Def A diffeo $f: M \circlearrowleft$ is *Anosov* if $\exists \lambda > 1$ and a Df-inv't splitting

$$TM = E^u \oplus E^s$$

such that

$$\| Df\, v \| > \lambda \| v \| \quad \text{for } v \in E^u$$

and

$$\| Df\, v \| < \frac{1}{\lambda} \| v \| \quad \text{for } v \in E^s.$$

Hopf argument: C^2 Anosov $\Rightarrow$ ergodic

$\phi : M \to \mathbb{R}$ continuous

Birkhoff Erg Thm $\Longrightarrow$ ϕ^+ const along W^s

ϕ^- const along W^u

$\phi^+ = \phi^-$ exists a.e.

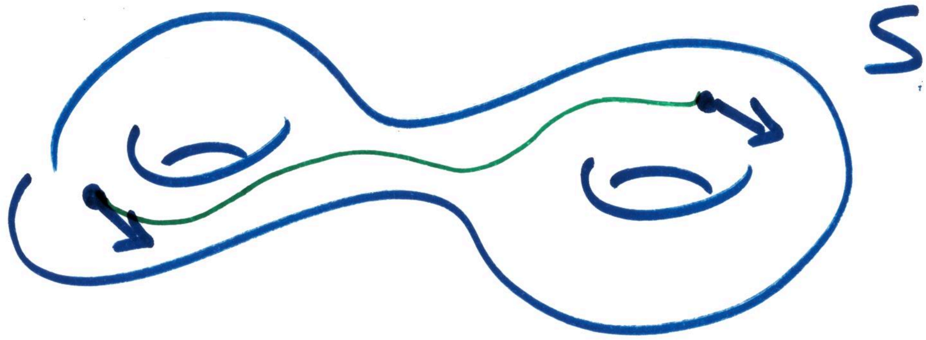
C^2 + Anosov

$\Downarrow$

W^s and W^u are absolutely continuous

$\phi^+ = \phi^-$ constant a.e

$\Rightarrow f$ is ergodic.



$M = T_1 S$

A Brief History of Smooth Ergodic Theory

1939 - Hopf

Geodesic flow of negatively curved surface is ergodic.

1967 - Anosov and Sinai

Anosov diffeos (e.g. cat map) and Anosov flows (e.g. geodesic flow above) are stably ergodic.

1974 - Brin and Pesin

~~Certain~~ Certain partially hyperbolic system are ergodic.

1994 - Grayson, Pugh, and Shub

The time-one map of the above geodesic flow is stably ergodic as a diffeo.

Grayson, Pugh, Shub

Two key ideas:

partial hyperbolicity

and

accessibility.

~~Def~~

φ_1 is the time-one map

All $f \underset{C^1}{\sim} \varphi_1$ meas pres.
are partially hyperbolic
and accessible
$\Longrightarrow$ ergodic.

Pugh Shub Conjectures.

<u>Conj 1</u> Ergodicity holds on an open and dense set of cons. partially hyperbolic (PH) diffeomorphisms.

Conservative = inv't meas equiv to Leb.

<u>Conj 2</u> Accessibility is open and dense

<u>Conj 3</u> Accessibility implies ergodicity.

Def A diffeo $f: M \circlearrowleft$ is partially hyperbolic if there is a Df-inv't splitting

$$TM = E^u \oplus E^c \oplus E^s$$

s.t.

$$\|Df v^s\| < 1 < \|Df v^u\|$$

$$\|Df v^s\| < \|Df v^c\| < \|Df v^u\|$$

for unit vectors $v^* \in E^*$.

