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# NONREPETITIVE COLOURING VIA ENTROPY COMPRESSION

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A vertex colouring of a graph is *nonrepetitive* if there is no path whose first half receives the same sequence of colours as the second half. A graph is nonrepetitively $\ell$-choosable if given lists of at least $\ell$ colours at each vertex, there is a nonrepetitive colouring such that each vertex is coloured from its own list. It is known that, for some constant $c$, every graph with maximum degree $\Delta$ is $c\Delta^2$-choosable. We prove this result with $c=1$ (ignoring lower order terms). We then prove that every subdivision of a graph with sufficiently many division vertices per edge is nonrepetitively 5-choosable. The proofs of both these results are based on the Moser-Tardos entropy-compression method, and a recent extension by Grytczuk, Kozik and Micek for the nonrepetitive choosability of paths. Finally, we prove that graphs with pathwidth $\theta$ are nonrepetitively $\mathcal{O}(\theta^2)$-colourable.

## 1. Introduction

A colouring of a graph[1] is *nonrepetitive* if there is no path $P$ such that the first half of $P$ receives the same sequence of colours as the second half of $P$.

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[1] We consider simple, finite, undirected graphs $G$ with vertex set $V(G)$, edge set $E(G)$, maximum degree $\Delta(G)$, and maximum clique size $\omega(G)$. The *neighbourhood* of a vertex $v \in V(G)$ is $N_G(v) := \{w \in V(G) : vw \in E(G)\}$. The *neighbourhood* of a set $S \subseteq V(G)$ is $N_G(S) := \bigcup\{N_G(x) : x \in S\} \setminus S$. The degree of a vertex $v \in V(G)$ is $\deg_G(v) := |N_G(v)|$. We use $N(v)$ and $N(S)$ and $\deg(v)$ if the graph $G$ is clear from the context.

More precisely, a *$k$-colouring* of a graph $G$ is a function $\psi$ that assigns one of $k$ colours to each vertex of $G$. A path is *even* if its order is even. An even path $v_1, v_2, \dots, v_{2t}$ of $G$ is *repetitively* coloured by $\psi$ if $\psi(v_i) = \psi(v_{t+i})$ for all $i \in [1,t] := \{1,2,\dots,t\}$. A colouring $\psi$ is *nonrepetitive* if no path of $G$ is repetitively coloured by $\psi$. The *nonrepetitive chromatic number* $\pi(G)$ is the minimum integer $k$ such that $G$ has a nonrepetitive $k$-colouring.

Observe that every nonrepetitive colouring is *proper*, in the sense that adjacent vertices receive distinct colours. Moreover, a nonrepetitive colouring has no 2-coloured $P_4$ (a path on four vertices). A proper colouring with no 2-coloured $P_4$ is called a *star colouring* since each bichromatic subgraph is a star forest; see [1, 8, 18, 21, 40, 48]. The *star chromatic number* $\chi_{\text{st}}(G)$ is the minimum number of colours in a star colouring of $G$. Thus

$$\chi(G) \leq \chi_{\text{st}}(G) \leq \pi(G).$$

The seminal result in this field is by Thue [47], who in 1906 proved[2] that every path is nonrepetitively 3-colourable. Nonrepetitive colourings have recently been widely studied [2–7, 9, 10, 12–15, 19, 22–27, 29–32, 34–38, 41, 43, 44]; see the surveys [12, 23–25].

The contributions of this paper concern three different generalisations of the result of Thue: bounded degree graphs, graph subdivisions, and graphs of bounded pathwidth.

### 1.1. Bounded Degree

In a sweeping generalisation of Thue's result, Alon et al. [3] proved[3] that for some constant $c$ and for every graph $G$ with maximum degree $\Delta \geq 1$,

$$\pi(G) \leq c\,\Delta^2. \tag{1}$$

Moreover, the bound in (1) is almost tight – Alon et al. [3] proved that there are graphs with maximum degree $\Delta$ that are not nonrepetitively $(c\Delta^2/\log\Delta)$-colourable for some constant $c$.

The bound in (1), in fact, holds in the stronger setting of nonrepetitive list colourings. A *list assignment* of a graph $G$ is a function $L$ that assigns

[2] A nonrepetitive 3-colouring of $P_n$ is obtained as follows. Given a nonrepetitive sequence over $\{1,2,3\}$, replace each 1 by the sequence 12312, replace each 2 by the sequence 131232, and replace each 3 by the sequence 1323132. It follows that the new sequence is nonrepetitive. Thus arbitrarily long paths can be nonrepetitively 3-coloured.

[3] The theorem of Alon et al. [3] was actually for edge colourings, but it is easily seen that the method works in the more general setting of vertex colourings.

a set $L(v)$ of colours to each vertex $v \in V(G)$. Then $G$ is *nonrepetitively $L$-colourable* if there is a nonrepetitive colouring of $G$, such that each vertex $v \in V(G)$ is assigned a colour in $L(v)$. And $G$ is *nonrepetitively $\ell$-choosable* if for every list assignment $L$ of $G$ such that $|L(v)| \geq \ell$ for each vertex $v \in V(G)$, there is a nonrepetitive $L$-colouring of $G$. The *nonrepetitive choice number* $\pi_{\text{ch}}(G)$ is the minimum integer $\ell$ such that $G$ is nonrepetitively $\ell$-choosable. Clearly, $\pi(G) \leq \pi_{\text{ch}}(G)$.

All the known proofs of (1) are based on the Lovász Local Lemma, and thus are easily seen to prove the stronger result that

$$\pi_{\text{ch}}(G) \leq c\,\Delta(G)^2. \tag{2}$$

Alon et al. [3] originally proved (2) with $c = 2e^{16}$, which was improved to 36 by Grytczuk [24] and then to 16 again by Grytczuk [23]. Prior to the present paper, the best bound was $\pi_{\text{ch}}(G) \leq 12.92(\Delta(G)-1)^2$ by Harant and Jendrol' [29] (assuming $\Delta(G) \geq 2$). We improve the constant $c$ to 1.

**Theorem 1.** *For every graph $G$ with maximum degree $\Delta$,*

$$\pi_{\text{ch}}(G) \leq (1 + o(1))\Delta^2.$$

The proof of Theorem 1 is based on the celebrated entropy-compression method of Moser and Tardos [39], and more precisely on an extension by Grytczuk et al. [26] for nonrepetitive sequences (or equivalently, nonrepetitive colourings of paths). The latter authors considered the following variant of the Moser-Tardos algorithm for nonrepetitively colouring paths. Start at the first vertex of the path and repeat the following step until a valid colouring is produced: Randomly colour the current vertex. If doing so creates a repetitively coloured subpath $P$, then uncolour the second half of $P$ and let the new current vertex be the first uncoloured vertex on the path. Otherwise, go to the next vertex in the path. Grytczuk et al. [26] used this algorithm to obtain a short proof that paths are nonrepetitively 4-choosable, which was first proved by Grytczuk et al. [27] using the Lovász Local Lemma. (It is open whether every path is nonrepetitively 3-choosable.) Our proof of Theorem 1 generalises this method for graphs of bounded degree. While the main conclusion of the Moser-Tardos method was a constructive proof of the Lovász Local Lemma, as Kolipaka and Szegedy [33] write, "variants of the Moser-Tardos algorithm can be useful in existence proofs". Our result is further evidence of this claim.

Note that the methods developed in the proof of Theorem 1 have subsequently been applied to other graph colouring problems [17, 45, 46] and also in pattern avoidance [42].

### 1.2. Subdivisions

A *subdivision* of a graph $G$ is a graph obtained from $G$ by replacing the edges of $G$ with internally disjoint paths, where the path replacing $vw$ has endpoints $v$ and $w$. In a beautiful generalisation of Thue's theorem, Pezarski and Zmarz [44] proved that every graph has a nonrepetitively 3-colourable subdivision (improving on analogous 5- and 4-colour results by Grytczuk [23] and Barát and Wood [7], respectively). For each of these theorems, the number of division vertices per edge is $\mathcal{O}(n)$ or $\mathcal{O}(n^2)$ for $n$-vertex graphs. Improving these bounds, Nešetřil et al. [41] proved that every graph has a nonrepetitively 17-colourable subdivision with $\mathcal{O}(\log n)$ division vertices per edge, and that $\Omega(\log n)$ division vertices are needed on some edge of a nonrepetitively $\mathcal{O}(1)$-colourable subdivision of $K_n$. Here we prove that every graph has a nonrepetitively $\mathcal{O}(1)$-*choosable* subdivision, which solves an open problem by Grytczuk et al. [27]. All logarithms are binary.

**Theorem 2.** *Let $G$ be a subdivision of a graph $H$, such that each edge $vw \in E(H)$ is subdivided at least $\lceil 10^5 \log(\deg(v)+1)\rceil + \lceil 10^5 \log(\deg(w)+1)\rceil + 2$ times in $G$. Then*

$$\pi_{\mathrm{ch}}(G) \leq 5.$$

Theorem 2 is stronger than the above subdivision results in the following respects: (1) it is for choosability not just colourability; (2) it applies to every subdivision with *at least* a certain number of division vertices per edge, and (3) the required number of division vertices per edge is asymptotically fewer than for the above results. Of course, Theorem 2 is weaker than the results in [7, 44] mentioned above in that the number of colours is 5.

Theorem 2 is also proved using the entropy-compression method mentioned above. An analogous theorem with more colours and $\mathcal{O}(\log \Delta(G))$ division vertices per edges can be proved using the Lovász Local Lemma; see [16, Appendix A].

