Average degree conditions forcing a minor

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June 8, 2015

Abstract

Mader first proved that high average degree forces a given graph as a minor. Often motivated by Hadwiger's Conjecture, much research has focused on the average degree required to force a complete graph as a minor. Subsequently, various authors have consider the average degree required to force an arbitrary graph H as a minor. Here, we strengthen (under certain conditions) a recent result by Reed and Wood, giving better bounds on the average degree required to force an H-minor when H is a sparse graph with many high degree vertices. This solves an open problem of Reed and Wood, and also generalises (to within a constant factor) known results when H is an unbalanced complete bipartite graph.

1   Introduction

Mader [13, 14] first proved that high average degree forces a given graph as a minor[1]. In particular, Mader [13, 14] proved that the following function is well-defined, where $d(G)$ denotes the average degree of a graph G:

$$f(H) := \inf\{D \in \mathbb{R} : \text{ every graph } G \text{ with } d(G) \geq D \text{ contains an } H\text{-minor}\}.$$

Often motivated by Hadwiger's Conjecture (see [18]), much research has focused on $f(K_t)$ where K_t is the complete graph on t vertices. For $3 \leq t \leq 9$, exact bounds on the number of edges due to Mader [14], Jørgensen [6], and Song and Thomas [19] imply that $f(K_t) = 2t - 4$. But this result does not hold for large t. In particular, $f(K_t) \in \Theta(t\sqrt{\ln t})$, where the lower bound is independently by Kostochka [7, 8] and de la Vega [4] (based on the work of Bollobás et al. [1]), and the upper bound is independently by Kostochka [7, 8] and Thomason [20]. Later, Thomason [21] determined the exact asymptotic constant.

Subsequently, other authors have considered $f(H)$ for arbitrary graphs H. Thomason [22] surveys some of these results. Myers and Thomason [16] determined an upper bound on $f(H)$ for all H, which is tight up to lower order terms when H is dense. Hence much recent work has focused on $f(H)$ for sparse graphs H. There have been two major approaches in this case.

Specific Sparse Graphs

The first approach is to consider specific sparse graphs H. For example, Myers [15] considered unbalanced complete bipartite graphs $K_{s,t}$ where $s \ll t$. It is easily seen that

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[1] A graph H is a *minor* of a graph G if a graph isomorphic to H can be constructed from G by vertex deletion, edge deletion and edge contraction.

$f(K_{1,t}) = t - 1$. Myers proved that $f(K_{2,t}) = t + 1$ for large t, and Chudnovsky et al. [3] proved the same result for all t. Kostochka and Prince [10] proved that $f(K_{3,t}) = t + 3$ for large t. In fact, for all these results, the authors determined the exact maximum number of edges in a $K_{s,t}$-minor-free graph (for $s \leq 3$).

Myers conjectured that $f(K_{s,t}) \leq c_s t$ for some constant c_s depending only on s. Strengthenings of this conjecture were independently proved by Kühn and Osthus [12] and Kostochka and Prince [9]. In particular, Kühn and Osthus [12] proved that for every $\epsilon \in (0, 10^{-16})$, if t is sufficiently large (with respect to ϵ) and $s \leq \epsilon^6 \frac{t}{\ln t}$, then

$$f(K_{s,t}) \leq (1 + \epsilon)t. \tag{1}$$

Kostochka and Prince [9] proved a sharper bound on $f(K_{s,t})$ under a stronger assumption: if $t > (180s \log_2 s)^{1+6s \log_2 s}$ then

$$t + 3s - 5\sqrt{s} \leq f(K_{s,t}) \leq t + 3s. \tag{2}$$

Later, Kostochka and Prince [11] proved an upper bound between (1) and (2) under a similar assumption to (1): if $s \leq \frac{t}{1000 \log_2 t}$ then

$$f(K_{s,t}) \leq t + 8s \log_2 s. \tag{3}$$

General Sparse Graphs

A second approach for sparse graphs is to determine upper bounds on $f(H)$ in terms of invariants of H. This is the approach of a recent paper by Reed and Wood [17]. Their main result is as follows: for every t-vertex graph H with average degree $d(H)$ at least some constant d_0, then

$$f(H) \leq 3.895\, t \, \sqrt{\ln d(H)}, \tag{4}$$

If $d(H)$ is very small, then this result is not applicable. For all H, Reed and Wood [17] proved that

$$f(H) \leq (1 + 3.146\, d(H))\, t. \tag{5}$$

Inequalities (4) and (5) imply that for some constant c, for every t-vertex graph H,

$$f(H) \leq ct\sqrt{\ln(d(H) + 2)}. \tag{6}$$

A lower bound of Myers and Thomason [16] shows that (6) is tight for random or close-to-regular graphs H. But for some graphs it is not tight. For example, for $K_{s,t}$ with $s \ll t$, inequality (6) says that $f(K_{s,t}) \leq ct\sqrt{\ln s}$ since $d(K_{s,t}) \approx 2s$, whereas $f(K_{s,t}) = \Theta(t)$ as discussed above.