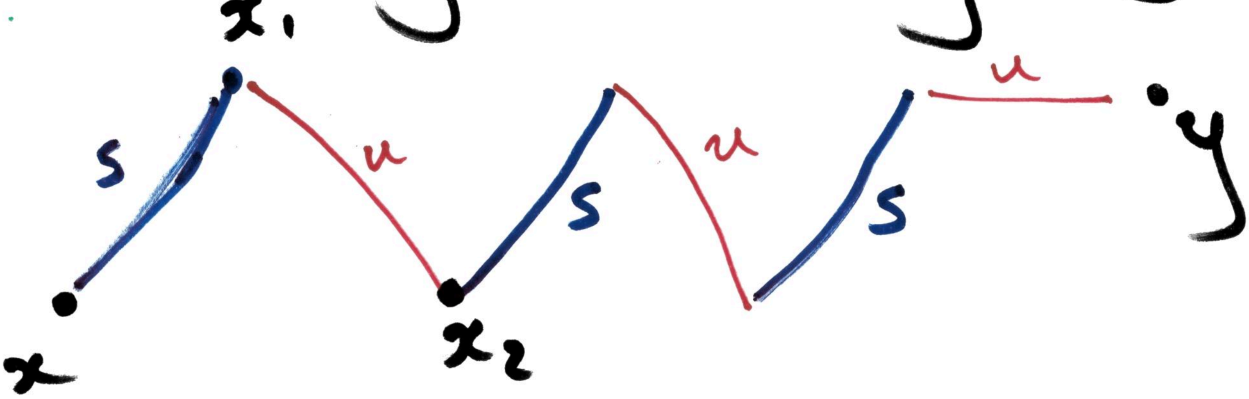
center bunched:

$$\|Df v^s\| < \frac{\|Df v_1^c\|}{\|Df v_2^c\|} < \|Df v^u\|$$

$$v_1^c, v_2^c \in E_x^c.$$

Accessibility: for any $x, y \in M$

$\exists$



"Proof" that PH + acc. implies ergodicity.

$\phi : M \to \mathbb{R}$ cts.

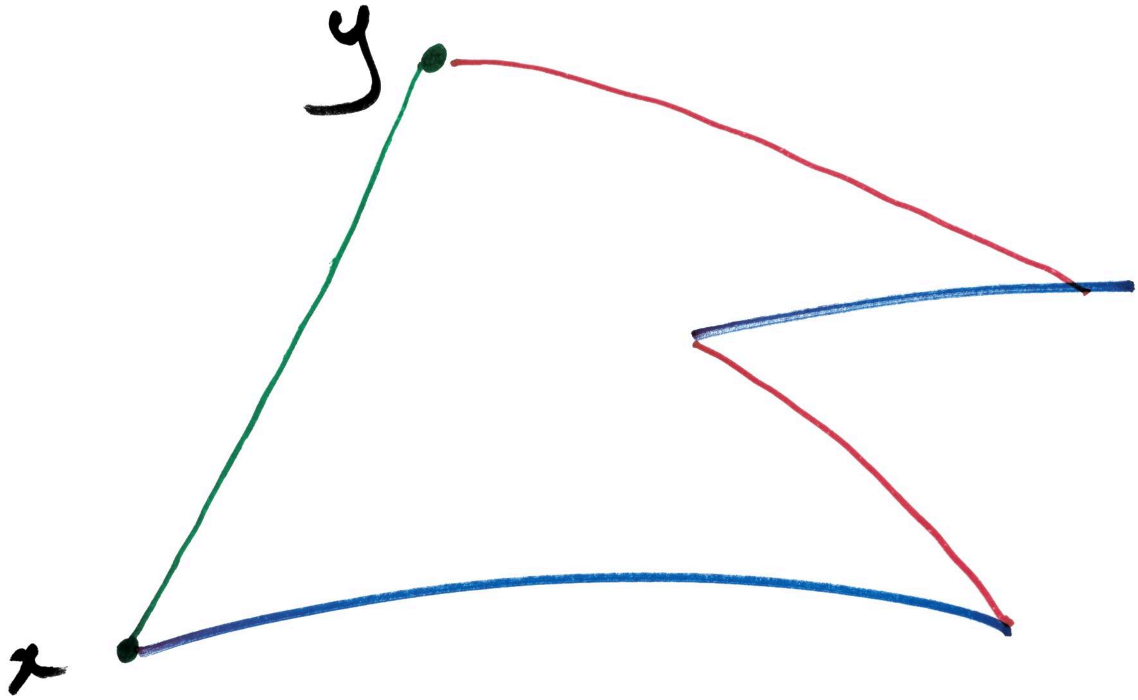
ϕ^+ is const on stable leaves.