### 1.3. Pathwidth

Thue's result was generalised in a different direction by Brešar et al. [9], who proved that every tree is nonrepetitively 4-colourable[4]. This result was

[4] No such result is possible for choosability – Fiorenzi et al. [19] proved that trees have arbitrarily high nonrepetitive choice number. On the other hand, Kozik and Micek [35] proved that $\pi_{\mathrm{ch}}(T) \leq \mathcal{O}(\Delta^{1+\varepsilon})$ for every tree $T$ with maximum degree $\Delta$.

further generalised by considering treewidth[5], which is a parameter that measures how similar a graph is to a tree. Kündgen and Pelsmajer [36] and Barát and Varjú [5] independently proved that graphs of bounded treewidth have bounded nonrepetitive chromatic number. The best upper bound is due to Kündgen and Pelsmajer [36], who proved that $\pi(G) \leq 4^{\theta}$ for every graph $G$ with treewidth $\theta$. The best lower bound is due to Albertson et al. [1], who described graphs $G$ with treewidth $\theta$ and $\pi(G) \geq \chi_{\rm st}(G) = \binom{\theta+2}{2}$. Thus there is a quadratic lower bound on $\pi$ in terms of treewidth. It is open whether $\pi$ is bounded from above by a polynomial function of treewidth. We prove the following related result.

**Theorem 3.** *For every graph $G$ with pathwidth $\theta$,*

$$\pi(G) \leq 2\theta^2 + 6\theta + 1.$$

It is open whether $\pi(G) \in \mathcal{O}(\theta)$ for every graph $G$ with pathwidth $\theta$. For treewidth, a quadratic lower bound on $\pi$ follows from the quadratic lower bound on $\chi_{\rm st}$, as explained above. However, we show that no such result holds for pathwidth.

**Theorem 4.** *For every graph $G$ with pathwidth $\theta$,*

$$\chi_{\rm st}(G) \leq 3\theta + 1.$$

## 2. An algorithm

This section presents and analyses an algorithm for nonrepetitively list colouring a graph. This machinery will be used to prove Theorems 1 and 2 in the following sections.

If a set $X$ is linearly ordered according to some fixed ordering and $e \in X$, then the *index* of $e$ in $X$ is the position of $e$ in this ordering of $X$. Such an ordering induces in a natural way an ordering of each subset $Y$ of $X$, so that the index of an element $e \in Y$ *in* $Y$ is well defined.

Let $G$ be a fixed $n$-vertex graph. Assume that $V(G)$ is ordered according to some arbitrary linear ordering. Let $L$ be a list assignment for $G$. Assume

[5] A *tree decomposition* of a graph $G$ consists of a tree $T$ and a set $\{B_x \subseteq V(G) \colon x \in V(T)\}$ of subsets of vertices of $G$, called *bags*, indexed by the vertices of $T$, such that (1) the endpoints of each edge of $G$ appear in some bag, and (2) for each vertex $v$ of $G$ the set $\{x \in V(T) \colon v \in B_x\}$ is nonempty and induces a connected subtree of $T$. The *width* of the tree decomposition is $\max\{|B_x| - 1 \colon x \in V(T)\}$. The *treewidth* of $G$ is the minimum width of a tree decomposition of $G$. A *path decomposition* is a tree decomposition whose underlying tree is a path. Thus a path decomposition is simply defined by a sequence of bags $B_1, \dots, B_p$. The *pathwidth* of $G$ is the minimum width of a path decomposition of $G$.

each list in $L$ has size $\ell$. Identify colours with nonnegative integers. Thus the colours in $L(v)$ are ordered in a natural way, for each $v \in V(G)$. Without loss of generality, the colour 0 is in none of these lists. In what follows, we consider an uncoloured vertex to be coloured 0. A *precolouring* of $G$ is a colouring $\psi$ of $G$ such that $\psi(v) \in L(v) \cup \{0\}$ for each $v \in V(G)$. If $\psi(v) \neq 0$, then $v$ is said to be *precoloured* by $\psi$.

For each path $P$ of $G$ with $2k$ vertices, for each subset $X \subseteq V(G) - V(P)$, and for each vertex $v \in V(P)$, define $\lambda(P,X,v)$ to be the sequence $(\lambda_1,\dots,\lambda_{2k})$ obtained as follows: Let $x,y$ be the two endpoints of $P$, with $v$ closer to $y$ than to $x$ in $P$. Let $v_1,\dots,v_p$ be the vertices of $P$ from $v_1 := v$ to $v_p := x$ defined by $P$, in order. (Observe that $p \geq 2$ since $v \neq x$.) Let $\lambda_1$ be the index of $v_2$ in $N(v_1) - X$, and for each $i \in [2,p-1]$, let $\lambda_i$ be the index of $v_{i+1}$ in $N(v_i) - (X \cup \{v_{i-1}\})$. Let $\lambda_p := -1$. If $v = y$, then $p = 2k$ and the sequence $(\lambda_1,\dots,\lambda_{2k})$ is completely defined. If $v \neq y$, then let $q := 2k - p + 1$ and let $w_1,\dots,w_q$ be the vertices of $P$ from $w_1 := v$ to $w_q := y$ defined by $P$, in order. Let $\lambda_{p+1}$ be the index of $w_2$ in $N(w_1) - (X \cup \{v_2\})$, and for each $i \in [2,q-1]$, let $\lambda_{p+i}$ be the index of $w_{i+1}$ in $N(w_i) - (X \cup \{w_{i-1}\})$.

An important feature of the above encoding of the triple $(P,X,v)$ as a sequence $\lambda(P,X,v)$ is that it can be reversed, as we now explain.

**Lemma 5.** *Suppose $\lambda = \lambda(P,X,v)$ for some even path $P$ of $G$ such that $X \subseteq V(G) - V(P)$, and $v \in V(P)$. Then, given $\lambda$, $X$, and $v$, one can uniquely determine the path $P$.*

**Proof.** Let $\lambda = (\lambda_1,\dots,\lambda_{2k})$ and let $p \in [2,2k]$ be the unique index such that $\lambda_p = -1$. Let $u_p := v$, let $u_{p-1}$ be the $\lambda_1$-th vertex in $N(u_p) - X$, and for $i = p-2,\dots,1$, let $u_i$ be the $\lambda_{p-i}$-th vertex in $N(u_{i+1}) - (X \cup \{u_{i+2}\})$. Next, for $j = p+1,\dots,2k$, let $u_j$ be the $\lambda_j$-th vertex in $N(u_{j-1}) - (X \cup \{u_{j-2}\})$. Then the vertices $u_1,u_2,\dots,u_{2k}$, in this order, determine a path $P$ of $G$ such that $\lambda(P,X,v) = \lambda$.

Observe that if $P'$ is an even path of $G$ such that $X \subseteq V(G) - V(P')$, $v \in V(P')$, and $P'$ is distinct from $P$, then $\lambda(P',X,v) \neq \lambda(P,X,v)$. Therefore, the path $P$ above is uniquely determined. ∎

Let $\Lambda$ be the set of all sequences $\lambda(P,X,v)$ where $P$ is an even path in $G$, $X \subseteq V(G) - V(P)$, and $v$ is a vertex of $P$. A *record* is a mapping $R\colon \mathbb{N} \to \Lambda \cup \{\emptyset\}$. The *empty record* is the record $R$ such that $R(i) = \emptyset$ for all $i \in \mathbb{N}$.

A *priority function* is a function $f$ that associates to each nonempty subset $X$ of $V(G)$ a vertex $f(X) \in X$. Consider Algorithm 1, which (for a fixed graph $G$, a list assignment $L$, a priority function $f$, and a precolouring $\psi$) takes as input a positive integer $t$ and a vector $(c_1,\dots,c_t) \in [1,\ell]^t$. Note that precoloured vertices and a specific priority function will only be needed when proving the result on subdivisions. We thus invite the reader to first consider the set $Q$ of precoloured vertices to be empty, and the priority function $f$ to be arbitrary (for instance, $f(X)$ could be the first vertex in $X$ in the fixed ordering of $V(G)$). Also note that the choice of the repetitively coloured path $P$ in the algorithm is assumed to be consistent; that is, according to some (arbitrary) fixed deterministic rule.

**Algorithm 1** $L$-colouring the graph $G$, where $f$ is a priority function, $\psi$ is a precolouring of $G$, and $Q$ is the set of precoloured vertices under $\psi$.

```
Input: (c_1,...,c_t) ∈ [1,ℓ]^t
Output: a (possibly invalid) colouring φ and a record R
  i ← 1
  φ ← ψ
  R ← empty record
  X ← V(G) − Q
  while i ≤ t and X ≠ ∅ do
      v ← f(X)
      φ(v) ← c_i-th colour in L(v)
      X ← X − {v}
      if G contains a repetitively coloured path P then
          divide P into first half P_1 and second half P_2 so that v ∈ V(P_2)
          for w ∈ V(P_2) − Q do
              φ(w) ← 0
          end for
          R(i) ← λ(P,X,v)
          X ← X ∪ (V(P_2) − Q)
      else
          R(i) ← ∅
      end if
      i ← i + 1
  end while
  return return φ, R
```

Say that the algorithm *succeeds* if it terminates with $X = \emptyset$, and *fails* otherwise. It is easily seen that if the algorithm succeeds, then the produced colouring $\phi$ is a nonrepetitive $L$-colouring of $G$. For a given integer $t \geq 1$, let $\mathcal{F}_t$ be the set of vectors $(c_1,\dots,c_t) \in [1,\ell]^t$ on which the algorithm fails. Let $\mathcal{A}_t$ be the set of pairs $(\phi, R)$ that are produced by the algorithm on vectors in $\mathcal{F}_t$. Let $\mathcal{R}_t$ be the set of distinct records $R$ that can be produced by the algorithm on vectors in $\mathcal{F}_t$. For $R \in \mathcal{R}_t$, let $\mathcal{F}_{t,R}$ be the set of vectors $(c_1,\dots,c_t) \in \mathcal{F}_t$ on which the algorithm produces record $R$. (Thus $\mathcal{F}_{t,R} \neq \emptyset$.) For a vector

$(c_1,\dots,c_t)\in\mathcal{F}_t$, let the *trace* $\mathrm{tr}(c_1,\dots,c_t)$ be the vector $(X_1,\dots,X_t)$ where $X_i$ is the set $X$ at the beginning of the $i$-th iteration of the while-loop of the algorithm on input $(c_1,\dots,c_t)$, for each $i\in[1,t]$. (Observe that $X_1$ always equals $V(G)-Q$.) Finally, let $\mathcal{T}_t:=\{\mathrm{tr}(c_1,\dots,c_t)\colon (c_1,\dots,c_t)\in\mathcal{F}_t\}$.

