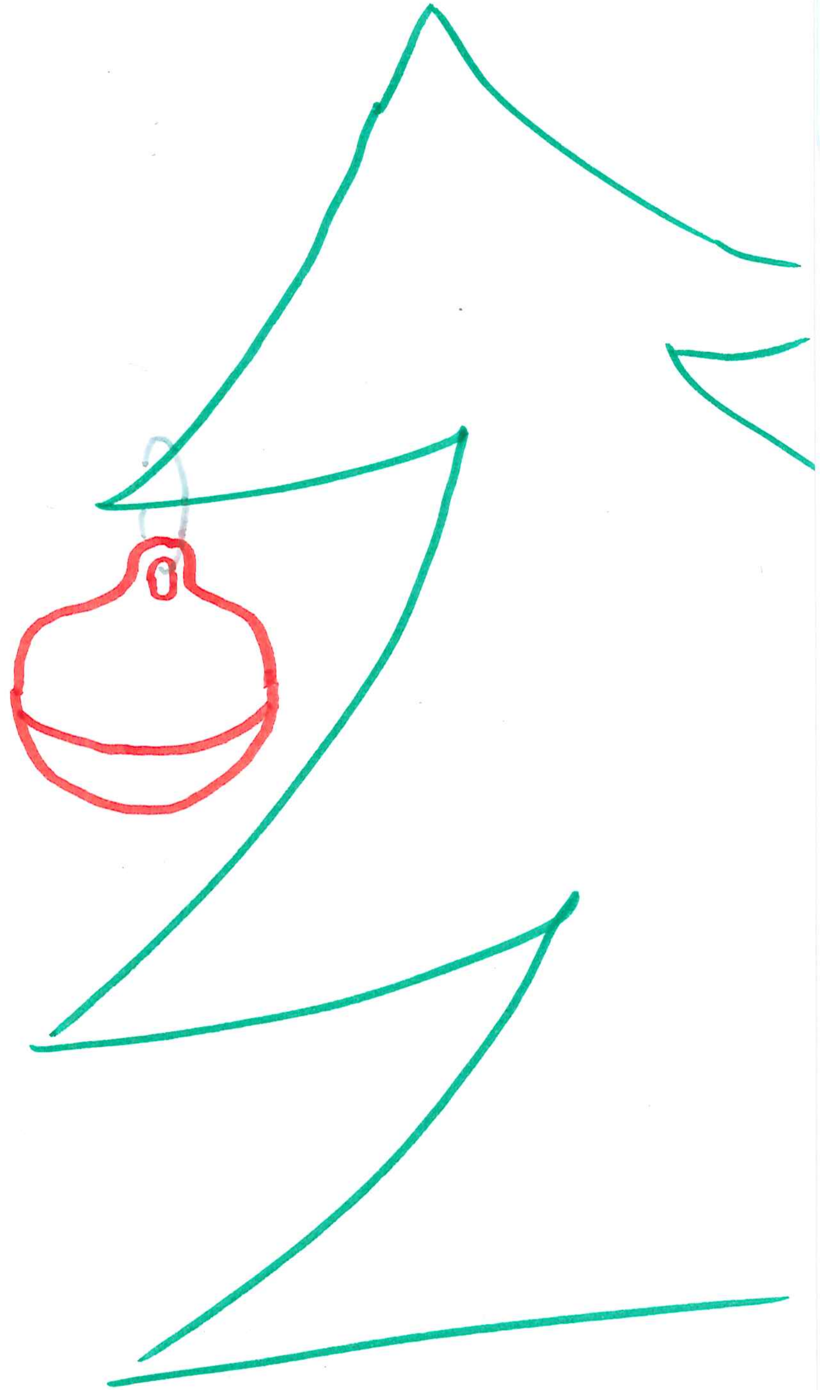
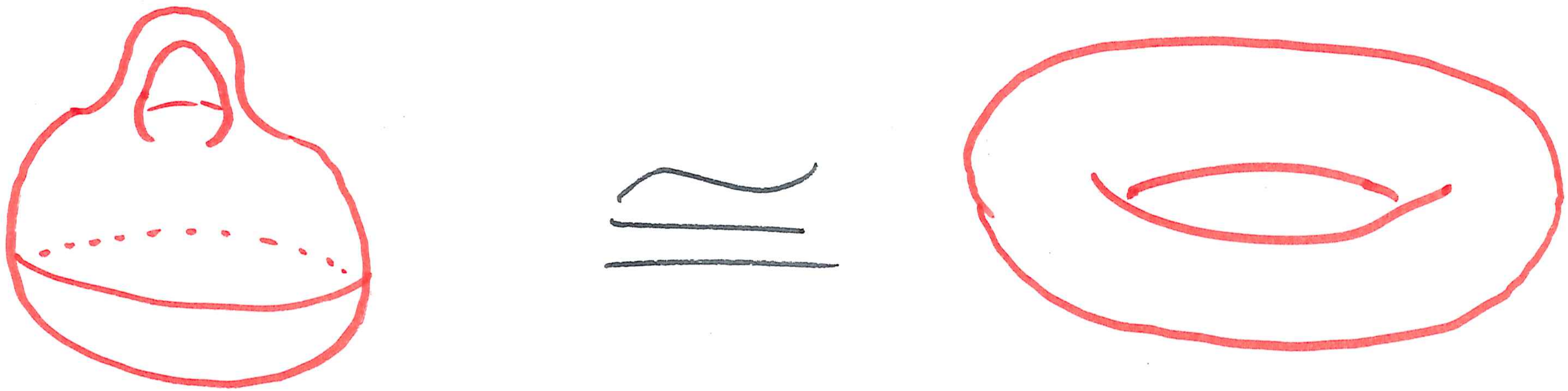
Can you hang a sphere from a Christmas tree?