W^s foln tangent E^s.

ϕ^- is const on unstable leaves.

$$\phi^+(x) = \phi^+(x_1) = \phi^-(x_1) = \phi^-(x_2)$$

$$= \cdots = \phi^+(y).$$



y
x

$f \in C^2$, partially hyperbolic

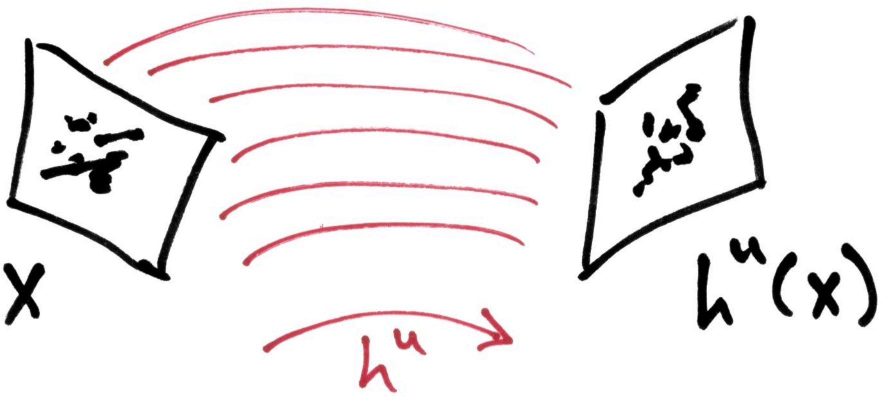
volume preserving

What still holds?

$\exists$ fol'n W^s tangent to E^s

— W^u ——— E^u

These fol'ns are abs cts



$$m(X) = 0 \iff m(h^u(X)) = 0$$

Birkhoff Erg Thm: $\phi : M \to \mathbb{R}$ cts

ϕ^+ const on stable leaves

ϕ^- const on unstable leaves

$\phi^+ = \phi^-$ a.e.

W^c is almost never absolutely continuous (if it even exists!)

<u>Thm</u> [Rodriguez Hertz - Rodriguez Hertz - Ures]

Pugh - Shub conjectures are true when $\dim E^c = 1$.

[Burns - Wilkinson]

<u>Thm</u> If $f \in C^2$ partially hyperbolic volume preserving accessible and "center bunched"

then

f is ergodic.

Pugh-Shub Conjs:

"PH leads to
stable ergodicity"

Does some form of
converse hold?

Yes ...
with the right def'n
of stable ergodicity.

Def A C^2-diffeo f is

stably ergodic

if every C^2-diffeo which C^1-close to f is also ergodic.

Using "a conservative pasting lemma" of Arbieto - Matheus:

Mañé:

In dimension 2,

Anosov $\iff$ stably ergodic

[and by Franks

Anosov $\implies$ top conj to a linear map]

Arbieto - Matheus:

In dim 3,
a stably ergodic <u>flow</u>
is Anosov.

Díaz - Pujals - Ures:

In dim 3,
a stably ergodic <u>diffeo</u>
is (weakly) partially hyperbolic

Bonatti - Díaz - Pujals:

In ANY dimension,
a stably ergodic diffeo
is "volume partially hyperbolic."

Consider symplectomorphisms of (M, ω)

$$f : M \to M \qquad f^* \omega = \omega$$

Horita - Tahzibi (2006):

stably ergodic $\Rightarrow$ partially hyperbolic

Ávila - Bochi - Wilkinson: (2009)

~~stable~~ ergodicity is C^1-generic among partially hyperbolic sympl.

not stable. Oops!