The next lemma shows that for a fixed record $R\in\mathcal{R}_t$, all the vectors in $\mathcal{F}_{t,R}$ have the same trace.

**Lemma 6.** *For every $t\geq 1$ there exists a function $h_t\colon\mathcal{R}_t\to\mathcal{T}_t$ such that for each $R\in\mathcal{R}_t$ and each $(c_1,\dots,c_t)\in\mathcal{F}_{t,R}$ we have* $\mathrm{tr}(c_1,\dots,c_t)=h_t(R)$.

**Proof.** We construct $h_t$ by induction on $t$. For $t=1$ simply let $h_t(R):=(V(G)-Q)$ for each $R\in\mathcal{R}_t$.

Now assume that $t\geq 2$. Let $R\in\mathcal{R}_t$. Let $R'$ be the record obtained from $R$ by setting $R'(i):=R(i)$ for each $i\in\mathbb{N}$ such that $i\neq t$, and $R'(t):=\emptyset$. Then $R'\in\mathcal{R}_{t-1}$, and by induction, $h_{t-1}(R')=(X_1,\dots,X_{t-1})$ for some $(X_1,\dots,X_{t-1})\in\mathcal{T}_{t-1}$. Let $v_{t-1}:=f(X_{t-1})$.

First suppose that $R(t-1)=\emptyset$. Let $X_t:=X_{t-1}-\{v_{t-1}\}$ and $h_t(R):=(X_1,\dots,X_{t-1},X_t)$. Consider a vector $(c_1,\dots,c_t)\in\mathcal{F}_{t,R}$. Then $(c_1,\dots,c_{t-1})\in\mathcal{F}_{t-1,R'}$, and by induction $\mathrm{tr}(c_1,\dots,c_{t-1})=(X_1,\dots,X_{t-1})$. Thus at the beginning of the $(t-1)$-th iteration of the while-loop in the algorithm on input $(c_1,\dots,c_t)$, the current record is $R'$, and $v=v_{t-1}$ and $X=X_{t-1}$. Since $R(t-1)=\emptyset$, the algorithm subsequently coloured $v_{t-1}$ without creating any repetitively coloured path, implying that $X=X_{t-1}-\{v_{t-1}\}=X_t$ at the beginning of the $t$-th iteration. Hence $\mathrm{tr}(c_1,\dots,c_t)=(X_1,\dots,X_{t-1},X_t)=h_t(R)$, as desired.

Now assume that $R(t-1)=\lambda$ for some $\lambda\in\Lambda$. Using Lemma 5, let $P$ be the path of $G$ determined by $\lambda$, $X_{t-1}$, and $v_{t-1}$. Let $P_1$ and $P_2$ denote the two halves of $P$, so that $v_{t-1}\in V(P_2)$. Let $X_t:=X_{t-1}\cup(V(P_2)-Q)$ and $h_t(R):=(X_1,\dots,X_{t-1},X_t)$. Consider a vector $(c_1,\dots,c_t)\in\mathcal{F}_{t,R}$. Then $(c_1,\dots,c_{t-1})\in\mathcal{F}_{t-1,R'}$, and by induction $\mathrm{tr}(c_1,\dots,c_{t-1})=(X_1,\dots,X_{t-1})$. As before, at the beginning of the $(t-1)$-th iteration of the while-loop in the algorithm on input $(c_1,\dots,c_t)$, the current record is $R'$, and $v=v_{t-1}$ and $X=X_{t-1}$. Then, after colouring $v$, the path $P$ is repetitively coloured, and all vertices in $P_2$ are subsequently uncoloured, except for those in $Q$. Hence we have $X=X_{t-1}\cup(V(P_2)-Q)=X_t$ at the beginning of the $t$-th iteration. Therefore $\mathrm{tr}(c_1,\dots,c_t)=(X_1,\dots,X_{t-1},X_t)=h_t(R)$. ∎

**Lemma 7.** *For every $(\phi,R)\in\mathcal{A}_t$ there is a unique vector $(c_1,\dots,c_t)\in\mathcal{F}_t$ such that the algorithm produces $(\phi,R)$ on input $(c_1,\dots,c_t)$.*

**Proof.** The proof is by induction on $t$. This claim is true for $t=1$ since in that case the unique vector $(c_1)\in\mathcal{F}_1$ yielding $(\phi,R)$ is the one where $c_1$ is the index of colour $\phi(v_1)$ in the list $L(v_1)$, where $v_1:=f(V(G)-Q)$.

Now assume that $t \geq 2$. Let $(X_1,\dots,X_t) := h_t(R)$, where $h_t$ is the function in Lemma 6. Let $v_t := f(X_t)$. (Recall that $X_t \neq \emptyset$.) Let $R'$ be the record obtained from $R$ by setting $R'(i) := R(i)$ for each $i \in \mathbb{N}$ such that $i \neq t$, and $R'(t) := \emptyset$. Then $R' \in \mathcal{R}_{t-1}$, and $h_{t-1}(R') = (X_1,\dots,X_{t-1})$.

First suppose that $R(t) = \emptyset$. Let $\phi'$ be the colouring obtained from $\phi$ by setting $\phi'(v_t) := 0$ and $\phi'(w) := \phi(w)$ for each $w \in V(G) - \{v_t\}$. Then $(\phi', R') \in \mathcal{A}_{t-1}$, and by induction there is a unique input vector $(c'_1,\dots,c'_{t-1}) \in \mathcal{F}_{t-1}$ for which the algorithm produces $(\phi', R')$. It follows that every vector $(c_1,\dots,c_t) \in \mathcal{F}_t$ resulting in the pair $(\phi, R)$ satisfies $c_i = c'_i$ for each $i \in [1, t-1]$. But $c_t$ is also uniquely determined, since it is the index of colour $\phi(v_t)$ in the list $L(v_t)$. Hence, there is a unique such vector $(c_1,\dots,c_t)$.

Now assume that $R(t) = \lambda$ for some $\lambda \in \Lambda$. Using Lemma 5, let $P$ be the path of $G$ determined by $\lambda$, $X_t$, and $v_t$. Let $P_1$ and $P_2$ denote the two halves of $P$, so that $v_t \in V(P_2)$. Let $w_1,\dots,w_{2k}$ denote the vertices of $P$, in order, so that $V(P_1) = \{w_1,\dots,w_k\}$ and $V(P_2) = \{w_{k+1},\dots,w_{2k}\}$. Let $j \in [1,k]$ be the index such that $w_{k+j} = v_t$. Let $\phi'$ be the colouring obtained from $\phi$ by setting $\phi'(v_t) := 0$, $\phi'(w_{k+i}) := \phi(w_i)$ for each $i \in [1,k]$ such that $i \neq j$ and $w_{k+i} \notin Q$, and $\phi'(w) := \phi(w)$ for each $w \in V(G) - (V(P_2) - Q)$. Then $(\phi', R') \in \mathcal{A}_{t-1}$, and by induction there is a unique vector $(c'_1,\dots,c'_{t-1}) \in \mathcal{F}_{t-1}$ on the input of which the algorithm produces $(\phi', R')$. It follows that every vector $(c_1,\dots,c_t) \in \mathcal{F}_t$ resulting in the pair $(\phi, R)$ satisfies $c_i = c'_i$ for each $i \in [1, t-1]$. Moreover, $c_t$ is the index of colour $\phi(v_t)$ in the list $L(v_t)$, and therefore is also uniquely determined. ∎

Lemma 7 implies that $|\mathcal{A}_t| = |\mathcal{F}_t|$ for all $t \geq 1$.

Recall that $\Lambda$ is the set of all sequences $\lambda(P,X,v)$ where $P$ is an even path in $G$, $X \subseteq V(G) - V(P)$, and $v \in V(P)$. Once we fix a precolouring $\psi$ of $G$ and a priority function $f$, as we did above, some triples $(P,X,v)$ will never be considered by the algorithm on any input. (For instance, this is the case if $X$ contains a precoloured vertex.) This leads to the following definition. A sequence $\lambda \in \Lambda$ is *realisable* (with respect to $\psi$ and $f$) if $R(i) = \lambda$ for some $t \geq 1$, $R \in \mathcal{R}_t$, and $i \in [1,t]$. For each $k \in [1, \lfloor \frac{n}{2} \rfloor]$, let $\alpha_k$ be the number of realisable sequences of length $2k$ in $\Lambda$. Define

$$\beta := \max\{1, \max\{(\alpha_k)^{1/k} \colon 1 \leq k \leq \lfloor \tfrac{n}{2} \rfloor\}\}.$$

Thus $\beta \geq 1$ and $\alpha_k \leq \beta^k$ for each $k \in [1, \lfloor \frac{n}{2} \rfloor]$.

A *substring* of some sequence or word is a subsequence of consecutive elements. A *prefix* of a sequence is a substring starting at the first element. A *Dyck word* of length $2t$ is a binary sequence with $t$ zeroes and $t$ ones such that the number of zeroes is at least the number of ones in every prefix of the sequence.