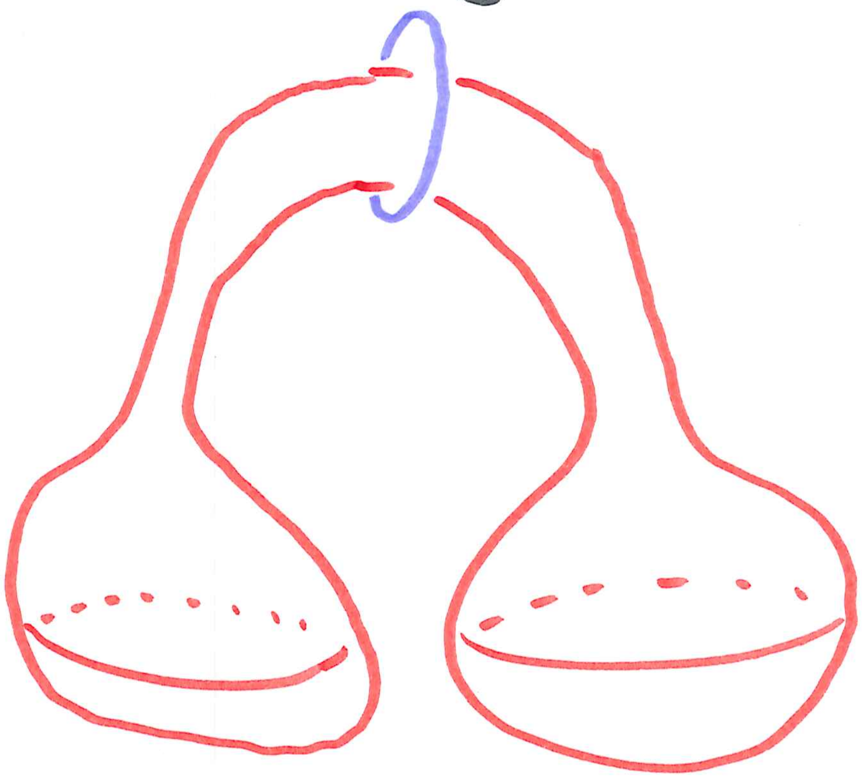


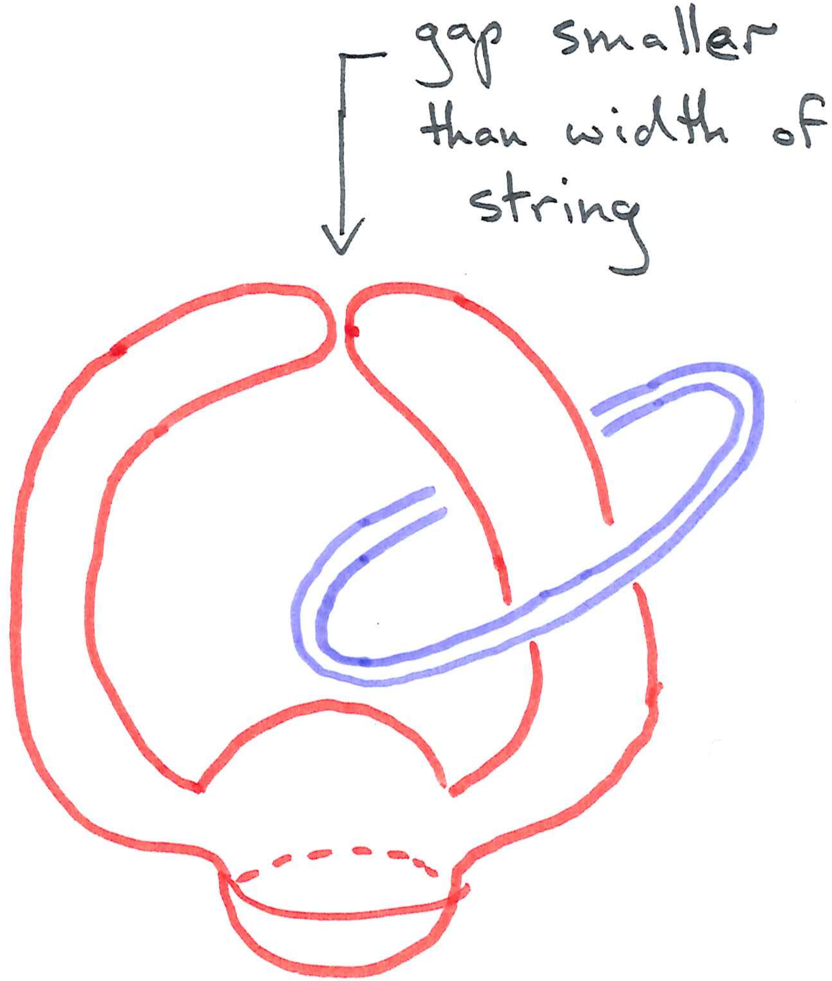
A sphere with a handle attached is topologically a torus.



Can we "hang" a topological sphere from a loop of string?

Cheating:

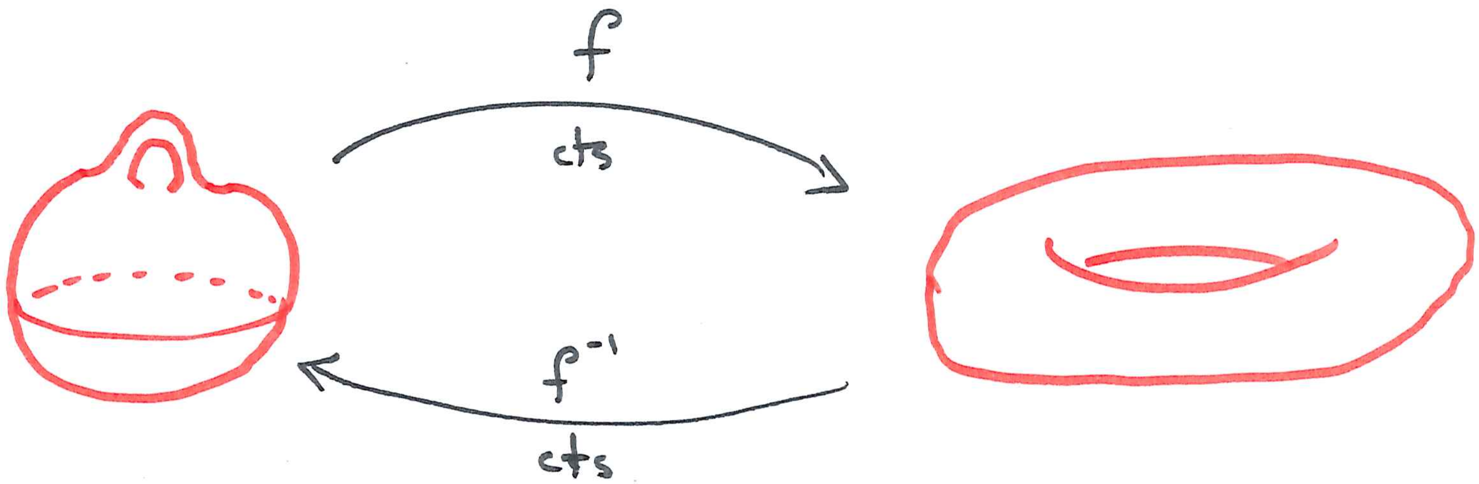




We want an infinitely thin, infinitely malleable string.

Def If a continuous function $f : X \to Y$ has a continuous inverse $f^{-1} : Y \to X$, then f is a **homeomorphism**.

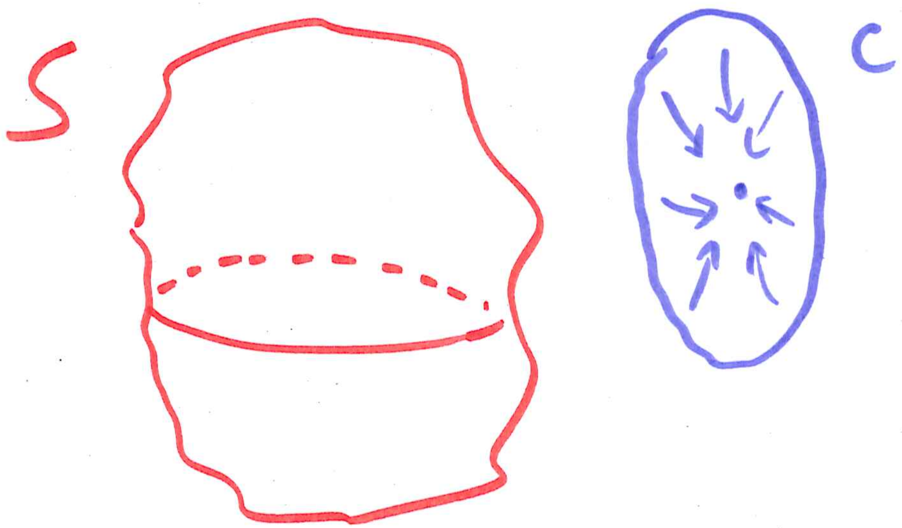
Ex



Def If $f : X \to Z$ is a homeomorphism to its image (i.e., $f : X \to f(X)$ is a homeo), then f is an **embedding**.

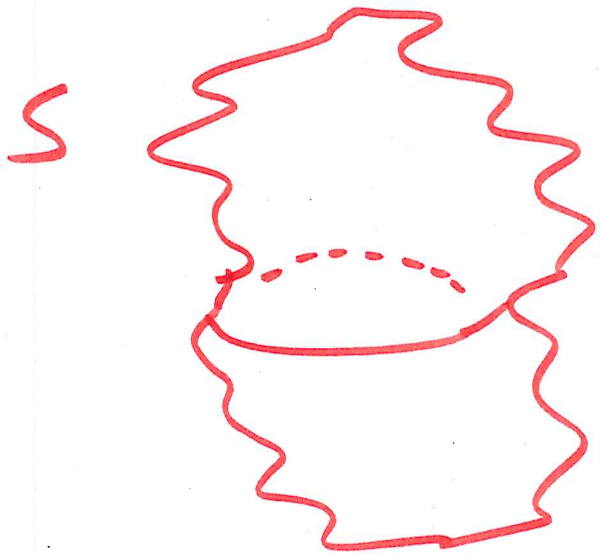
Suppose S is an embedded sphere in $\mathbb{R}^3$
(i.e., $S = f(S^2)$ for an embedding $f: S^2 \to \mathbb{R}^3$)

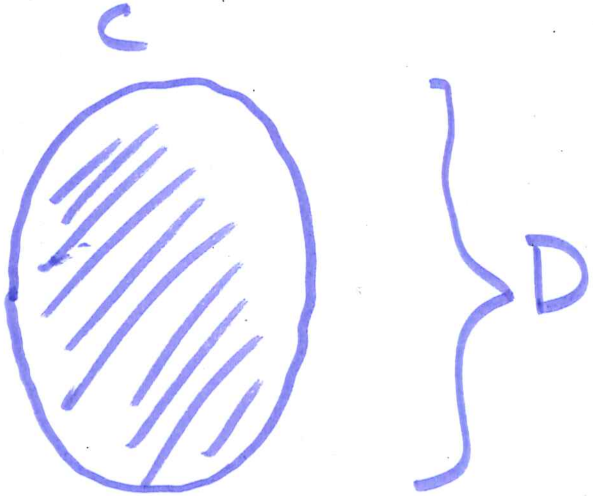
<u>Question</u>: Can any circle C embedded in $\mathbb{R}^3 \setminus S$ be shrunk down to a point?