Our Results

In this paper, we make a first attempt at unifying these two approaches, both improving the bounds of Reed and Wood [17] in certain cases, and generalising the above mentioned results on unbalanced complete bipartite graphs up to a constant factor.

Theorem 1. *There is a constant d_0 such that for integers $s \geq 0$ and $t \geq 3$ with $s \leq 10^{-5} \frac{t}{\ln t}$, for every $(s+t)$-vertex graph H and set S of s vertices in H such that $d(H-S) \geq d_0$,*

$$f(H) \leq 3.895\, t\sqrt{\ln d(H-S)}.$$

Theorem 1 solves the second open problem of Reed and Wood [17], and is an improvement over (4) when H contains a set S of vertices with high degree (compared to the average degree). Then Theorem 1 forces H as a minor in a graph G as long as $d(G) > 3.895\, t\sqrt{\ln d}$, where d is the average degree of $H - S$, instead of H itself. Since vertices in S have high degree, one expects that $d(H-S)$ is significantly less than $d(H)$.

Our second contribution relates $f(H)$ directly to $f(H-S)$.

Theorem 2. *For $\epsilon \in (0,1]$ and integers $s \geq 0$ and $t \geq 3$ with $s \leq \frac{\epsilon}{100} \frac{t}{\ln t}$, for every graph H and set S of s vertices in H,*

$$f(H) \leq 4\lceil (1+\epsilon)f(H-S) \rceil.$$

Theorem 2 may allow a better result that Theorem 1 if we happen to know a good upper bound on $f(H-S)$. It also makes no explicit assumption on $d(H-S)$. Theorems 1 and 2 together imply the following general result.

Theorem 3. *There is a constant c such that for integers $s \geq 0$ and $t \geq 3$ with $s \leq 10^{-5} \frac{t}{\ln t}$, for every $(s+t)$-vertex graph H and set S of vertices in H of size s,*

$$f(H) \leq ct\sqrt{\ln(d(H-S)+2)}.$$

Consider Theorem 3 with $H = K_{s,t}$, where S is the smaller colour class. Thus $d(H - S) = 0$ and Theorem 3 says that $f(H) \leq ct$, for some constant c independent of s, which is within a constant factor of the bounds in (1), (2) and (3). Note however that these older results are still stronger, since Theorem 3 has a large multiplicative constant. On the other hand, Theorem 3 applies more generally when $H - S$ is not an independent set.

The following section contains a few preliminary lemmas. Section 3 contains proofs of Theorems 1, 2 and 3. Section 4 contains an observation about $f(H)$ when H is series-parallel. Section 5 considers some future directions.

2 Preliminaries

We first prove a few useful preliminaries before proving our theorems.

A *connected dominating set* A of a graph G is a set of vertices such that the induced subgraph $G[A]$ is connected, and each vertex of $V(G)$ is either in A or adjacent to a vertex in A. Lemma 4 is well known, and weaker than other previous results such as that by Caro et al. [2]. We present it here for completeness and because the upper bound on the order of the connected dominating set has no lower order terms, which simplifies the calculations in Section 3. Similarly, Lemma 5 resembles a previous result of Kostochka and Prince [9].