Let $R \in \mathcal{R}_t$. That is, $R$ is a record that can be produced by the algorithm on some vector $(c_1,\dots,c_t) \in [1,\ell]^t$ on which the algorithm fails. By Lemma 6, $\operatorname{tr}(c_1,\dots,c_t) = h_t(R)$. That is, the vector $(X_1,\dots,X_t)$ is determined by $R$, where $X_i$ is the set $X$ at the beginning of the $i$-th iteration of the while-loop of the algorithm on input $(c_1,\dots,c_t)$. For each $i \in [1,t]$, let $r_i := |X_{i+1}| - |X_i| + 1$. Note that $r_i$ is the number of vertices that are uncoloured at step $i$. In particular, at step $i$, if $G$ contained no repetitively coloured path, then $r_i = 0$, and if $G$ contained a repetitively coloured path $P$ with second half $P_2$, then $r_i = |V(P_2) - Q|$. We emphasise, however, that $r_i$ is determined by $R$. Define $z(R) := t - \sum_{i=1}^{t} r_i$. Observe that $z(R)$ equals the number of coloured vertices in $V(G) - Q$ at the end of any execution of the algorithm that produces the record $R$. (Recall that a vertex of colour 0 is interpreted as being uncoloured.) In particular, $z(R) \geq 1$, since there is always at least one coloured vertex, and $z(R) \leq n$. Associate with $R$ the word

$$D(R) := 01^{r_1}01^{r_2}\dots 01^{r_t}1^{z(R)}.$$

Then $D(R)$ is a Dyck word of length $2t$. A 0 in $D(R)$ corresponds to the colouring of a vertex in the algorithm, while a 1 corresponds to the uncolouring of a vertex, *except* for the last $z(R)$ 1's, which are added in order to ensure that the number of 0's and 1's in $D(R)$ is the same.

Conversely, a Dyck word $d$ is *realisable* if there exist $t \geq 1$ and $R \in \mathcal{R}_t$ such that $D(R) = d$. The set of realisable Dyck words of length $2t$ is denoted $\mathcal{D}_t$.

## 3. Bounded degree proof

The next result is a precise version of Theorem 1. The proof makes use of the symbolic approach to combinatorial enumeration via generating functions. We refer the reader to the book by Flajolet and Sedgewick [20] for background on this topic, as well as for undefined terms and notations. We postpone these technicalities until the end of the section. Note that we do not attempt to optimize the lower order terms in the proof of Theorem 8.

**Theorem 8.** *For every graph $G$ with maximum degree $\Delta > 1$,*

$$\pi_{\mathrm{ch}}(G) \leq \left\lceil \left(1 + \frac{1}{\Delta^{1/3} - 1} + \frac{1}{\Delta^{1/3}}\right) \Delta^2 \right\rceil .$$

**Proof.** Let $G$ be a graph with maximum degree $\Delta$. Fix an ordering of $V(G)$. Let $n := |V(G)|$ and let $L$ be a list assignment of $G$. Assume each list in $L$ has size $\ell := \lceil d\Delta^2 \rceil$, where

$$d := 1 + \frac{1}{\Delta^{1/3} - 1} + \frac{1}{\Delta^{1/3}}.$$

Let $f$ be an arbitrary priority function. Consider the algorithm on $G$, where none of the vertices of $G$ are precoloured (thus $Q=\emptyset$). We will prove that $|\mathcal{A}_t|=o(d^t\Delta^{2t})$. It suffices to show that $|\mathcal{R}_t|=o(d^t\Delta^{2t})$ since the number of distinct colourings that can be produced by the algorithm is at most $(\ell+1)^n$ (taking into account the extra colour 0).

Let $\lambda=(\lambda_1,\dots,\lambda_{2k})$ be a realisable sequence in $\Lambda$. Observe that $\lambda_j\neq\Delta$ for each $j\in[2,2k]$. Also, there is a unique index $p\in[k+1,2k]$ such that $\lambda_p=-1$. Thus $\lambda_1\in[1,\Delta]$ and $\lambda_p=-1$ and $\lambda_j\in[1,\Delta-1]$ for each $j\in[1,2k]\setminus\{1,p\}$. Hence, there are at most $k\Delta(\Delta-1)^{2(k-1)}$ realisable sequences of length $2k$ in $\Lambda$. Therefore $\alpha_k<k\Delta^{2k-1}$. Let $d=(d_1,\dots,d_{2t})$ be a realisable Dyck word of length $2t$. Suppose that $d$ has the form $0^{l_1}1^{k_1}0^{l_2}1^{k_2}\dots 0^{l_q}1^{k_q}1$, for some positive integers $q,l_1,\dots,l_q,k_1,\dots,k_q$. Note that $\sum_{j=1}^{q}k_j=t-1$. Associate with the word $d$ a weight $w(d):=k_1k_2\dots k_q\Delta^{-q}$. Clearly, for every $i\in[0,k_q]$ the number of distinct records $R\in\mathcal{R}_t$ with $z(R)=i+1$ such that $D(R)=d$ does not exceed

$$\begin{aligned}\alpha_{k_1}\cdots\alpha_{k_{q-1}}\alpha_{k_q-i} &< k_1\Delta^{2k_1-1}\cdots k_{q-1}\Delta^{2k_{q-1}-1}(k_q-i)\Delta^{2(k_q-i)-1}\\ &\le k_1\Delta^{2k_1-1}\cdots k_q\Delta^{2k_q-1}\\ &\le w(d)\Delta^{2t}.\end{aligned}$$

Therefore

$$|\mathcal{R}_t|<n\cdot\Delta^{2t}\cdot\sum_{d\in\mathcal{D}_t}w(d).$$

**Claim 9.**

$$\sum_{d\in\mathcal{D}_t}w(d)=o(d^t).$$

**Proof.** Let $D'$ be the set of words on the alphabet $\{0,1,2\}$ that

- do not contain substrings 21 and 02,
- in every nonempty prefix the number of nonzero elements is strictly less than the number of zeroes, and
- the number of ones and twos in the whole word is one less than the number of zeroes.

Let $\gamma$ be the function that, given a word in $D'$, replaces each 2 by 1 and appends 1. Observe that the image of $\gamma$ is a Dyck word. Let $d$ be a realisable Dyck word of length $2t$. Then for every proper, nonempty prefix of $d$, the number of ones is strictly less than the number of zeroes. In particular, $d$ belongs to the image of $\gamma$. We are interested in the size of the preimage of $d$. Suppose that $d$ has the form $0^{l_1}1^{k_1}0^{l_2}1^{k_2}\dots 0^{l_q}1^{k_q}1$, for some positive integers

$q, l_1, \dots, l_q, k_1, \dots, k_q$. By the definition of $\gamma$, the elements of the preimage of $d$ are exactly the words of the form $0^{l_1}1^{a_1}2^{b_1}0^{l_2}1^{a_2}2^{b_2}\dots 0^{l_q}1^{a_q}2^{b_q}$ where for every $i \in [1,q]$ we have $a_i + b_i = k_i$ and $a_i > 0$ and $b_i \geq 0$. Hence, the size of this preimage is $k_1 k_2 \dots k_q$, which equals $w(d)\Delta^q$. Moreover, every element of the preimage of $d$ has $t$ zeroes and exactly $q$ substrings 01. Let $F_{t,q}$ be the number of words from $D'$ with exactly $t$ zeroes and exactly $q$ substrings 01. It follows from the above observations that

$$\sum_{d \in \mathcal{D}_t} w(d) \leq \sum_{q=0}^{\infty} F_{t,q} \Delta^{-q}.$$

Define the formal power series

$$F(z,y) := \sum_{t=0}^{\infty} \sum_{q=0}^{\infty} F_{t,q} z^t y^q.$$

Then

$$B(z) := F(z, \Delta^{-1}) = \sum_{t=0}^{\infty} z^t \left( \sum_{q=0}^{\infty} F_{t,q} \Delta^{-q} \right).$$

Recall that $[z^t]B(z)$ is the coefficient of $z^t$ in $B(z)$. Hence

$$\sum_{d \in \mathcal{D}_t} w(d) \leq [z^t]B(z).$$

We now derive a functional equation defining $F(z,y)$ by decomposing elements of $D'$ recursively along the last sequence of nonzero letters. For $(d_1, \dots, d_{2t-1}) \in D'$ we say that *position $j$ visits level $k$* if the number of zeroes in $(d_1, \dots, d_j)$ exceeds the number of nonzero symbols by $k$. The sequence $(0)$ is the unique sequence in $D'$ that contains only one zero. Consider some other sequence $d = (d_1, \dots, d_{2t-1})$ from $D'$ with $t$ zeroes that ends with exactly $p > 0$ nonzero symbols. Define $\delta_1$ to be the substring of $d$ starting at $d_1$ and ending at the last position that visits level 1. For $i = 2, 3, \dots, p$, define $\delta_i$ to be the substring of $d$ starting at the position immediately after the last position that visits level $i-1$ and ending at the last position that visits level $i$. In this way, $d$ is uniquely decomposed into $p$ sequences $\delta_1, \dots, \delta_p \in D'$, of total length $2t - p - 2$, and the remaining sequence of the form $01^a 2^b$ with $a + b = p$ and $a > 0$ and $b \geq 0$.

Let $\textsc{Seq}(D')$ denote the set of finite sequences of sequences from $D'$. Let $\textsc{Seq}_{\geq 1}(D')$ denote the set of nonempty finite sequences of sequences from $D'$. Let

$$D'' := \{(0)\} \times \textsc{Seq}_{\geq 1}(D') \times \textsc{Seq}(D').$$

Let $h$ be the function that maps a sequence $d=(d_1,\dots,d_{2t-1})\in D'\setminus\{(0)\}$ to the triple $((0),(\delta_1,\dots,\delta_a),(\delta_{a+1},\dots,\delta_{a+b}))$, where $a,b$, and the $\delta_i$'s are defined as above. Observe that $h$ is a bijection between $D'\setminus\{(0)\}$ and $D''$.