Equivalently:

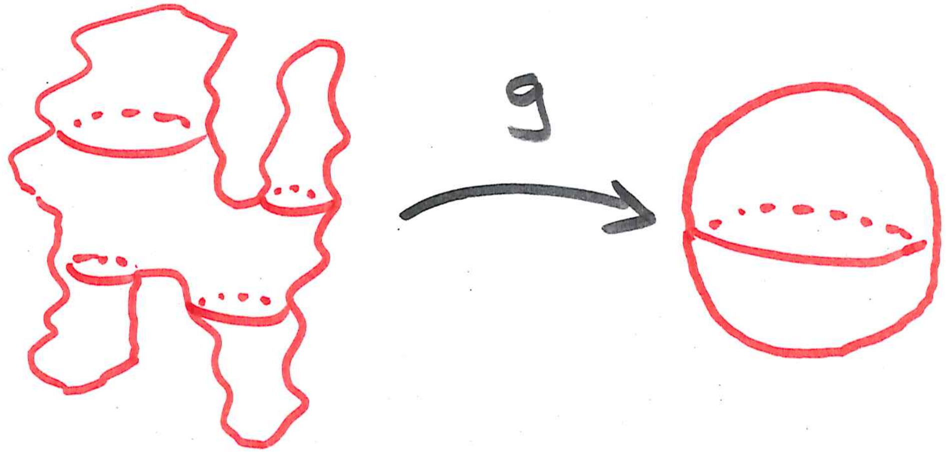
If S is an embedded sphere in $\mathbb{R}^3$ and C is an embedded circle in $\mathbb{R}^3 \setminus S$, does C bound an embedded disk $D \subset \mathbb{R}^3 \setminus S$?





Related Question:

If $f: S^{d-1} \to \mathbb{R}^d$ is an embedding, is there a homeomorphism $g: \mathbb{R}^d \to \mathbb{R}^d$ so $g(f(S^{d-1}))$ is the standard sphere in $\mathbb{R}^d$?

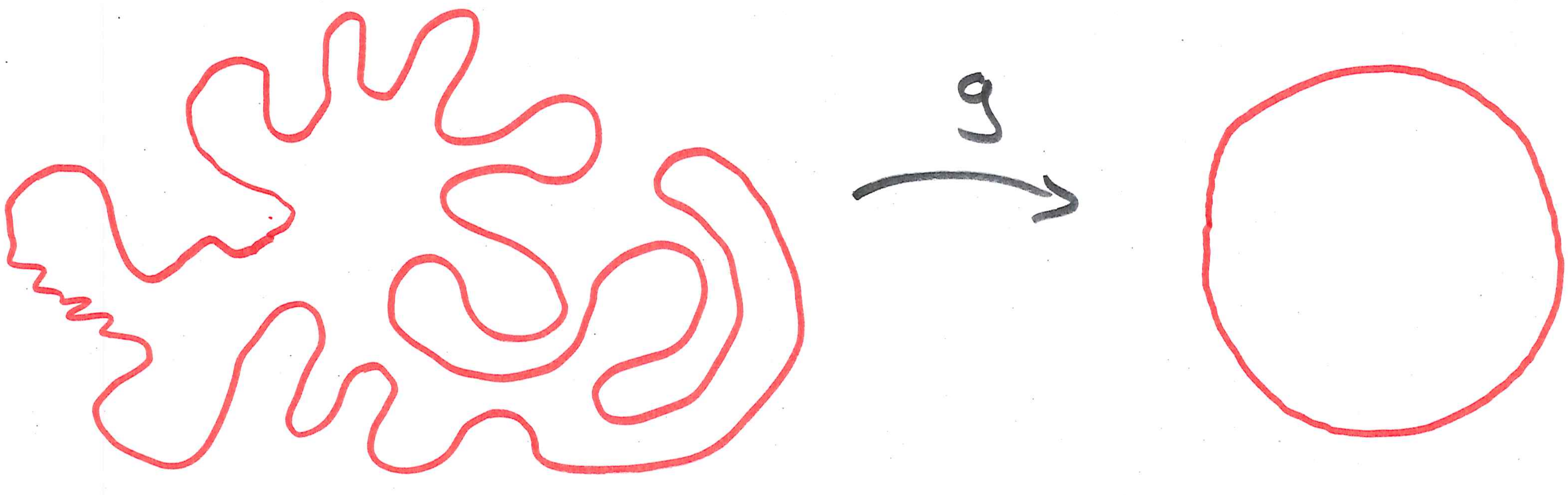


This is called the Schönflies Problem.

True for $d=2$.

Jordon - Schönflies Theorem

For any simple closed curve $C \subset \mathbb{R}^2$, there is a homeomorphism $g : \mathbb{R}^2 \to \mathbb{R}^2$ so that $g(C)$ is the unit circle.

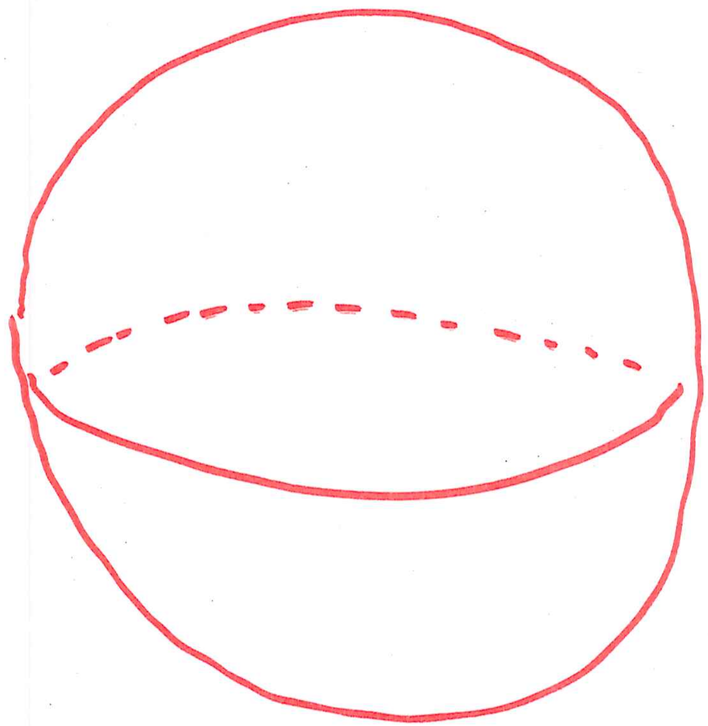


In 1922, Alexander announced a proof for $d = 3$.

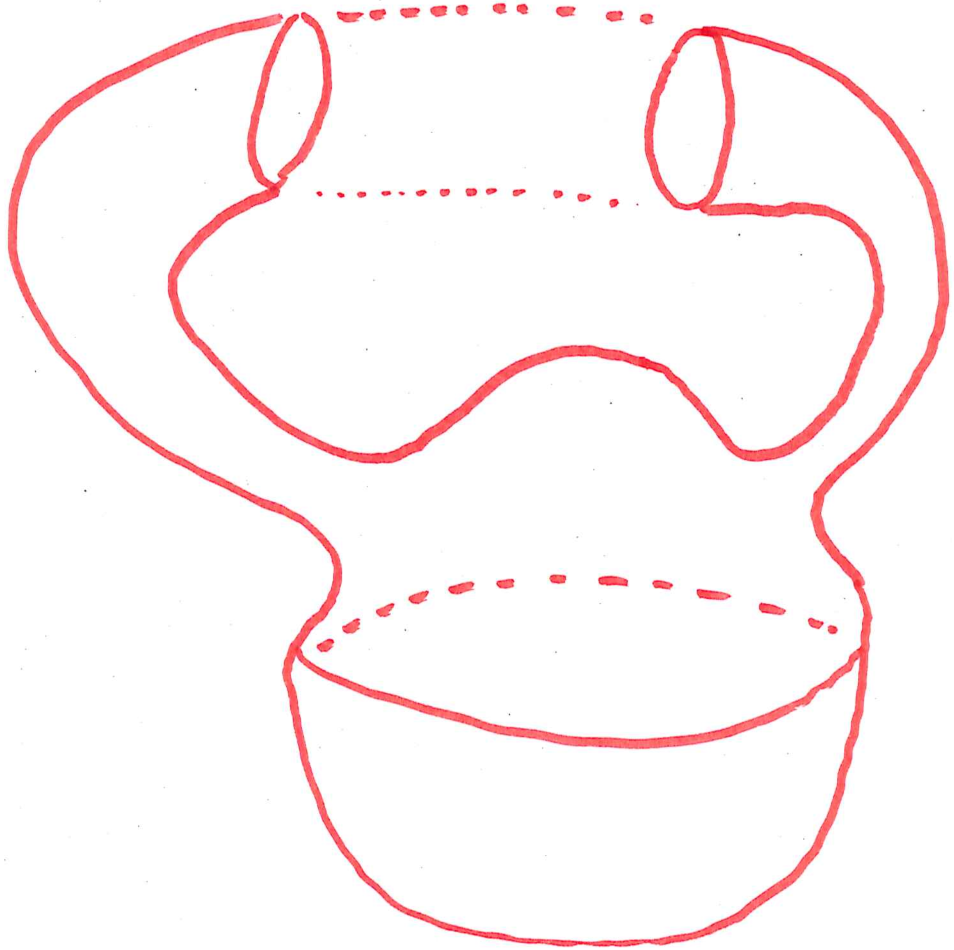
In 1924, he published a counterexample.