Lemma 4. *Every graph G with n vertices and minimum degree at least $\frac{1}{2}n$ has a connected dominating set of order less than $2\log_2 n$.*

Proof. Let A be a set of $\lfloor \log_2 n \rfloor$ vertices in G chosen uniformly at random. For each vertex x of $G - A$, since $\deg_G(x) \geq \frac{n}{2}$, the probability that x has no neighbour in A is less than $(\frac{1}{2})^{\log_2 n} = \frac{1}{n}$, and so the probability that A does not dominate G is less than 1. Hence there is some choice of A that does dominate G. Label the vertices of this A by $x_1, \ldots, x_{\lfloor \log_2 n \rfloor}$. For $i = 1, \ldots, \lfloor \log_2 n \rfloor - 1$, let $v_i = x_{i+1}$ if x_i and x_{i+1} are adjacent, otherwise let v_i be a common neighbour of x_i and x_{i+1}, which exists since x_i and x_{i+1} each have at least $\frac{n}{2}$ neighbours in a set of $n - 2$ vertices. Then $A \cup \{v_1, \ldots, v_{\lfloor \log_2 n \rfloor - 1}\}$ is a connected dominating set of G with less than $2 \log_2 n$ vertices. □

Lemma 5. *For every integer $s \geq 1$ and $\rho \geq \frac{1}{2}$, every graph G with n vertices and minimum degree at least $\rho n + 2s \log_2 n$ contains vertex sets $A_1, \ldots, A_s$ and subgraphs $G_0, G_1, \ldots, G_s$ such that $G_0 = G$, $G_i = G - (A_1 \cup \cdots \cup A_i)$ for $i \in \{1, \ldots, s\}$, and*

(a) *A_i is a connected dominating set in G_{i-1},*

(b) *$|A_i| < 2 \log_2 n$,*

(c) *G_i has minimum degree at least $\rho n + 2(s - i) \log_2 n$.*

(d) *G_i has at least $n - 2i \log_2 n$ vertices.*

Proof. We use induction on i. Say $i = 1$. Let A_1 be a connected dominating set in $G_0 = G$ from Lemma 4. This satisfies (a) and (b). The minimum degree of $G_1 = G - A_1$ is at least the minimum degree of G minus $|A_1|$, which proves (c). The number of vertices of G_1 is at least $n - |A_1|$, which proves (d).

Assume our lemma holds for $i - 1$. Let A_i be a connected dominating set in G_{i-1} from Lemma 4; such a set exists since G_{i-1} has minimum degree greater than $\rho n \geq \frac{1}{2} n$. This satisfies (a) and (b). Finally, G_i has minimum degree at least $\rho n + 2(s - (i-1)) \log_2 n - |A_i| > \rho n - 2(s - i) \log_2 n$ and $|V(G_i)| \geq n - 2(i-1) \log_2 n - |A_i| > n - 2i \log_2 n$, as required. □

Finally, we cite two key results of Reed and Wood [17].

Lemma 6 (Reed and Wood [17], Lemma 2.5). *For every integer $k \geq 1$, every graph with average degree at least $4k$ contains a complete graph K_k as a minor or contains a minor with n vertices and minimum degree δ, where $\delta \geq 0.6518n$, and $k \leq \delta < n \leq 4k$.*

Lemma 7 (Reed and Wood [17], Lemma 5.1). *For all $\lambda \in (\frac{1}{2}, 1)$ and $\epsilon \in (0, \lambda)$ there exists $d_0(\lambda, \epsilon)$ such that for every graph H with t vertices and average degree $d \geq d_0$ every graph G with $n \geq (1 + \epsilon) \lceil \sqrt{\log_b d} \rceil t$ vertices and minimum degree at least λn contains H as a minor, where $b = (1 - \lambda + \epsilon)^{-1}$.*

3 Proofs of Theorems

Proof of Theorem 1. Let G be a graph with $d(G) \geq 3.895 \, t \sqrt{\ln d}$ where $d := d(H - S) \geq d_0$. Let $k := \lfloor \frac{1}{4}(3.895 \, t \sqrt{\ln d}) \rfloor$. Our goal is to show that G contains an H-minor. We can take d_0 large enough so that $k \geq 2t > 1$. By Lemma 6, G contains either K_{2t} or G' as a minor, where G' is a graph with n vertices and minimum degree at least $0.6518n$ such that $k + 1 \leq n \leq 3.895 \, t \sqrt{\ln d}$. If G contains a K_{2t} minor, then G contain a K_{s+t} minor and we are done. Otherwise apply Lemma 5 to G' where $\rho = 0.6517$. Since

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$s \le 10^{-5}\frac{t}{\ln t} < 5 \times 10^{-5}\frac{t}{\log_2 t} \le 5 \times 10^{-5}\frac{n}{\log_2 n}$, we have $2s \log_2 n \le 10^{-4}n$, and it follows that G' has minimum degree at least $\rho n + 2s \log_2 n$, as required. Let $G'' := G_s$ from Lemma 5. Then G'' has minimum degree at least $0.6517n > 0.6517|V(G'')|$.