Let $C_{t,q}$ be the number of elements of $D''$ with $t$ zeroes and $q$ substrings 01. Define the formal power series

$$C(z,y) := \sum_{t=0}^{\infty}\sum_{q=0}^{\infty} C_{t,q} z^t y^q.$$

Observe that $d$ and $h(d)$ have the same number of zeroes, for every $d \in D'\setminus\{(0)\}$. (Indeed, this is the reason for the leading (0) in the definition of $D''$.) Also, the total number of occurrences of 01 substrings in $h(d)$ is one less than in $d$. Thus $F_{t,q}=C_{t,q-1}$ for every $t\geq 1$, and $F(z,y)-z=y\cdot C(z,y)$. On the other hand, it follows from the definition of $D''$ that

$$\begin{aligned}C(z,y) &= z\left(\sum_{i\geq 1} F(z,y)^i\right)\left(\sum_{i\geq 0} F(z,y)^i\right)\\ &= z\left(\frac{F(z,y)}{1-F(z,y)}\right)\left(\frac{1}{1-F(z,y)}\right).\end{aligned}$$

This justifies that $F(z,y)$ satisfies the following equation:

$$F(z,y) = z + zy\frac{F(z,y)}{(1-F(z,y))^2}.$$

In particular,

$$B(z) = z\left(1+\Delta^{-1}\frac{B(z)}{(1-B(z))^2}\right).$$

Let $\phi(u)=1+\Delta^{-1}u/(1-u)^2$. Then $B(z)$ is the formal solution of the equation $B(z)=z\phi(B(z))$. It is straightforward to check that $\phi(u)$ satisfies the assumptions of Corollary 11 below. The radius of convergence of $\phi(u)$ is 1. Choose $\tau_0=1-\Delta^{-1/3}$. Then $\tau_0\frac{\phi'(\tau_0)}{\phi(\tau_0)}\neq 1$ and $\frac{\phi(\tau_0)}{\tau_0}=d$. Hence, Corollary 11 below implies that $[z^t]B(z)=o(d^t)$. This completes the proof of the claim. ∎

Returning to the proof of Theorem 8, Claim 9 implies $|\mathcal{R}_t|=o(d^t\Delta^{2t})$. Thus, if $t$ is large enough, then $|\mathcal{A}_t|$ is strictly smaller than $\ell^t$, implying that there is at least one vector $(c_1,\dots,c_t)$ among the $\ell^t$ vectors in $[1,\ell]^t$ on which the algorithm succeeds. Therefore, $G$ admits a nonrepetitive $L$-colouring. ∎

The following results of Flajolet and Sedgewick [20] were used above.

**Theorem 10 (Proposition IV.5 from [20]).** *Let $\phi$ be a function analytic at 0, having nonnegative Taylor coefficients, and such that $\phi(0) \neq 0$. Let $R \leq +\infty$ be the radius of convergence of the series representing $\phi$ at 0. Under the condition*

$$\lim_{x \to R^-} \frac{x\phi'(x)}{\phi(x)} > 1$$

*there exists a unique solution $\tau \in (0, R)$ of the characteristic equation $\frac{\tau\phi'(\tau)}{\phi(\tau)} = 1$. Then, the formal solution $y(z)$ of the equation $y(z) = z\phi(y(z))$ is analytic at 0 and has radius of convergence $\rho = \frac{\tau}{\phi(\tau)}$.*

For a function $\phi$ with nonnegative coefficients, the function $\frac{\tau}{\phi(\tau)}$ is concave in $(0, R)$. Then $\tau$ is the point in the interval $(0, R)$ that maximizes $\frac{\tau}{\phi(\tau)}$. Thus, for any $\tau_0 \in (0, R)$ for which $\frac{\tau_0\phi'(\tau_0)}{\phi(\tau_0)} \neq 1$, we have $\rho > \frac{\tau_0}{\phi(\tau_0)}$ which implies that $[z^n]y(z) = o((\frac{\tau_0}{\phi(\tau_0)})^{-n})$.

**Corollary 11.** *Let $\phi$ be a function analytic at 0, having nonnegative Taylor coefficients, and such that $\phi(0) \neq 0$. Let $R \leq +\infty$ be the radius of convergence of $\phi$. Assume that $\phi(R^-) = +\infty$ and let $y(z)$ be the formal solution of the equation $y(z) = z\phi(y(z))$. Then for any $\tau_0 \in (0, R)$ for which $\frac{\tau_0\phi'(\tau_0)}{\phi(\tau_0)} \neq 1$,*

$$[z^n]y(z) = o\Big(\Big(\frac{\phi(\tau_0)}{\tau_0}\Big)^n\Big).$$

## 4. Subdivision proof

We now begin the proof of Theorem 2. A sequence $(s_1, \ldots, s_q)$ of positive integers is *$c$-spread* if each entry equal to 1 can be mapped to an entry greater than 1 such that for each $i \in [1, q]$ with $s_i \geq 2$, there are at least $\lceil c \log s_i \rceil$ entries, either all immediately before $s_i$ or all immediately after $s_i$, that are equal to 1 and are mapped to $s_i$.

**Lemma 12.** *Fix $\varepsilon > 0$. Let $w := (1 + \varepsilon)^{-1/2} < 1$. Let $c \in \mathbb{N}$ be such that $2^{2/c} \leq 1 + \varepsilon$ and $w^c \leq \frac{\varepsilon}{2}(1 - w)$. Then for each $q \geq 1$ the number of distinct $c$-spread sequences of length $q$ is at most $(1 + \varepsilon)^q$.*

**Proof.** The proof is by induction on $q$. Let $f(q)$ be the number of $c$-spread sequences of length $q$. The claim holds when $q \leq c$ since the sequence $(1, \ldots, 1)$ of length $q$ is the only $c$-spread sequence of length $q$ in that case.

Now assume that $q \geq c + 1$. Here are three ways of obtaining $c$-spread sequences of length $q$ from shorter ones:

1. If $(s_1,\dots,s_{q-1})$ is $c$-spread, then so is $(1,s_1,\dots,s_{q-1})$.
2. If $r\in\mathbb{N}$ such that $r\geq 2$ and $\lceil c\log r\rceil = q-1$, then the two sequences $(1,\dots,1,r)$ and $(r,1,\dots,1)$ of length $q$ are $c$-spread.
3. If $r\in\mathbb{N}$ such that $r\geq 2$ and $z:=\lceil c\log r\rceil \leq q-2$, and if $(s_1,\dots,s_{q-z-1})$ is a $c$-spread sequence, then the two sequences $(1,\dots,1,r,s_1,\dots,s_{q-z-1})$ and $(r,1,\dots,1,s_1,\dots,s_{q-z-1})$ of length $q$ are $c$-spread.

It is not difficult to see that each $c$-spread sequence of length $q$ can be obtained using the three constructions above. Notice that if $z,r\in\mathbb{N}$ are such that $r\geq 2$ and $z=\lceil c\log r\rceil$, then in particular $z\geq c$ and $r\leq 2^{z/c}$. Letting $f(0):=1$, we deduce that

$$\begin{aligned}
f(q) &\leq f(q-1)+2\sum_{z=c}^{q-1}2^{z/c}f(q-z-1)\\
&\leq (1+\varepsilon)^{q-1}+2\sum_{z=c}^{q-1}(1+\varepsilon)^{z/2}(1+\varepsilon)^{q-z-1}\\
&= (1+\varepsilon)^{q-1}+2(1+\varepsilon)^{q-1}\sum_{z=c}^{q-1}w^z\\
&\leq (1+\varepsilon)^{q-1}+2(1+\varepsilon)^{q-1}\sum_{z=c}^{\infty}w^z\\
&= (1+\varepsilon)^{q-1}+2(1+\varepsilon)^{q-1}\frac{w^c}{1-w}\\
&\leq (1+\varepsilon)^{q-1}+\varepsilon(1+\varepsilon)^{q-1}\\
&= (1+\varepsilon)^q.
\end{aligned}$$

∎

A Dyck word $d$ is *special* if $d$ does not contain 0110110 as a substring. The following crude upper bound on the number of such words will be used in our proof of Theorem 2.

**Lemma 13.** *The number of special Dyck words of length $2t$ is at most $3.992^{t+1}$.*

**Proof.** For $q\geq 1$, let $g(q)$ be the number of binary words of length $q$ not containing 0110110. Let $\xi:=(2^7-1)^{1/7}$. Then $g(q)\leq 2^q\leq \xi^{q+1}$ for $q\in[1,7]$, and $g(q)\leq \xi^7\cdot g(q-7)$ for $q\geq 8$, since such binary words cannot start with 0110110. Thus $g(q)\leq \xi^{q+1}$ for all $q\geq 1$. Since every special Dyck word of length $2t$ is a binary word not containing 0110110, it follows that the number of such Dyck words is at most $\xi^{2t+1}<3.992^{t+1}$. ∎

**Theorem 2.** *Let $G$ be a subdivision of a graph $H$, such that each edge $vw\in E(H)$ is subdivided at least $\lceil 10^5\log(\deg(v)+1)\rceil+\lceil 10^5\log(\deg(w)+1)\rceil+2$ times in $G$. Then $\pi_{\rm ch}(G)\leq 5$.*

**Proof.** Let $n:=|V(G)|$. Let $L'(v)$ denote a list of available colours for each vertex $v\in V(G)$, and assume all these lists have size 5. Let $\psi$ be an arbitrary precolouring of $G$ with precoloured set $Q:=V(H)$ and with $\psi(v)\in L'(v)$ for each $v\in Q$. Fix an ordering of $V(G)$ such that $V(G)-Q$ precedes $Q$.