The Alexander Horned Sphere

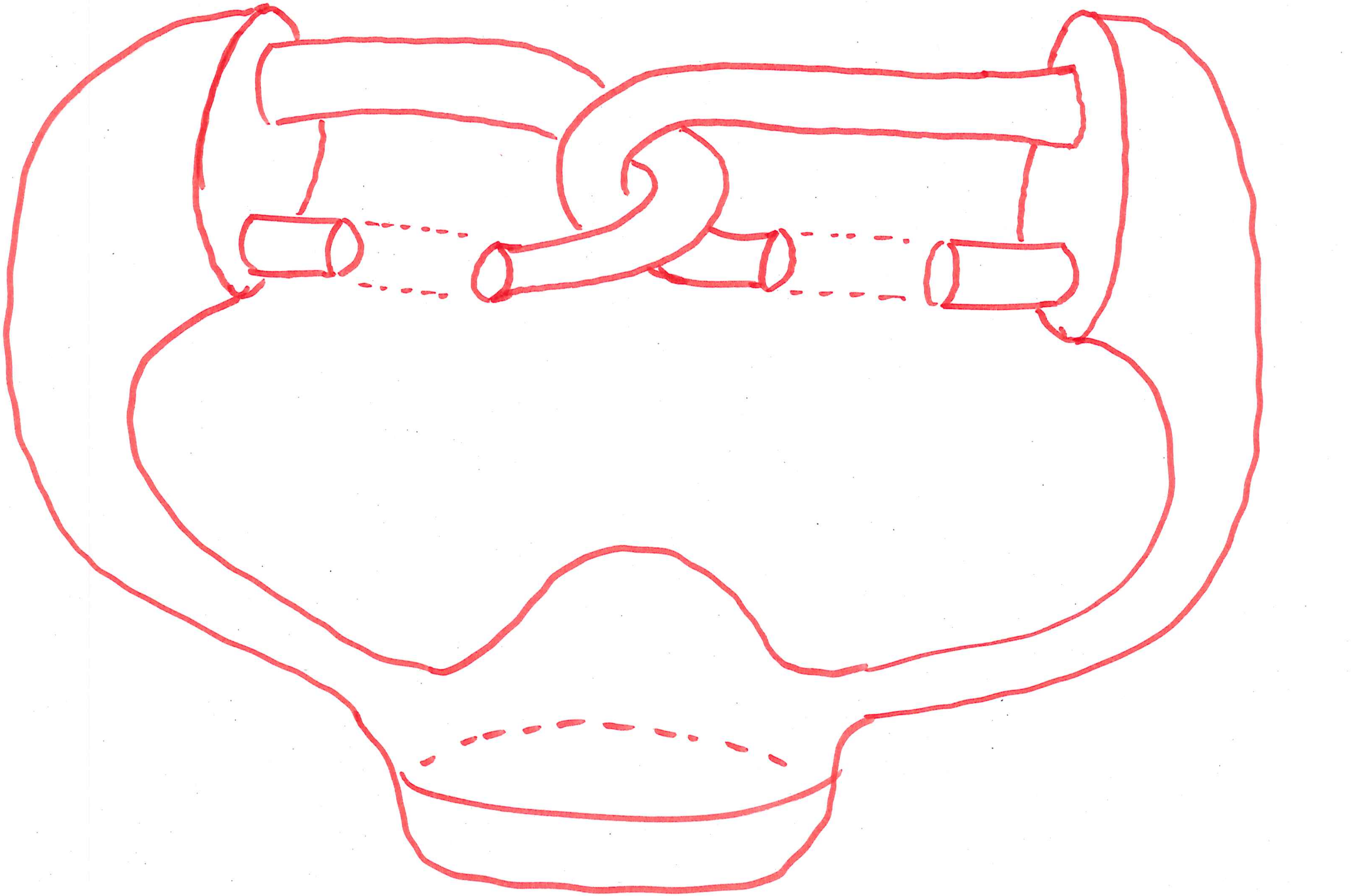
Step 0



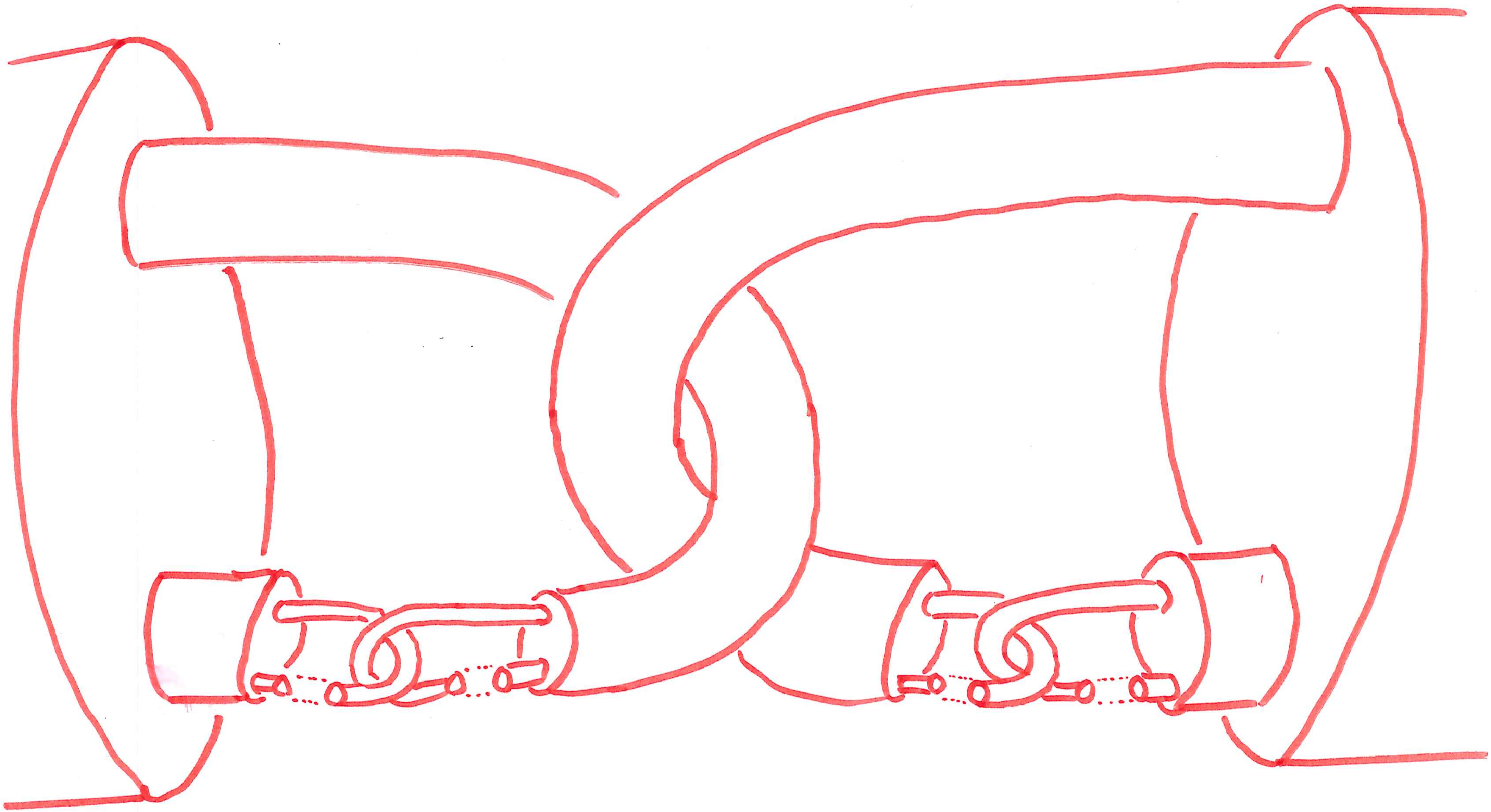
Step 1



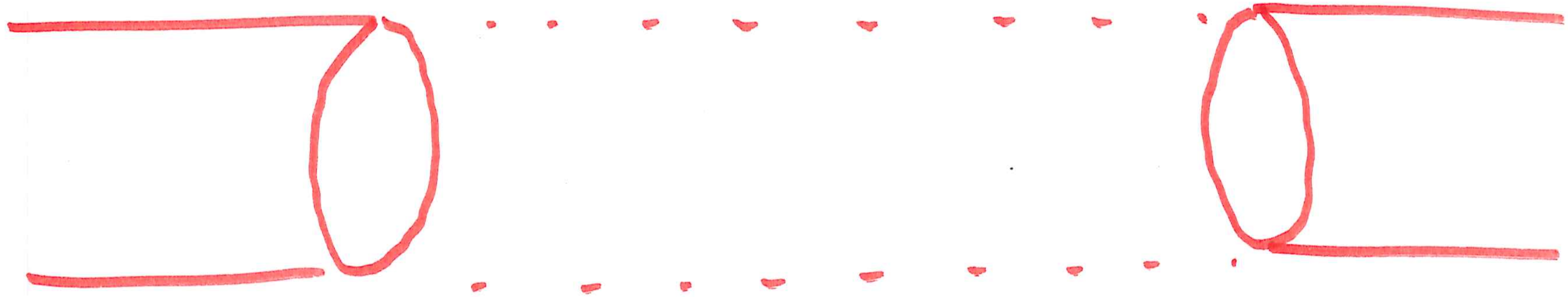
