + \frac{m^2}{\rho_0^2}\left(c_s^2 - \frac{2P_0}{\rho_0}\right)\left[\sum_b \sin k(x_a - x_b) \frac{\partial W}{\partial x}(x_a - x_b, h)\right]^2, \end{aligned} \tag{B.16}$$

For an isothermal equation of state this can be simplified further by setting $c_s^2 = P_0/\rho_0$. An adiabatic equation of state corresponds to setting $c_s^2 = \gamma P_0/\rho_0$.

Appendix C
Linear waves in MHD

In this section we describe the setup used for the MHD waves described in §4.6.4. The MHD equations in continuum form may be written as

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{v}, \tag{C.1}$$

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} - \frac{\mathbf{B}\times(\nabla\times\mathbf{B})}{\mu_0\rho}, \tag{C.2}$$

$$\frac{d\mathbf{B}}{dt} = (\mathbf{B}\cdot\nabla)\mathbf{v} - \mathbf{B}(\nabla\cdot\mathbf{v}), \tag{C.3}$$

together with the divergence constraint $\nabla\cdot\mathbf{B} = 0$. We perturb according to

$$\begin{aligned} \rho &= \rho_0 + \delta\rho, \\ \mathbf{v} &= \mathbf{v}, \\ \mathbf{B} &= \mathbf{B}_0 + \delta\mathbf{B}, \\ \delta P &= c_s^2\delta\rho. \end{aligned} \tag{C.4}$$

where $c_s^2 = \gamma P_0/\rho_0$ is the sound speed. Considering only linear terms, the perturbed equations are therefore given by

$$\frac{d(\delta\rho)}{dt} = -\rho_0(\nabla\cdot\mathbf{v}), \tag{C.5}$$

$$\frac{d\mathbf{v}}{dt} = -c_s^2\frac{\nabla(\delta\rho)}{\rho_0} - \frac{\mathbf{B}_0\times(\nabla\times\delta\mathbf{B})}{\mu_0\rho_0}, \tag{C.6}$$

$$\frac{d(\delta\mathbf{B})}{dt} = (\mathbf{B}_0\cdot\nabla)\mathbf{v} - \mathbf{B}_0(\nabla\cdot\mathbf{v}). \tag{C.7}$$

Specifying the perturbation according to

$$\begin{aligned} \delta\rho &= De^{i(\mathbf{k}x-\omega t)}, \\ \mathbf{v} &= \mathbf{v}e^{i(\mathbf{k}x-\omega t)}, \\ \delta\mathbf{B} &= \mathbf{b}e^{i(\mathbf{k}x-\omega t)}, \end{aligned} \tag{C.8}$$

we have

$$-\omega D = -\rho_0(\mathbf{v}\cdot\mathbf{k}) \tag{C.9}$$

$$
\begin{aligned}
-\omega \mathbf{v} &= -c_s^2 \frac{D\mathbf{k}}{\rho_0} - \frac{1}{\mu_0 \rho_0} \left[(\mathbf{B}_0 \cdot \mathbf{b})\mathbf{k} - (\mathbf{B}_0 \cdot \mathbf{k})\mathbf{b} \right] && \text{(C.10)} \\
-\omega \mathbf{b} &= (\mathbf{B}_0 \cdot \mathbf{k})\mathbf{v} - \mathbf{B}_0 (\mathbf{k} \cdot \mathbf{v}). && \text{(C.11)}
\end{aligned}
$$

Considering only waves in the x-direction (ie. $\mathbf{k} = [\mathrm{k}_x, 0, 0]$), defining the wave speed $v = \omega/\mathrm{k}$ and using (C.9) to eliminate D, equation (C.10) gives

$$
\begin{aligned}
v_x \left(v - \frac{c_s^2}{v} \right) &= \left(\frac{B_{y0} b_y + B_{z0} b_z}{\mu_0 \rho_0} \right), && \text{(C.12)} \\
v v_y &= -\frac{B_{x0} b_y}{\mu_0 \rho_0}, && \text{(C.13)} \\
v v_z &= -\frac{B_{x0} b_z}{\mu_0 \rho_0}, && \text{(C.14)}
\end{aligned}
$$

where $b_x = 0$ since $\nabla \cdot \mathbf{B} = 0$. Using these in (C.11) we have

$$
\begin{aligned}
v b_y &= -B_{x0} v_y + B_{y0} v_x, && \text{(C.15)} \\
v b_z &= -B_{x0} v_z + B_{z0} v_x. && \text{(C.16)}
\end{aligned}
$$

We can therefore solve for the perturbation amplitudes v_x, v_y, v_z, b_y and b_z in terms of the amplitude of the density perturbation D and the wave speed v. We find

$$
\begin{aligned}
v_x &= \frac{vD}{\rho} && \text{(C.17)} \\
v_y \left(v^2 - \frac{B_x^2}{\mu_0 \rho} \right) &= \frac{B_x B_y}{\mu_0 \rho} v_x && \text{(C.18)} \\
v_z \left(v^2 - \frac{B_x^2}{\mu_0 \rho} \right) &= \frac{B_x B_z}{\mu_0 \rho} v_x && \text{(C.19)} \\
b_y \left(v^2 - \frac{B_x^2}{\mu_0 \rho} \right) &= v B_y v_x && \text{(C.20)} \\
b_z \left(v^2 - \frac{B_x^2}{\mu_0 \rho} \right) &= v B_z v_x && \text{(C.21)}
\end{aligned}
$$

where we have dropped the subscript 0. The wave speed v is found by eliminating these quantities from (C.12), giving

$$
\frac{v_x}{(v^2 - B_x^2/\mu_0 \rho)} \left[v^4 - v^2 \left(c_s^2 + \frac{B_x^2 + B_y^2 + B_z^2}{\mu_0 \rho} \right) + \frac{c_s^2 B_x^2}{\mu_0 \rho} \right] = 0, \tag{C.22}
$$

which reveals the three wave types in MHD. The Alfvén waves are those with

$$
v^2 = \frac{B_x^2}{\mu_0 \rho}, \tag{C.23}
$$

These are transverse waves which travel along the field lines. The term in square brackets in (C.22) gives a quartic for v (or a quadratic for v^2), with roots

$$v^2 = \frac{1}{2}\left[\left(c_s^2 + \frac{B_x^2 + B_y^2 + B_z^2}{\mu_0 \rho}\right) \pm \sqrt{\left(c_s^2 + \frac{B_x^2 + B_y^2 + B_z^2}{\mu_0 \rho}\right)^2 - 4\frac{c_s^2 B_x^2}{\mu_0 \rho}}\right], \tag{C.24}$$

which are the fast(+) and slow(-) magnetosonic waves.