Announced by

Ávila - Crovisier - Wilkinson:

Ergodicity is C^1-open and C^1-dense among C^2 conservative partially hyperbolic diffeomorphisms.

[A C^1- answer to the Pugh - Shub conjectures]

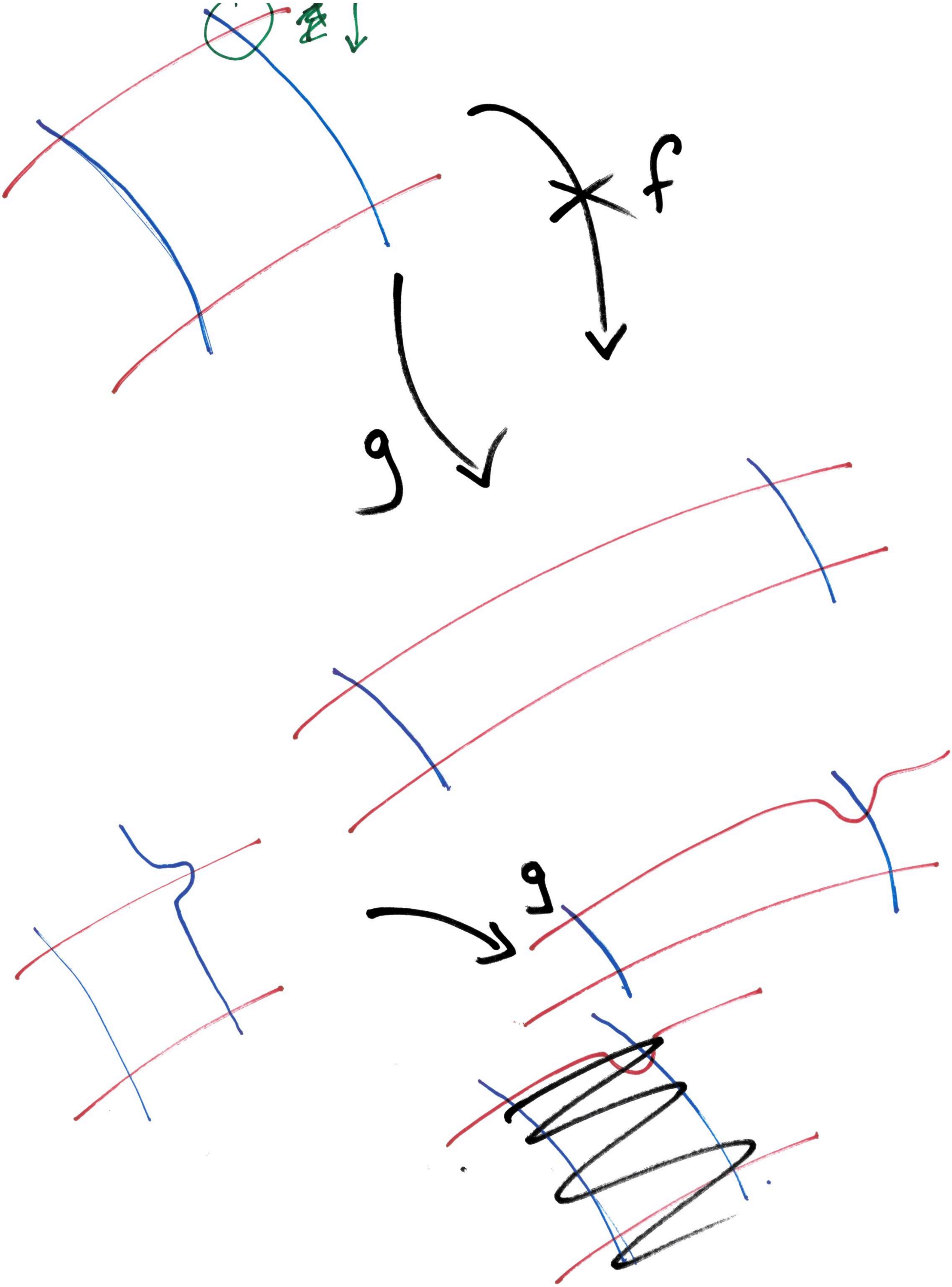
Density of accessibility if $\dim E^c = 1$.