Step 2



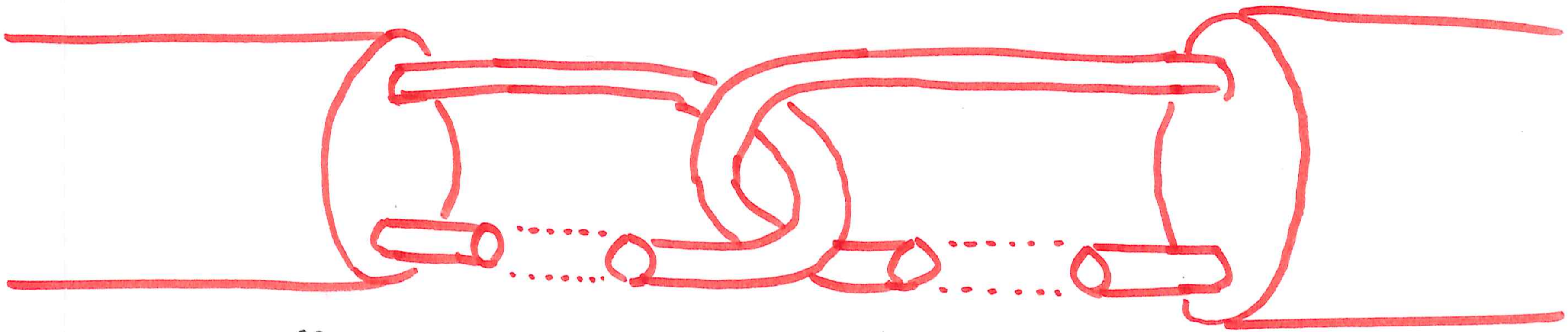
Step 3



At each step, we have 2^{n-1} cylinders

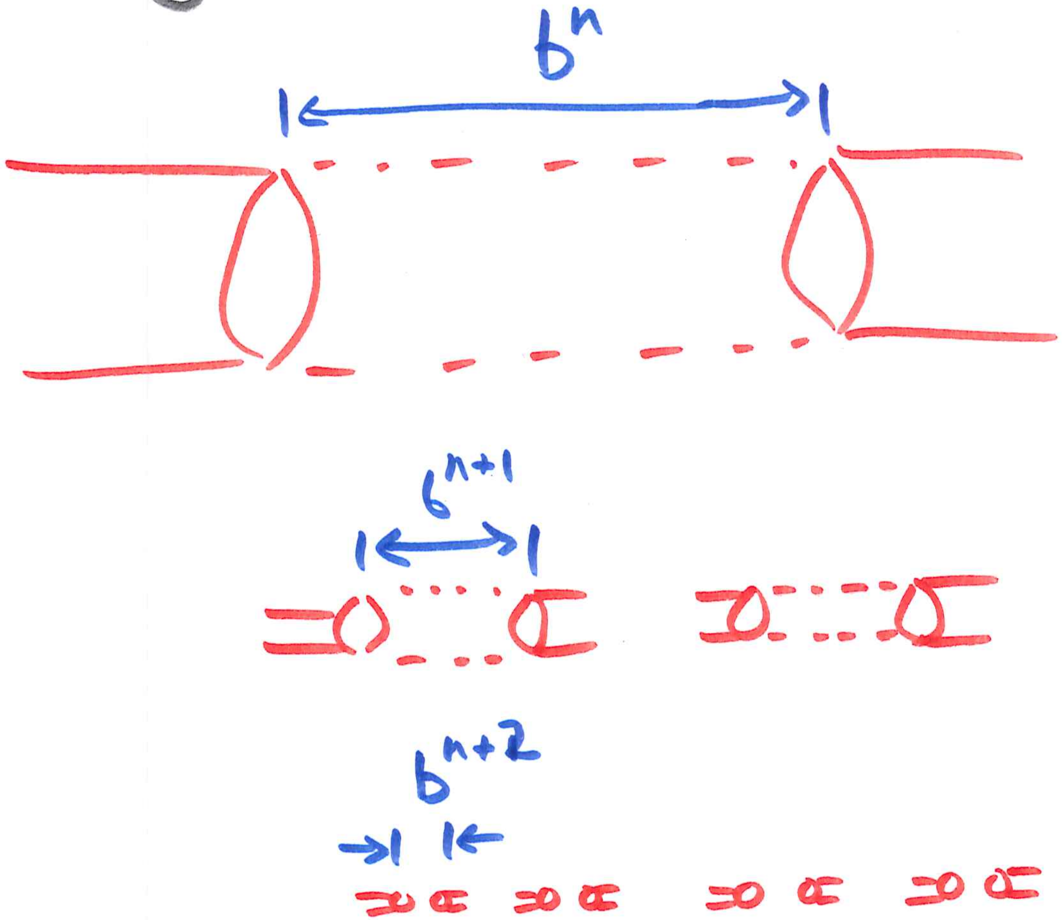


Add hooks to each cylinder.