Let $c:=10^5$. For each $v\in Q$, let $g(v):=\lceil c\log(\deg(v)+1)\rceil$ and let $M(v)$ be the set of vertices of $G$ at distance at most $g(v)+1$ from $v$. Thus $M(v)\cap M(w)=\emptyset$ for distinct vertices $v,w\in Q$; we say that $u\in V(G)-Q$ *belongs to* $v$ if $u\in M(v)$ and $v\in Q$.

For each edge $vw\in E(H)$, let $P_{vw}$ denote the path of $G$ induced by the subdivision vertices introduced on the edge $vw$ in $G$. Note that $v,w\notin V(P_{vw})$. A set $X\subseteq V(G)-Q$ is *nice* if $X\neq\emptyset$ and, for each edge $vw\in E(H)$, the graph $P_{vw}-X$ is either connected or empty. The *boundary* $\partial(X)$ of a nice set $X$ is the set of vertices $y\in X$ such that $X-\{y\}$ is either nice or empty. Observe that $\partial(X)$ is always nonempty.

Fix an arbitrary ordering of the edges in $E(H)$. For each edge $vw\in E(H)$, orient the path $P_{vw}$ from an arbitrarily chosen endpoint to the other. If $Y$ is a set of consecutive vertices of a path $P_{vw}$ and $x\in V(P_{vw})-Y$, then $x$ is either *before* $Y$ or *after* $Y$, depending on the orientation of $P_{vw}$. Let $f$ be a priority function defined as follows: For every nice set $X$, let $vw$ be the first edge in the ordering of $E(H)$ such that $V(P_{vw})\cap X\neq\emptyset$. If $V(P_{vw})\subseteq X$, then $V(P_{vw})\subseteq\partial(X)$, and we let $f(X)$ be an arbitrary vertex in $V(P_{vw})$. If $V(P_{vw})-X\neq\emptyset$ and there is a vertex $x\in\partial(X)\cap V(P_{vw})$ before $V(P_{vw})-X$ on $P_{vw}$, then $x$ is uniquely determined, and we let $f(X):=x$. If $V(P_{vw})-X\neq\emptyset$ but there is no such vertex $x$, then we let $f(X)$ be the unique vertex in $\partial(X)\cap V(P_{vw})$ that is after $V(P_{vw})-X$ on $P_{vw}$.

For each $u\in V(G)-Q$, let $L(u)$ be the list $L'(u)-\{\phi(v)\}$ if $u$ belongs to $v\in Q$ and $\phi(v)\in L'(u)$, otherwise let $L(u)$ be obtained from $L'(u)$ by removing one arbitrary colour from $L'(u)$. This defines a list $L(u)$ of available colours for each vertex $u\in V(G)-Q$, and all these lists have size $\ell:=4$. Consider the algorithm on $G$ with the latter lists, with priority function $f$, and with precolouring $\psi$. By the definition of the lists $L(u)$, if the algorithm succeeds on some input $(c_1,\dots,c_t)\in[1,\ell]^t$ then it produces a nonrepetitive $L'$-colouring of $G$.

**Claim 14.** *Let $t\geq 1$. For each vector $(c_1,\dots,c_t)\in\mathcal{F}_t$, all the sets appearing in the trace $\mathrm{tr}(c_1,\dots,c_t)$ are nice.*

**Proof.** The proof is by induction on $t$. The claim is true for $t=1$ since $X_1 = V(G)-Q$ is nice. Now assume that $t \geq 2$. Let $(c_1,\dots,c_t)\in\mathcal{F}_t$ and let $\mathrm{tr}(c_1,\dots,c_t)=(X_1,\dots,X_t)$. Then $(c_1,\dots,c_{t-1})\in\mathcal{F}_{t-1}$, and by induction the sets $X_1,\dots,X_{t-1}$ are nice. Let $v_{t-1} := f(X_{t-1})$.

First suppose that $R(t-1)=\emptyset$. Then $X_t = X_{t-1}-\{v_{t-1}\}$, which is a nice set since $v_{t-1}\in\partial(X_{t-1})$ and $X_t\neq\emptyset$.

Now assume that $R(t-1)=\lambda$ for some $\lambda\in\Lambda$. Using Lemma 5, let $P$ be the path of $G$ determined by $\lambda$, $X_{t-1}$, and $v_{t-1}$. Let $P_1$ and $P_2$ denote the two halves of $P$, so that $v_{t-1}\in V(P_2)$. Then $X_t = X_{t-1}\cup(V(P_2)-Q)$. Arguing by contradiction, suppose that $X_t$ is not nice. Then there exists $vw\in E(H)$ such that $P_{vw}-X_t$ has at least two components. Let $x,y$ be two vertices in distinct components of $P_{vw}-X_t$ that are as close as possible on the path $P_{vw}$. Then the set $Z$ of vertices strictly between $x$ and $y$ on $P_{vw}$ is a subset of $X_t$. On the other hand, $Z\cap X_{t-1}=\emptyset$ since otherwise $x$ and $y$ would be in distinct components of $P_{vw}-X_{t-1}$, contradicting the fact that $X_{t-1}$ is nice. Thus $Z\subseteq V(P_2)-Q$, and also $\partial(X_{t-1})\cap Z=\emptyset$. Since $P_2$ is connected and avoids $x$ and $y$, we deduce that $Z=V(P_2)$ (and thus $Q\cap V(P_2)=\emptyset$). However, $v_{t-1}\in\partial(X_{t-1})$ and $v_{t-1}\in V(P_2)=Z$, contradicting $\partial(X_{t-1})\cap Z=\emptyset$. ∎

**Claim 15.** $\beta \leq 1.001$.

**Proof.** We need to show that $\alpha_k \leq 1.001^k$ for each $k\in[1,\lfloor\frac{n}{2}\rfloor]$. Fix $k\in[1,\lfloor\frac{n}{2}\rfloor]$. Let $\mathcal{W}$ be the set of triples $(P,X,v)$ that may be considered by the algorithm in the uncolouring step, over all $t\geq 1$ and vectors $(c_1,\dots,c_t)\in\mathcal{F}_t$, such that $P$ has exactly $2k$ vertices.

Observe that if $(P,X,v)\in\mathcal{W}$, then $X$ is a nice subset of $V(G)-(Q\cup V(P))$ by Claim 14; also, $v\in V(P)$ and $X\cup\{v\}$ is again a nice set. By the definition of $\mathcal{W}$, every realisable sequence $\lambda\in\Lambda$ of length $2k$ is 'produced' by at least one triple in $\mathcal{W}$, in the sense that there exists $(P,X,v)\in\mathcal{W}$ such that $\lambda=\lambda(P,X,v)$. We may assume that $\mathcal{W}$ is not empty, since otherwise $\alpha_k=0$, and we are trivially done. Let $(P,X,v)\in\mathcal{W}$ and let $\lambda(P,X,v)=(\lambda_1,\dots,\lambda_{2k})$. Let $v_1,\dots,v_{2k}$ be the vertices of $P$. Note that $P$ may contain vertices of $Q$. Since $v$ is not in $Q$, it has degree 2, and thus $\lambda_1\in\{1,2\}$. (Note that we could have $\lambda_1=2$ if the neighbour of $v$ in $P$ is in $Q$.) We have $\lambda_p=-1$ for a unique $p\in[2,2k]$. We claim that the sequence $\lambda' := (\lambda_{p-1},\dots,\lambda_2,1,\lambda_{p+1},\dots,\lambda_{2k})$ obtained from $(\lambda_1,\dots,\lambda_{2k})$ by removing the $p$-th entry, reversing the $(\lambda_1,\dots,\lambda_{p-1})$ prefix, and replacing $\lambda_1$ by 1, is $c$-spread.

**Case 1.** $2k\leq c+1$: Then $P$ has no vertex $u$ in $Q$, since otherwise $P$ would have at least $g(u)+2\geq\lceil c\log 2\rceil+2>c+1$ vertices. It follows that there is an edge $xy\in E(H)$ such that $P$ is a subpath of $P_{xy}$. Then $v$ must be an endpoint of $P$. Indeed, if not, then the two neighbours of $v$ in $P$ are in distinct components of $P_{xy}-(X\cup\{v\})$, contradicting the fact that $X\cup\{v\}$ is nice. Clearly $\lambda_i=1$ for each $i\in[2,2k-1]$ and $\lambda_{2k}=-1$. If $v$ is an internal vertex of $P_{xy}$, then one of the two neighbours of $v$ is in $X$, and it follows that $\lambda_1=1$. If $v$ is an endpoint of $P_{xy}$, then $\lambda_1$ is the index in the set $N(v)$ of the only neighbour $w$ of $v$ that is in $P_{xy}$. This index is always 1 by our choice of the ordering of $V(G)$ (since elements of $V(H)$ come last in the order). Hence, we again have $\lambda_1=1$. Therefore, $(\lambda_1,\dots,\lambda_{2k})=(1,\dots,1,-1)$ and $\lambda'$ is the sequence of $2k-1$ ones, which is $c$-spread.

**Case 2.** $2k\geq c+2$: If $\lambda_i>1$ for some $i\in[2,p]$, then $v_i$ is in $Q$; in this case, our goal is to show that $\lambda'$ contains $g(v_i)$ ones immediately before or after $\lambda_i$ that can be mapped to $\lambda_i$. Similarly, if $\lambda_{p+j}>1$ for some $j\in[1,q]$, then $w_j$ is in $Q$; in this case, our goal is to show that $\lambda'$ contains $g(w_j)$ ones immediately before or after $\lambda_{p+j}$ that can be mapped to $\lambda_{p+j}$.