We wish to find an $(H-S)$-minor of G''. Then contracting each A_i to a single vertex gives an H-minor in G'. We now verify that Lemma 7 gives the desired $(H-S)$-minor in G'', where $\lambda := 0.6517$ and $\epsilon := 10^{-6}$ and $b := (1 - \lambda + \epsilon)^{-1}$. Clearly G'' has sufficient minimum degree (by our choice of λ) and the average degree d of $H-S$ is sufficiently large (since we may assume d_0 is sufficiently large in terms of absolute constants), and so all that remains is to ensure that G'' has sufficiently many vertices. Note $|V(G'')| \ge n - 2s \log_2 n$ from Lemma 5(d). We may choose d_0 large enough so that $\frac{3.895\, t\sqrt{\ln d}}{\log_2(3.895\, t\sqrt{\ln d})} \ge 100\frac{t}{\ln t}$ and so $s \le 10^{-5}\frac{t}{\ln t} \le 10^{-7}\frac{3.895\, t\sqrt{\ln d}}{\log_2(3.895\, t\sqrt{\ln d})}$. Thus it follows that

$$
\begin{aligned}
|V(G'')| &\ge n - 2(10^{-7})\frac{3.895\, t\sqrt{\ln d}}{\log_2(3.895\, t\sqrt{\ln d})}\log_2 n \\
&\ge \lfloor \tfrac{1}{4}(3.895\, t\sqrt{\ln d})\rfloor + 1 - 2(10^{-7})(3.895\, t\sqrt{\ln d}) \\
&\ge \tfrac{1}{4}(3.895\, t\sqrt{\ln d}) - 2(10^{-7})(3.895\, t\sqrt{\ln d}) \\
&= (\tfrac{1}{4} - 2(10^{-7}))(3.895\, t\sqrt{\ln d}) \\
&= (\tfrac{1}{4} - 2(10^{-7}))(3.895\, t\sqrt{\ln b}\sqrt{\log_b d}) \\
&\ge 1.00002\sqrt{\log_b d}\, t \\
&= (1 + 20\epsilon)\sqrt{\log_b d}\, t \\
&\ge (1 + \epsilon)(1 + \sqrt{\log_b d})t \qquad \text{(taking d_0 large enough)} \\
&\ge (1 + \epsilon)\lceil \sqrt{\log_b d}\rceil\, t.
\end{aligned}
$$

Hence it follows that G'' contains an $(H-S)$-minor. This is an $(H-S)$-minor in G' that avoids the sets $A_1, \ldots, A_s$. Hence G' (and also G) contains our desired H-minor. $\qquad\square$

Proof of Theorem 2. Let G be a graph with average degree $d(G) \ge 4k$, where $k := \lceil(1+\epsilon)f(H-S)\rceil$. If $s = 0$ this result is trivial, so assume $s \ge 1$. It follows from Lemma 6 that G contains either a K_k-minor or a minor G' with n vertices and minimum degree $\delta(G') \ge 0.6518n$ such that $k \le \delta(G') < n \le 4k$. Since $f(H-S) \ge t-2$ (as K_{t-1} has no $(H-S)$-minor), it follows that $k \ge (1+\epsilon)f(H-S) \ge (1+\epsilon)(t-2) \ge t - 2 + 100s\ln t - 2\epsilon \ge t + 100s - 4 > t + s$. Hence a K_k-minor contains an H-minor. Now assume Lemma 6 finds G' as a minor.

We wish to apply Lemma 5 to G' with $\rho = \frac{1}{2}$. It is sufficient to show that $0.1518n \ge 2s \log_2 n$. Since $n \ge k+1 \ge (1+\epsilon)f(H-S)+1 \ge t-1+\epsilon f(H-S)$, and n and t are integers, it follows that $n \ge t$. Thus $s \le \frac{\epsilon}{100}\frac{t}{\ln t} \le \frac{\epsilon}{100}\frac{n}{\ln n}$ and so $2s \log_2 n \le \frac{n}{50 \ln 2} < 0.1518n$ as required. Let $G'' := G_s$ from Lemma 5. By Lemma 5(b), it follows that G'' has minimum degree at least $\delta(G') - 2s \log_2 n \ge k - 2s \log_2 n$. From our upper bound on s and since $k < n \le 4k$, it follows that $k - 2s \log_2 n \ge k - \epsilon\frac{n}{50 \ln n}\log_2 n \ge k(1 - \frac{2}{25\ln 2}\epsilon) \ge (1+\epsilon)(1 - 0.1155\epsilon)f(H-S) > f(H-S)$. Thus $d(G'') > f(H-S)$ and G'' contains an $(H-S)$-minor. Contracting the sets $A_1, \ldots, A_s$ to single vertices gives an H-minor in G', and thus also in G. $\qquad\square$