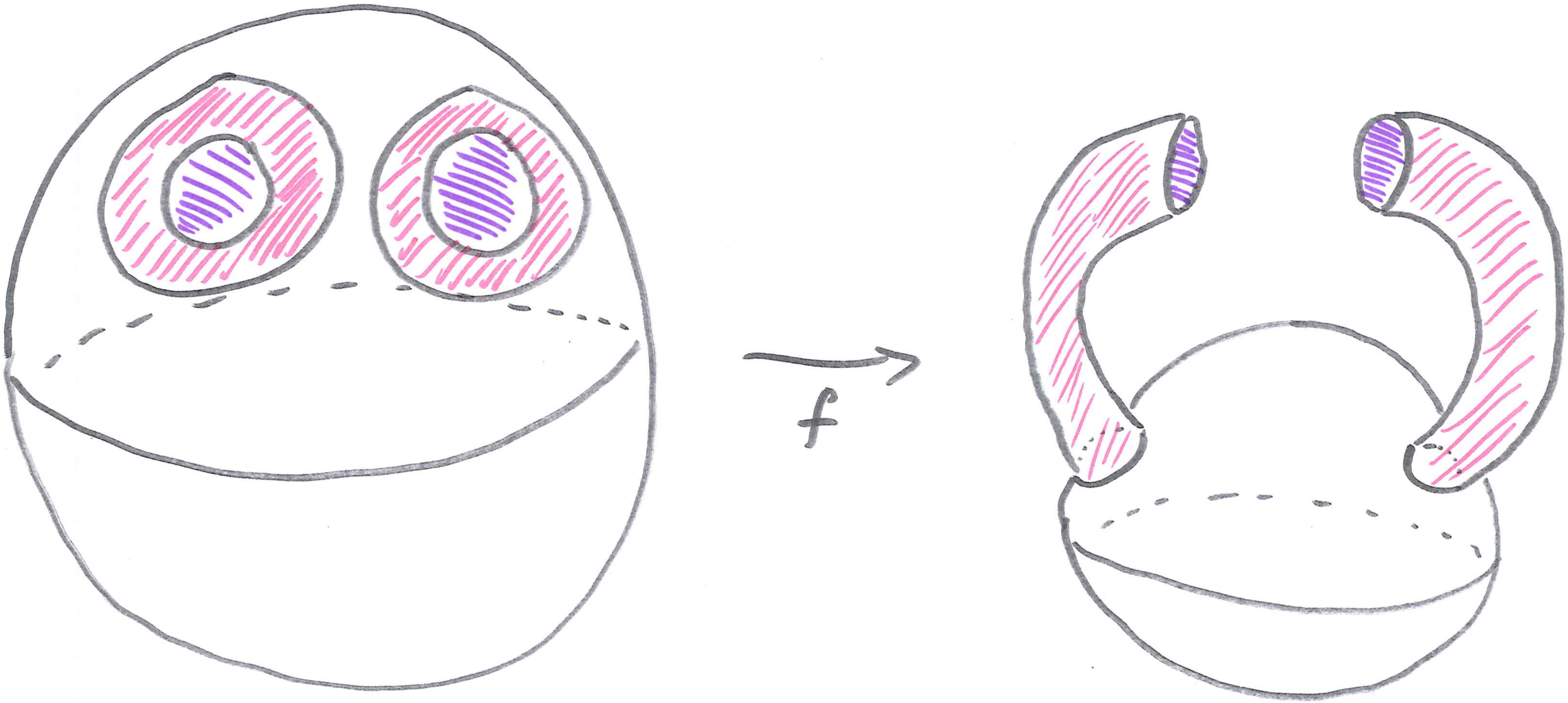


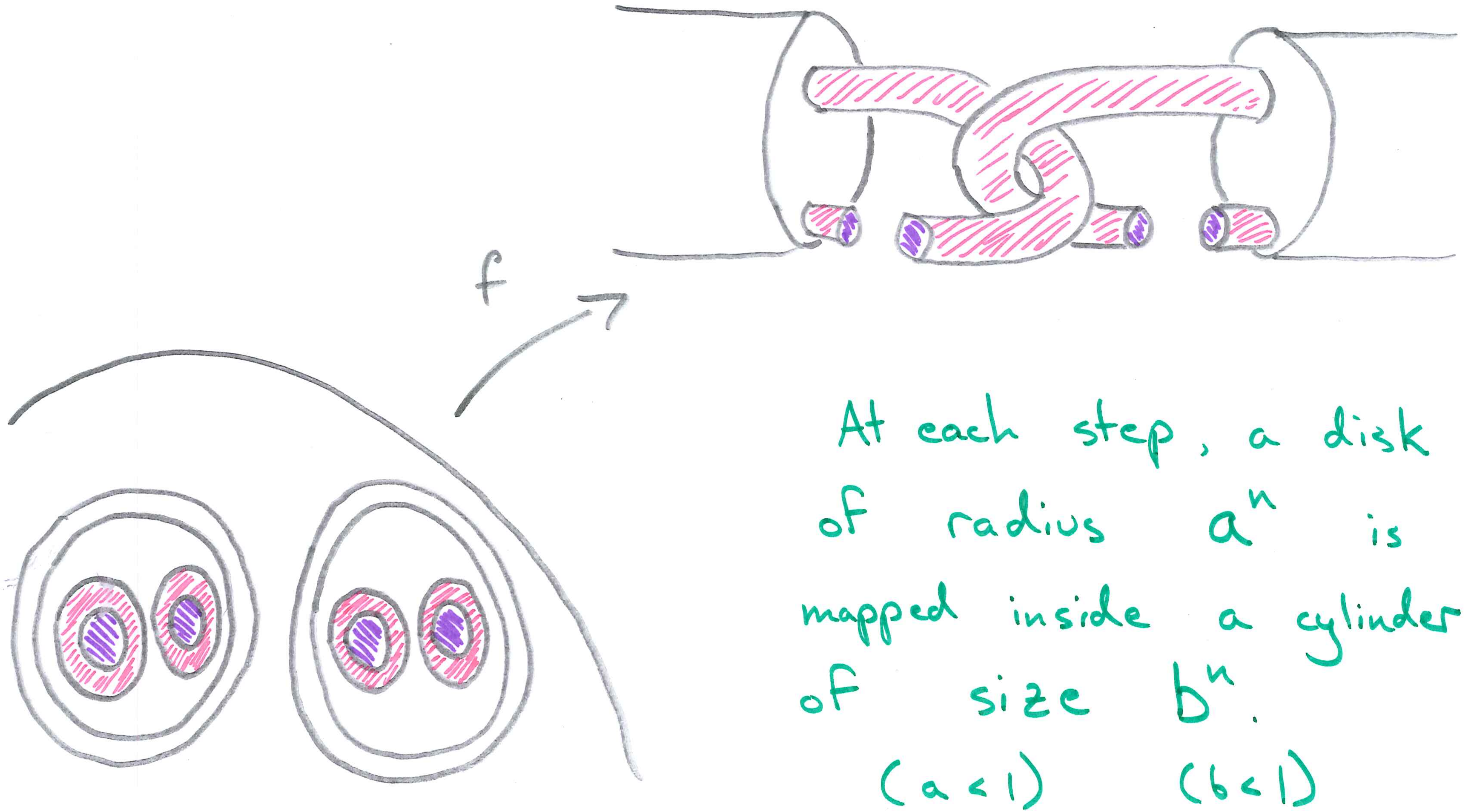
Get 2^n new cylinders.

At each step, the size of the cylinders shrinks by some factor $b < 1$.



The embedding $f : S^2 \longrightarrow \mathbb{R}^3$



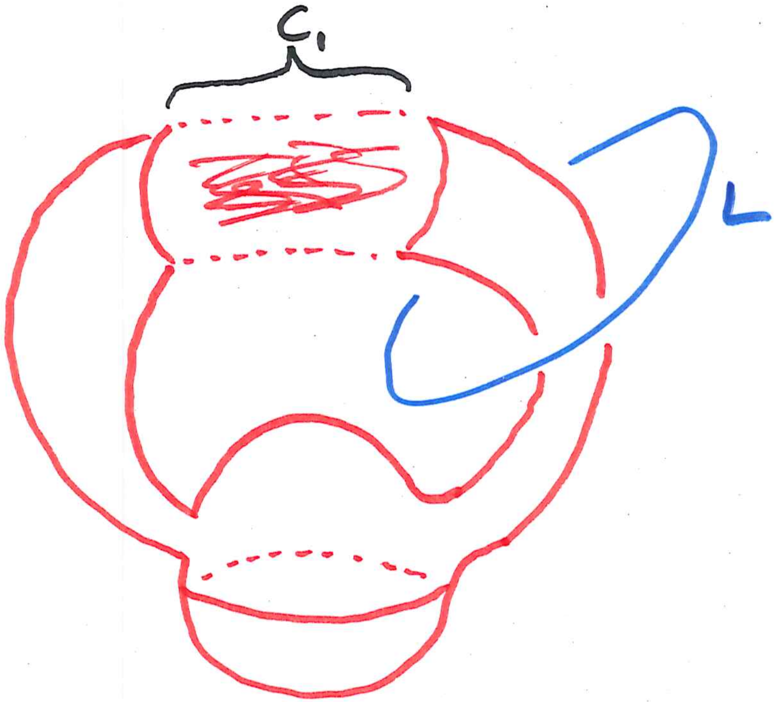


At each step, a disk of radius a^n is mapped inside a cylinder of size b^n.

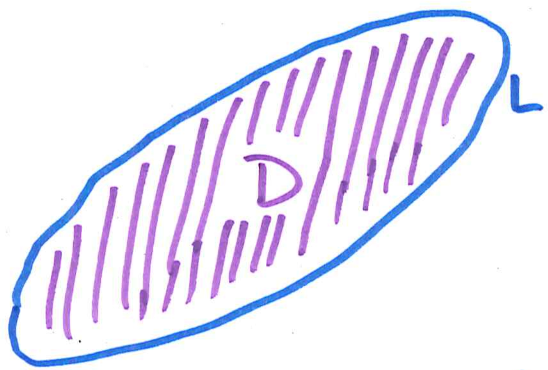
$(a<1)$ $(b<1)$

Hanging the Horned Sphere

Consider a loop L in $\mathbb{R}^3 \setminus f(S^2)$.