Consider a vertex $u\in V(P)\cap Q$. By the definition of $L$, the colour assigned to $u$ is assigned to no vertex that belongs to $u$ (those in the set $M(u)$) when the algorithm considers the triple $(P,X,v)$. At that stage, $P$ is repetitively coloured. Let $x$ be the unique vertex at distance $k$ from $u$ in $P$. Then $u$ and $x$ are assigned the same colour, and $x$ is not in $M(u)$. Walk from $u$ towards $x$ and stop after $g(u)+1$ steps. This defines a subpath $P'$ of $P$ consisting of exactly $g(u)+1$ vertices that belong to $u$, either all immediately before $u$ or all immediately after $u$ in $P$. Consider the following six possible values of $u$ and $P'$:

- If $u=v_i$ and $P'=(v_{i+g(u)+1},v_{i+g(u)},\dots,v_{i+1})$, then $\lambda_{i+g(u)}=\lambda_{i+g(u)-1}\dots=\lambda_{i+1}=1$ and $\lambda'$ contains $g(u)$ ones immediately before $\lambda_i$ that can be mapped to $\lambda_i$.
- If $u=v_i$ and $P'=(v_{i-1},v_{i-2},\dots,v_{i-g(u)-1})$ and $i-g(u)-1\neq 1$, then $\lambda_{i-1}=\lambda_{i-2}=\dots=\lambda_{i-g(u)}=1$ and $\lambda'$ contains $g(u)$ ones immediately after $\lambda_i$ that can be mapped to $\lambda_i$.
- If $u=w_j$ and $P'=(w_{j+1},w_{j+2},\dots,w_{j+g(u)+1})$, then $\lambda_{p+j+1}=\lambda_{p+j+2}=\dots=\lambda_{p+j+g(u)}=1$ and $\lambda'$ contains $g(u)$ ones immediately after $\lambda_{p+j}$ that can be mapped to $\lambda_{p+j}$.
- If $u=w_j$ and $P'=(w_{j-g(u)-1},w_{j-g(u)},\dots,w_{j-1})$, then $\lambda_{p+j-g(u)}=\lambda_{p+j-g(u)+1}=\dots=\lambda_{p+j-1}=1$ and $\lambda'$ contains $g(u)$ ones immediately before $\lambda_{p+j}$ that can be mapped to $\lambda_{p+j}$.
- If $u=v_i$ and $P'=(v_{i-1},v_{i-2},\dots,v_1=w_1,w_2,\dots,w_{g(u)-i+3})$, then $\lambda_{i-1}=\lambda_{i-2}=\dots=\lambda_2=1$ and $\lambda_{p+1}=\lambda_{p+2}=\dots=\lambda_{p+g(u)-i+2}=1$, implying that
$$\lambda_{i-1},\lambda_{i-2},\dots,\lambda_2,1,\lambda_{p+1},\lambda_{p+2},\dots,\lambda_{p+g(u)-i+1}$$
is a sequence of $g(u)$ ones immediately after $\lambda_i$ in $\lambda'$ that can be mapped to $\lambda_i$.

- If $u = w_j$ and $P' = (v_{g(u)-j+3}, v_{g(u)-j+2}, \dots, v_1 = w_1, w_2, \dots, w_{j-1})$, then $\lambda_{g(u)-j+2} = \lambda_{g(u)-j+1} = \dots = \lambda_2 = 1$ and $\lambda_{p+1} = \lambda_{p+2} = \dots = \lambda_{p+j-1} = 1$, implying that

$$\lambda_{g(u)-j+2}, \lambda_{g(u)-j+1}, \dots, \lambda_2, 1, \lambda_{p+1}, \lambda_{p+2}, \dots, \lambda_{p+j-1}$$

is a sequence of $g(u)$ ones immediately before $\lambda_{p+j}$ in $\lambda'$ that can be mapped to $\lambda_{p+j}$.

Hence, $\lambda'$ is $c$-spread, as claimed.

Therefore, $(\lambda_1, \dots, \lambda_{2k})$ is obtained from a $c$-spread sequence $\lambda'$ of length $2k-1$ by choosing an index $p \in [1, 2k-1]$, inserting $-1$ between the $p$-th and $(p+1)$-th entries, reversing the prefix of $\lambda'$ up to the $p$-th entry, and possibly changing the first entry to a 2. Hence, the number of realisable sequences in $\Lambda$ of length $2k$ is at most $2(2k-1)$ times the number of $c$-spread sequences of length $2k-1$. Let $\varepsilon := 0.0002$. Then $\varepsilon$ and $c$ satisfy the hypotheses of Lemma 12, and we deduce from that lemma that

$$\alpha_k \le (4k-2) \cdot (1+\varepsilon)^{2k-1} \le 4k(1+\varepsilon)^{2k} \le 1.001^k.$$

(The rightmost inequality holds because $2k \ge c$ and $2c(1+\varepsilon)^c \le 1.001^{c/2}$.) ∎

Next we show that every realisable Dyck word is special. Consider a word $d \in \mathcal{D}_t$ for some $t \ge 1$, and let $R \in \mathcal{R}_t$ be a record such that $D(R) = d$. Suppose that $d$ contains 0110110 as a subsequence. Then there is an index $i \in [1, t-2]$ such that $|R(i)| = |R(i+1)| = 4$. Fix an arbitrary vector $(c_1, \dots, c_t) \in \mathcal{F}_{t,R}$, and let $(P, X, v)$ and $(P', X', v')$ be the triples such that $\lambda(P, X, v) = R(i)$ and $\lambda(P', X', v') = R(i+1)$, respectively, in the execution of the algorithm on input $(c_1, \dots, c_t)$. Then $P$ contains no vertex from $Q$, since otherwise $P$ would need to have at least $c+2 > 4$ vertices, as explained in Case 1 of the proof of Claim 15. Since our ordering of $V(G)$ puts vertices in $V(G) - Q$ before those in $Q$, and since $X$ is nice, it follows that $\lambda(P, X, v) = R(i) = (1, 1, 1, -1)$. By the same argument $\lambda(P', X', v') = R(i+1) = (1, 1, 1, -1)$. Let $v_1, \dots, v_4$ denote the vertices of $P$, with $v_4 = v$. Then in the $i$-th iteration of the while-loop of the algorithm, immediately after colouring $v_4$, we have $\phi(v_1) = \phi(v_3)$ and $\phi(v_2) = \phi(v_4)$. Vertices $v_3$ and $v_4$ are subsequently uncoloured. Thus $X' = X \cup \{v_3\}$.

By our choice of the priority function $f$, we have $f(X') = v' = v_3$. Indeed, $f(X) = v_4$ and $v_1, v_2 \in V(P_{vw}) - X'$, where $vw \in E(H)$ is the edge such that $P \subseteq P_{vw}$. In particular, $v_3 \in \partial(X')$ and $V(P_{vw}) - X' \neq \emptyset$. Thus either $v_4$ is before $V(P_{vw}) - X$ on $P_{vw}$, in which case $v_3$ is before $V(P_{vw}) - X'$ on $P_{vw}$, implying $f(X') = v_3$; or $v_4$ is after $V(P_{vw}) - X$ on $P_{vw}$, in which case $v_3$ is after $V(P_{vw})-X'$ *and* there is no vertex in $\partial(X')$ before $V(P_{vw})-X'$ on $P_{vw}$, implying again $f(X')=v_3$.

It follows that the vertices of $P'$ are $v_0,v_1,v_2,v_3$ in order, where $v_0 \in V(P_{vw})$ (and obviously $v_0 \neq v_4$). In the $(i+1)$-th iteration of the while-loop, immediately after colouring $v_3$ $(=v')$, we have $\phi(v_0)=\phi(v_2)$ and $\phi(v_1)=\phi(v_3)$. However, the colours of $v_0,v_1,v_2$ have not changed since the beginning of the $i$-th iteration, and $\phi(v_1)=\phi(v_3)$ at that point. This imples that $P'$ was already repetitively coloured at the start of the $i$-th iteration, a contradiction. Therefore, realisable Dyck words are special, as claimed.

Let $t\geq 1$, let $m:=\min\{n,t\}$, let $i\in[1,m]$ and let $(d_1,\dots,d_{2t})$ be a realisable Dyck word of length $2t$ such that $d_{2t-j}=1$ for each $j\in[0,i-1]$. Let $q$ be the number of maximal subsequences of consecutive ones in $(d_1,\dots,d_{2t-i})$. If $q\geq 1$, then let $k_1,\dots,k_q$ be the lengths of these sequences, in order. If $q\geq 1$, then $\sum_{j=1}^{q} k_j \leq t-i$, and we deduce that there are at most

$$\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_q} \leq \beta^{k_1}\beta^{k_2}\cdots\beta^{k_q} \leq \beta^{t-i} \leq \beta^t$$

distinct records $R\in\mathcal{R}_t$ with $z(R)=i$ such that $D(R)=(d_1,\dots,d_{2t})$. If $q=0$, then there is at most $1\leq\beta^t$ records $R\in\mathcal{R}_t$ with $z(R)=i$ such that $D(R)=(d_1,\dots,d_{2t})$. (Note that here we use the fact that $\beta\geq 1$.)

Since for each $i\in[1,m]$, there are at most $\beta^t$ distinct records $R\in\mathcal{R}_t$ with $z(R)=i$ that have the same Dyck word $D(R)$, and since there are exactly $|\mathcal{D}_t|$ distinct realisable special Dyck words of length $2t$, it follows that $|\mathcal{R}_t|\leq m|\mathcal{D}_t|\beta^t\leq n|\mathcal{D}_t|\beta^t$. Using Claim 15 and Lemmas 7 and 13, we obtain

$$|\mathcal{F}_t| = |\mathcal{A}_t| \leq n(\ell+1)^n|\mathcal{D}_t|\beta^t \leq n(\ell+1)^n 3.992^{t+1}1.001^t < n(\ell+1)^n 3.996^{t+1}.$$

Hence, if $t$ is sufficiently large then $|\mathcal{F}_t|<\ell^t$, implying that there is at least one vector $(c_1,\dots,c_t)$ among the $\ell^t$ many vectors in $[1,\ell]^t$ on which the algorithm succeeds. Therefore, $G$ admits a nonrepetitive $L'$-colouring. ∎

Note that we made no effort to optimise the constant $10^5$ in the proof of Theorem 2.