Proof of Theorem 3. First suppose that $d(H-S) \ge d_0$, where d_0 is from Theorem 1. Let $c \ge 3.895$. By Theorem 1, $f(H) \le 3.895\, t\sqrt{\ln d(H-S)} \le ct\sqrt{\ln(d(H-S)+2)}$ as

required. Alternatively, $d(H-S) < d_0$. By (5) we have $f(H-S) \leq (1+3.146d(H-S))t \leq (1+3.146d_0)t$. Let $\epsilon = 10^{-3}$. It follows from Theorem 2 that $f(H) \leq 4\lceil(1+\epsilon)(1+3.146d_0)t\rceil$. Setting c large enough, this proves $f(H) \leq ct\sqrt{\ln(d(H-S)+2)}$, as required. $\qquad\square$

4   Series-Parallel Graphs

A graph is *series-parallel* if it contains no K_4 minor (or equivalently, it has treewidth at most 2). Reed and Wood [17, Lemma 3.3] proved that $f(H) \leq 6.929t$ for every t-vertex 2-degenerate graph H. Every series-parallel graph H is 2-degenerate, implying $f(H) \leq 6.929t$. Here we make the following improvement.

Proposition 8. *For every t-vertex series-parallel graph H,*

$$f(H) \leq 2t - 4.$$

Proof. A *2-tree* is a graph that can be constructed by starting with K_3 and repeatedly selecting an edge and adding a new vertex adjacent to exactly the endpoints of that edge. It is well known that 2-trees are exactly the edge-maximal series-parallel graphs [5]. Hence it suffices to prove that $f(H) \leq 2t-4$ for every t-vertex 2-tree H. Say a graph G is *minor-minimal* (with respect to t) if $d(G) \geq 2t-4$ but every proper minor of G has average degree less than $2t-4$. Considering the effect of contracting an edge on the average degree, it is easily seen that every edge of a minor-minimal graph is in at least $t-2$ triangles; see [17, Lemma 2.1]. It suffices to prove that every minor-minimal graph G contains an H-minor. In fact, we prove that every 2-tree H_0 on at most t vertices can be embedded in G as a subgraph.

We do this by induction on $|V(H_0)|$ with G fixed. Suppose $|V(H_0)| = 3$. Then every edge of G is in at least $t-2 \geq 1$ triangles, and it is trivial to embed H_0. Now suppose $|V(H_0)| = i > 3$. There is a vertex $v \in V(H_0)$ with neighbours x and y such that $H_0 - v$ is also a 2-tree. By induction, $H_0 - v$ can be embedded in G. Let x', y' denote the vertices of G where x, y are respectively embedded. Then $x'y'$ is an edge of G in at least $t-2 \geq |V(H_0) - \{v, x, y\}| + 1$ triangles. Hence there exists a common neighbour w of x' and y' in G where no vertex of $H_0 - \{v, x, y\}$ is embedded. Clearly, neither x nor y are embedded at w. Embedding v at w, we obtain H_0 as a subgraph of G, as required. $\qquad\square$

5   Open Problems

An obvious question is whether the upper bound on $f(H)$ provided by Theorem 3 is within a constant factor of optimal for all H. If S is a small (perhaps empty) set of vertices in H and $H - S$ is sufficiently large and either random or close-to-regular, then it follows from the lower bound of Myers and Thomason [16] that

$$f(H) \geq f(H-S) \geq c|V(H-S)|\sqrt{\ln(d(H-S))},$$

and Theorem 3 is within a constant factor of optimal.

Is it possible that Theorem 3 always gives a result within a constant factor of optimal (for the best possible choice of S)? If not, for which graphs is Theorem 3 not within a constant factor of optimal, and what approach should we take for such graphs? Is it

possible to determine a small set of "techniques" for upper bounding $f(H)$ such that for each graph H, one of these bounds is within a constant factor of optimal? Similarly, is there a constant factor approximation algorithm for computing $f(H)$?

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