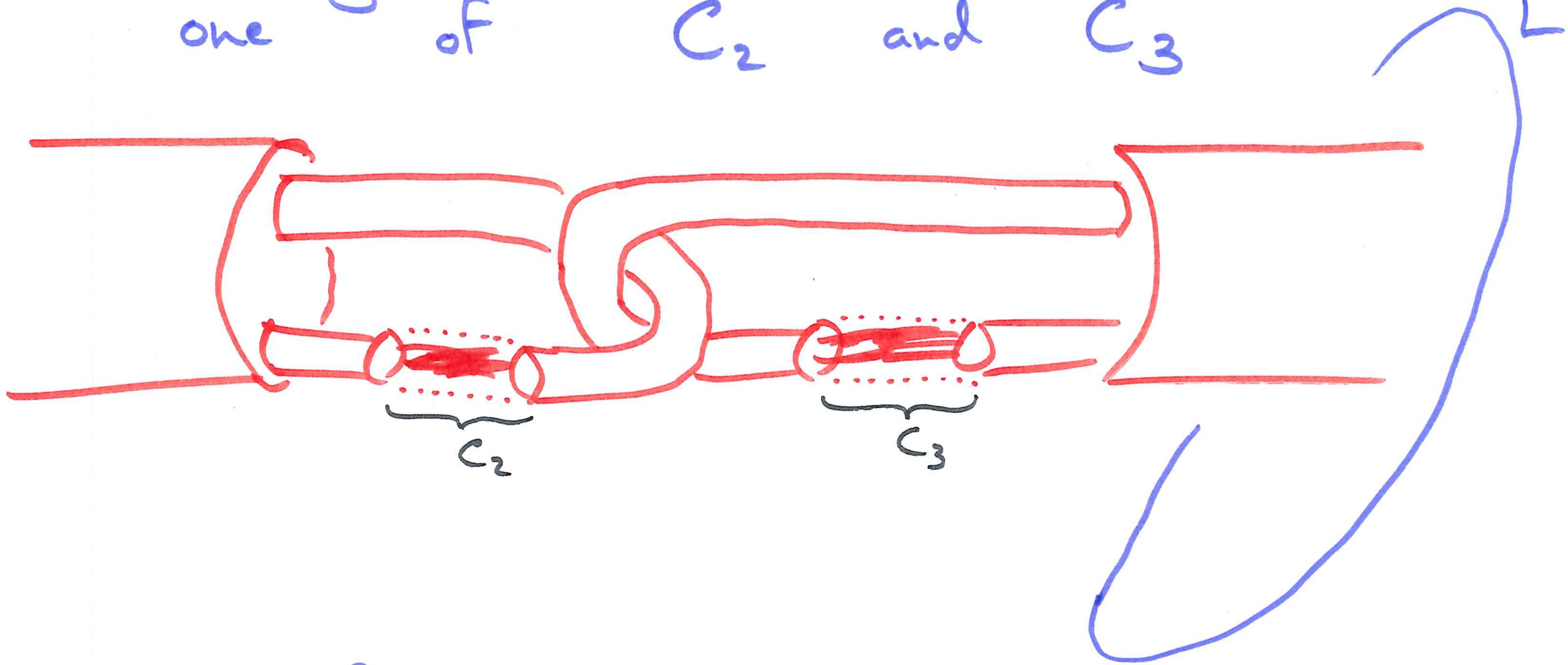


If L bounds a disc D

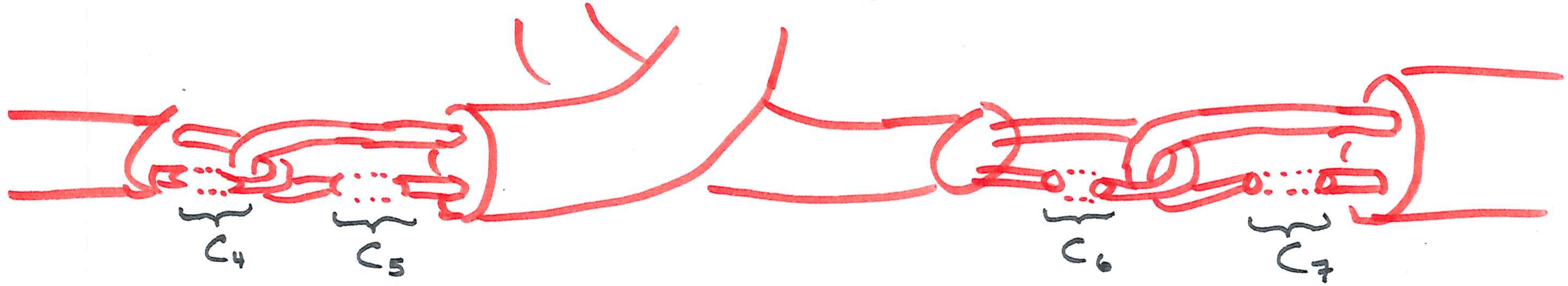


then D must intersect the cylinder C_1.

Similarly, D must intersect at least one of C_2 and C_3



and one of C_4, C_5, C_6, C_7



We can show D intersects each cylinder C_{n_k} in a sequence where the size of C_{n_k} goes to 0.

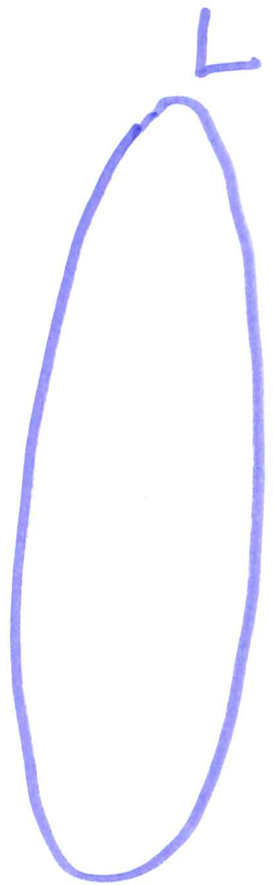
Then,

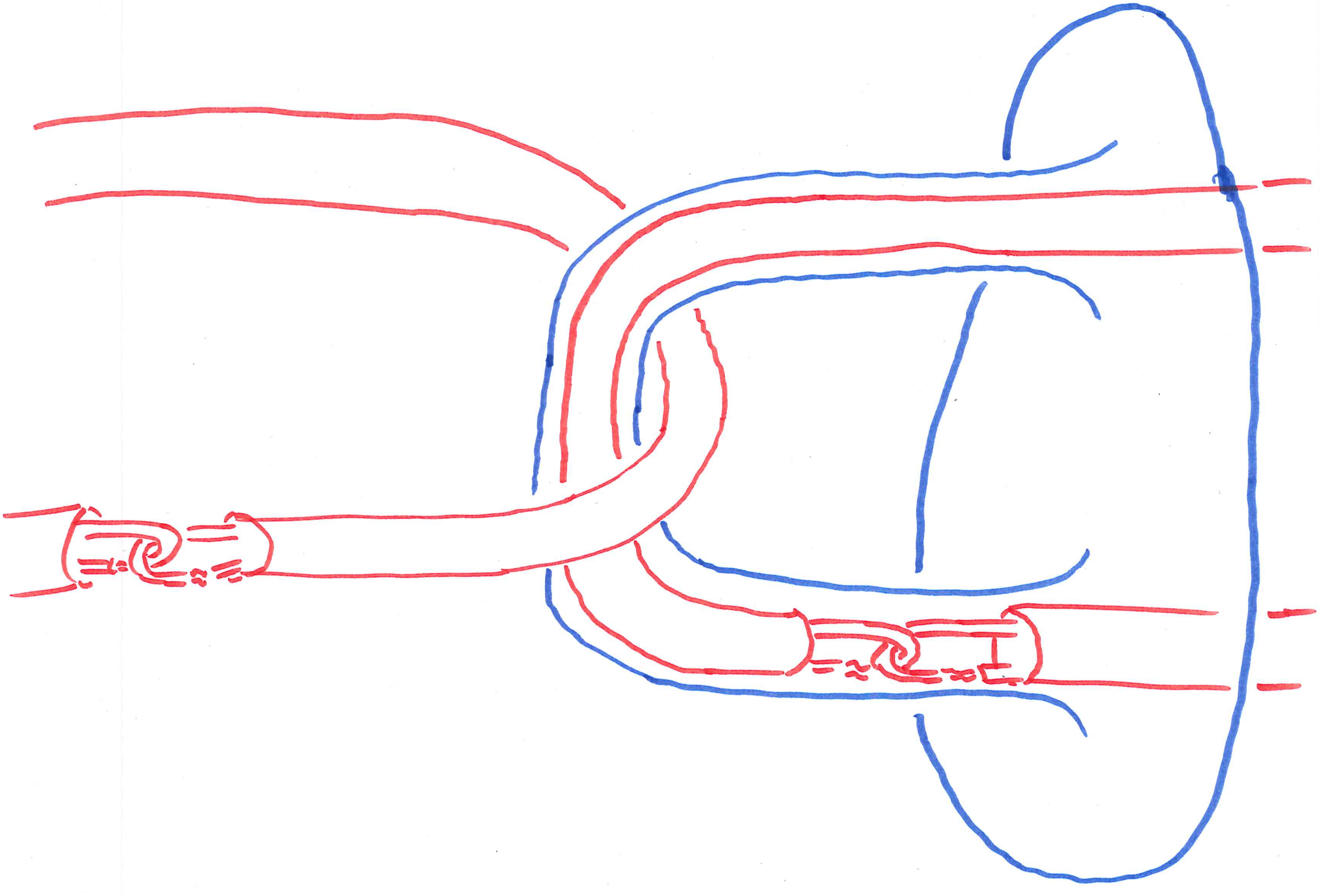
$$\operatorname{dist}\big(D\ ,\ \underbrace{f(S^2)}_{\text{the horned sphere}}\big) = 0$$