## 5. Pathwidth proofs

The proof of Theorem 3 depends on the following lemma of independent interest.

**Lemma 16.** *Let $B_1,\dots,B_m$ be pairwise disjoint sets of vertices in a graph $G$, such that no two vertices in distinct $B_i$ are adjacent. Let $H$ be the graph* *obtained from $G$ by deleting $B_i$ and adding a clique on $N_G(B_i)$ for each $i \in [1,m]$. Then*

$$\pi(G) \leq \pi(H) + \max_i \pi(G[B_i]).$$

**Proof.** Nonrepetitively colour $G[B_1 \cup \cdots \cup B_m]$ with $\max_i \pi(G[B_i])$ colours. Nonrepetitively colour $H$ with a disjoint set of $\pi(H)$ colours. Suppose on the contrary that $G$ contains a repetitively coloured path $P$. Let $P'$ be the set of vertices in $P$ that are in $H$, ordered according to $P$. Then $P' \neq \emptyset$, as otherwise $P$ is contained in some $B_i$, implying $B_i$ contains a repetitively coloured path. Consider a maximal subpath $S$ in $P$ that is not in $H$. So $S$ was deleted from $P$ in the construction of $P'$. Since no two vertices in distinct $B_i$ are adjacent, $S$ is contained in a single set $B_i$. Thus the vertices in $P$ immediately before and after $S$ (if they exist) are in $N_G(B_i)$, and are thus adjacent in $H$. Hence $P'$ is a path in $H$. Since the vertices in $B_1 \cup \cdots \cup B_m$ receive distinct colours from the vertices in $H$, the path $P'$ is repetitively coloured. This contradiction proves that $G$ is nonrepetitively coloured. ∎

The next lemma provides a useful way to think about graphs of bounded pathwidth. Let $G \cdot K_\theta$ denote the *lexicographical product* of a graph $G$ and the complete graph $K_\theta$. That is, $G \cdot K_\theta$ is obtained by replacing each vertex of $G$ by a copy of $K_\theta$, and replacing each edge of $G$ by a copy of $K_{\theta,\theta}$.

**Lemma 17.** *Every graph $G$ with pathwidth $\theta$ contains pairwise disjoint sets $B_1, \dots, B_m$ of vertices, such that:*

- *no two vertices in distinct $B_i$ are adjacent,*
- *$G[B_i]$ has pathwidth at most $\theta - 1$ for each $i \in [1,m]$, and*
- *if $H$ is the graph obtained from $G$ by deleting $B_i$ and adding a clique on $N_G(B_i)$ for each $i \in [1,m]$, then $H$ is a subgraph of $P_m \cdot K_{\theta+1}$.*

**Proof.** Consider a path decomposition $\mathcal{D}$ of $G$ with width $\theta$. Let $X_1, \dots, X_m$ be the set of bags in $\mathcal{D}$, such that $X_1$ is the first bag in $\mathcal{D}$, and for each $i \geq 2$, the bag $X_i$ is the first bag in $\mathcal{D}$ that is disjoint from $X_{i-1}$. Thus $X_1, \dots, X_m$ are pairwise disjoint. For $i \in [1,m]$, let $B_i$ be the set of vertices that only appear in bags strictly between $X_i$ and $X_{i+1}$ (or strictly after $X_m$ if $i = m$). By construction, each such bag intersects $X_i$. Hence, $G[B_i]$ has pathwidth at most $\theta - 1$. Since each $X_i$ separates $B_{i-1}$ and $B_{i+1}$ (for $i \neq m$), no two vertices in distinct $B_i$ are adjacent. Moreover, the neighbourhood of $B_i$ is contained in $X_i \cup X_{i+1}$ (or $X_i$ if $i = m$). Hence, the graph $H$ (defined above) has vertex set $X_1 \cup \cdots \cup X_m$ where $X_i \cup X_{i+1}$ is a clique for each $i \in [1, m-1]$. Since $|X_i| \leq \theta + 1$, the graph $H$ is a subgraph of $P_m \cdot K_{\theta+1}$. ∎

**Proof of Theorem 3.** We proceed by induction on $\theta \geq 0$. Every graph with pathwidth 0 is edgeless, and is thus nonrepetitively 1-colourable, as desired. Now assume that $G$ is a graph with pathwidth $\theta \geq 1$. Let $B_1, \dots, B_m$ be the sets that satisfy Lemma 17. Let $H$ be the graph obtained from $G$ by deleting $B_i$ and adding a clique on $N_G(B_i)$ for each $i \in [1, m]$. Then $H$ is a subgraph of $P_{m+1} \cdot K_{\theta+1}$, which is nonrepetitively $4(\theta+1)$-colourable by a theorem of Kündgen and Pelsmajer [36][6]. By induction, $\pi(G[B_i]) \leq 2(\theta-1)^2+6(\theta-1)-4$. By Lemma 16, $\pi(G) \leq \pi(H)+\max_i \pi(G[B_i]) \leq 4(\theta+1)+2(\theta-1)^2+6(\theta-1)-4 = 2\theta^2+6\theta-4$. This completes the proof. ∎

**Proof of Theorem 4.** We proceed by induction on $\theta \geq 0$. Every graph with pathwidth 0 is edgeless, and is thus star 1-colourable, as desired. Now assume that $G$ is a graph with pathwidth $\theta \geq 1$. We may assume that $G$ is connected. Let $G'$ be an interval graph containing $G$ as a spanning subgraph and with $\omega(G') = \theta+1$. Let $I(v)$ be the interval representing each vertex $v$. Let $X$ be an inclusion-wise minimal set of vertices in $G'$ such that for every vertex $w$,

$$I(w) \subseteq \bigcup \{I(v) \colon v \in X\}. \tag{3}$$

The set $X$ exists since $X = V(G)$ satisfies (3). It is easily seen that $G[X]$ is an induced path, say $(x_1, \dots, x_n)$. Colour $x_i$ by $i \bmod 3$ (in $\{0,1,2\}$). Observe that $G'[X]$ is star 3-coloured. By (3), the subgraph $G' - X$ is an interval graph with $\omega(G'-X) \leq \theta$. Thus, by induction, $G'-X$ is star-colourable with colours $\{3, 4, \dots, 3\theta\}$. Suppose on the contrary that $G'$ contains a 2-coloured path $(u, v, w, x)$. First suppose that $u$ is in $X$. Then $w$ is also in $X$. If $v$ is also in $X$, then so is $x$, which contradicts the fact that $G'[X]$ is star-coloured. So $v \notin X$. Since $u$ and $w$ receive the same colour, there are at least two vertices $p$ and $q$ between $u$ and $w$ in the path $G'[X]$. Thus replacing $p$ and $q$ by $v$ gives a shorter path that satisfies (3). This contradiction proves that $u \notin X$. By symmetry $x \notin X$. Since $X$ and $G'-X$ are assigned disjoint sets of colours, $v \notin X$ and $w \notin X$. Hence, $(u, v, w, x)$ is a 2-coloured path in $G' - X$, which is the desired contradiction. Hence, $G'$ is star-coloured with $3\theta+1$ colours. ∎

[6] Say $V_1, \dots, V_t$ is a partition of $V(G)$ such that for all $i \in [1, t]$, we have $N_G(V_i) \subseteq V_{i-1} \cup V_{i+1}$ and $N_G(V_i) \cap V_{i-1}$ is a clique. Kündgen and Pelsmajer [36] proved that $\pi(G) \leq 4 \max_i \pi(G[V_i])$. Clearly $P_m \cdot K_{\theta+1}$ has such a partition with each $V_i$ a $(\theta+1)$-clique. Thus $\pi(P_m \cdot K_{\theta+1}) \leq 4(\theta+1)$.

## 6. Open problems

We conclude with a number of open problems:

- Whether there is a relationship between nonrepetitive choosability and pathwidth is an interesting open problem. The graphs with pathwidth 1 (i.e., caterpillars) are nonrepetitively $\ell$-choosable for some constant $\ell$; see [16, Appendix B]. Is every graph (or tree) with pathwidth 2 nonrepetitively $\ell$-choosable for some constant $\ell$?
- Except for a finite number of examples, every cycle is nonrepetitively 3-colourable [13]. Every cycle is nonrepetitively 5-choosable. (**Proof.** Precolour one vertex, remove this colour from every other list, and apply the nonrepetitive 4-choosability result for paths.) Is every cycle nonrepetitively 4-choosable? Which cycles are nonrepetitively 3-choosable?
- Does every graph have a nonrepetitively 4-choosable subdivision? Even 3-choosable might be possible.
- Is there a function $f$ such that every graph $G$ has a nonrepetitively $\mathcal{O}(1)$-colourable subdivision with $f(\pi(G))$ division vertices per edge?
- Is there a function $f$ such that $\pi(G/M) \leq f(\pi(G))$ for every graph $G$ and for every matching $M$ of $G$, where $G/M$ denotes the graph obtained from $G$ by contracting the edges in $M$? This would generalise a result of Nešetřil et al. [41] about subdivisions (when each edge in $M$ has one endpoint of degree 2).
- Is there a polynomial-time Monte Carlo algorithm that nonrepetitively $\mathcal{O}(\Delta^2)$-colours a graph with maximum degree $\Delta$? Haeupler et al. [28] show that $\mathcal{O}(\Delta^{2+\varepsilon})$ colours suffice for all fixed $\varepsilon > 0$; also see [11, 33] for related results. Note that testing whether a given colouring of a graph is nonrepetitive is co-NP-complete, even for 4-colourings [38].

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