and so

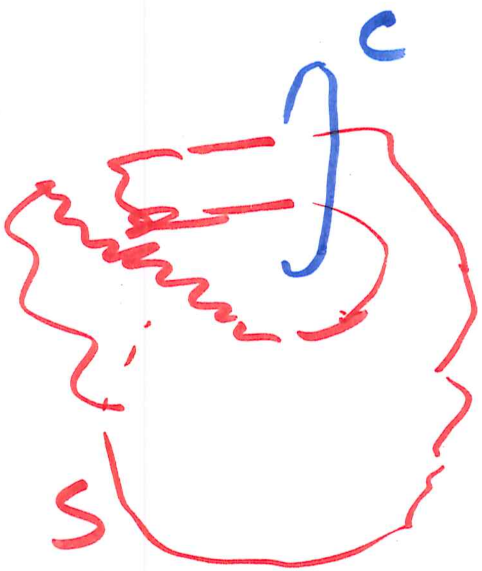
$$D \text{ intersects } f(S^2).$$

The loop L is the boundary of a surface

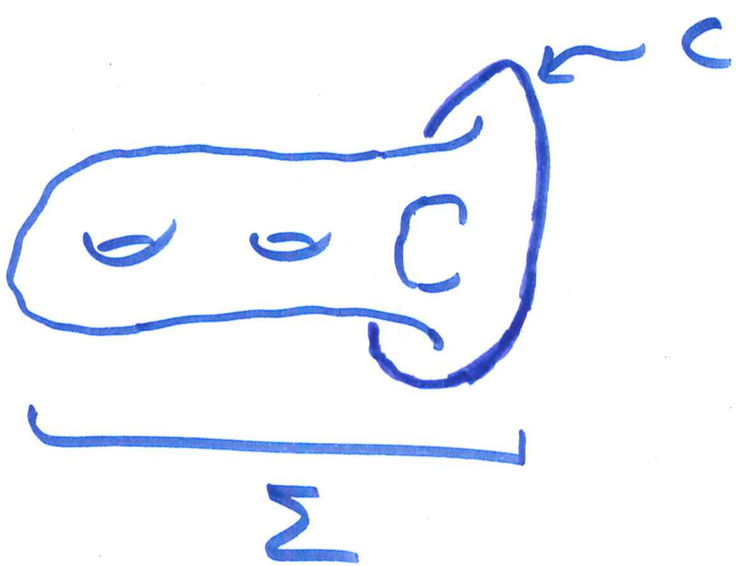




Alexander Duality shows that for any embedded sphere $S \subset \mathbb{R}^3$ and embedded circle $C \subset \mathbb{R}^3 \setminus S$, C is the boundary of some surface $\Sigma \subset \mathbb{R}^3 \setminus S$



$\Longrightarrow$



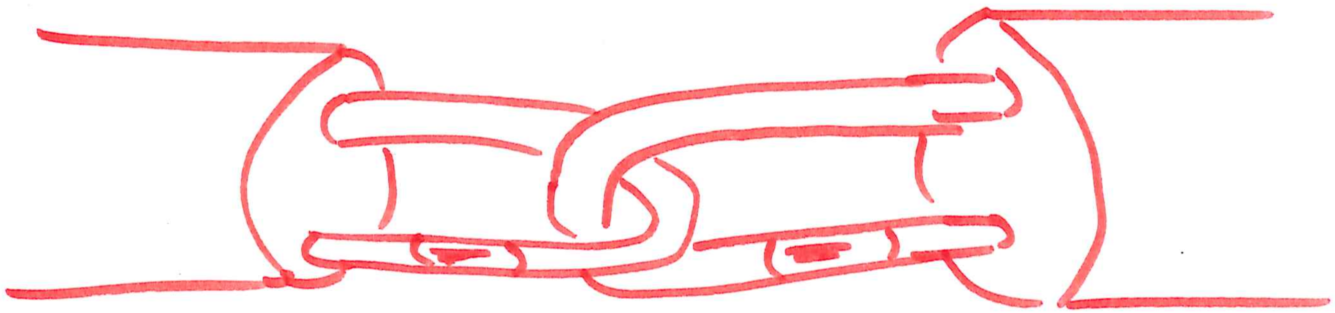
From

ALGEBRA

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix} \neq \begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

to

TOPOLOGY



Matrix multiplication is not abelian.

For $d \times d$ matrices

$$XY \neq YX \quad \text{in general.}$$

Let's MAKE IT ABELIAN!

Write $A \approx B$ if there are X and Y so that $A = XY$ and $B = YX$.

Also, if $A \approx B$ and $B \approx C$, then $A \approx C$.

If $A = XY$ and $B = YX$, then

$$\det(A) = \det(X)\det(Y) = \det(B).$$

It turns out $A \approx B$ if and only if $\det(A) = \det(B)$.

By <u>abelianizing</u> matrix multiplication, we have turned it into multiplication of real numbers (the determinants).

Alexander showed that if $f : S^2 \to \mathbb{R}^3$ is an embedding, then the fundamental group of $\mathbb{R}^3 \setminus f(S^2)$ has a trivial abelianization.

That is $A \approx B$ for any A and B in the group.

How can you define multiplication so that the abelianization is trivial?

Step 0: Include the identity element 1.

Step 1: Include an element A.

Step 2: Include two new elements B and C and require

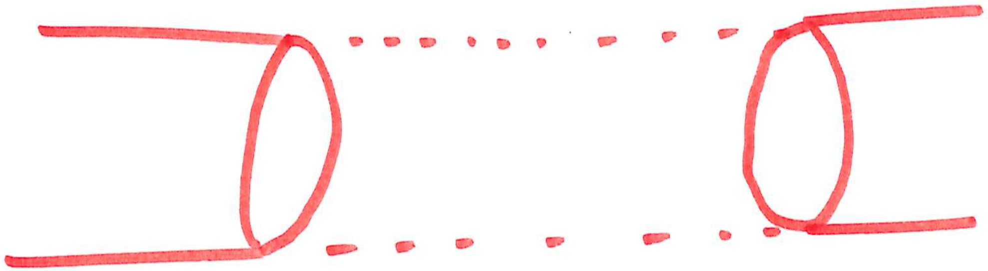
$$A = BCB^{-1}C^{-1}.$$

Step 3: Include new elements D, E, F, G, and require

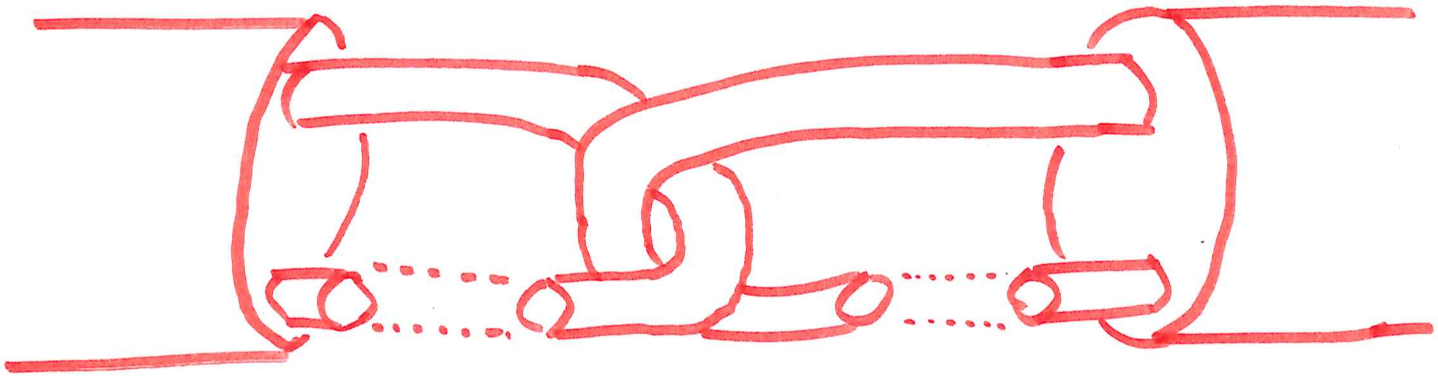
$$B = DED^{-1}E^{-1} \quad \text{and} \quad C = FGF^{-1}G^{-1}.$$

The algebraic step of adding B and C with $A = BCB^{-1}C^{-1}$

corresponds to the topological step of replacing



with



Continuous

v.

Smooth

Smooth Schönflies Problem:

If $f: S^{d-1} \to \mathbb{R}^d$ is a smooth embedding, is there a smooth change of coordinates $g: \mathbb{R}^d \to \mathbb{R}^d$, so that $g(f(S^{d-1}))$ is the standard sphere in $\mathbb{R}^d$?

Alexander (1924): Yes, if $d \leq 3$.

Smale (1961): Yes, if $d \geq 5$.

Open question for $d = 4$.