[Hertz - Hertz - Ures]

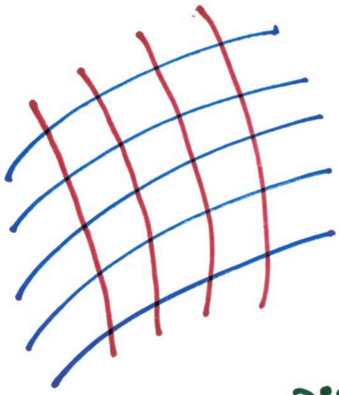
If f is not acc. there is a lamination Λ of leaves tangent to

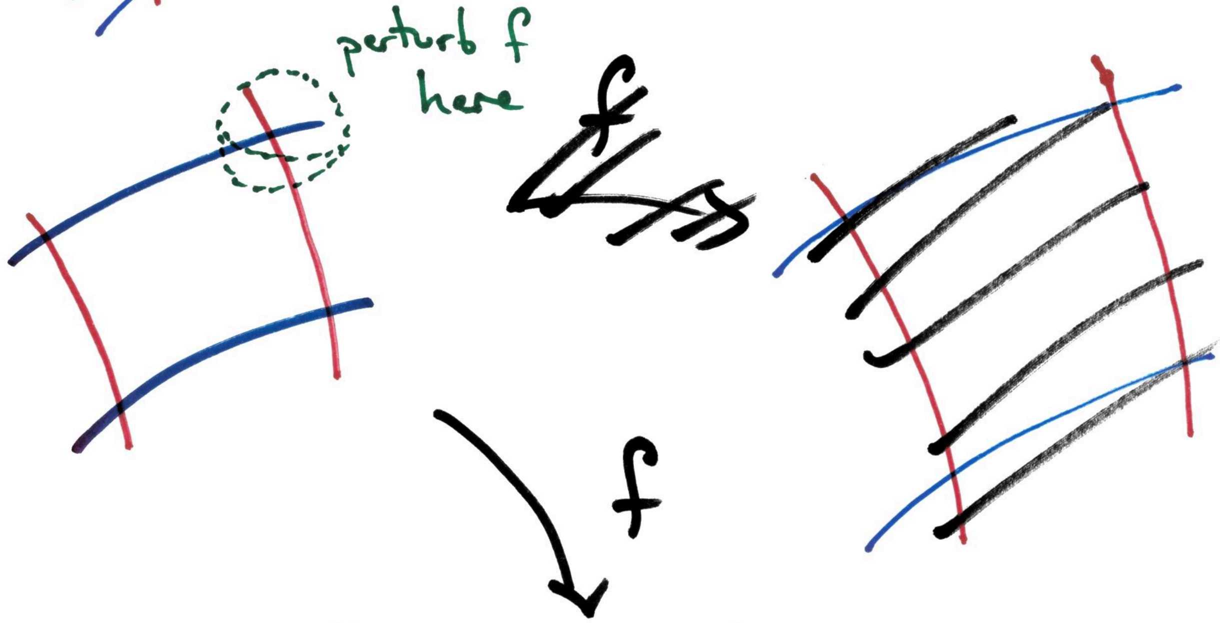
$$E^u \oplus E^s$$

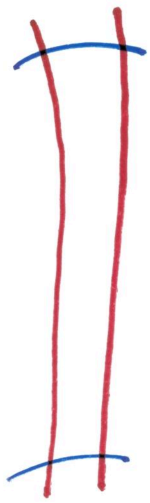
Perturb to "break" all the leaves.


f
g
g

2D leaf in 3D space





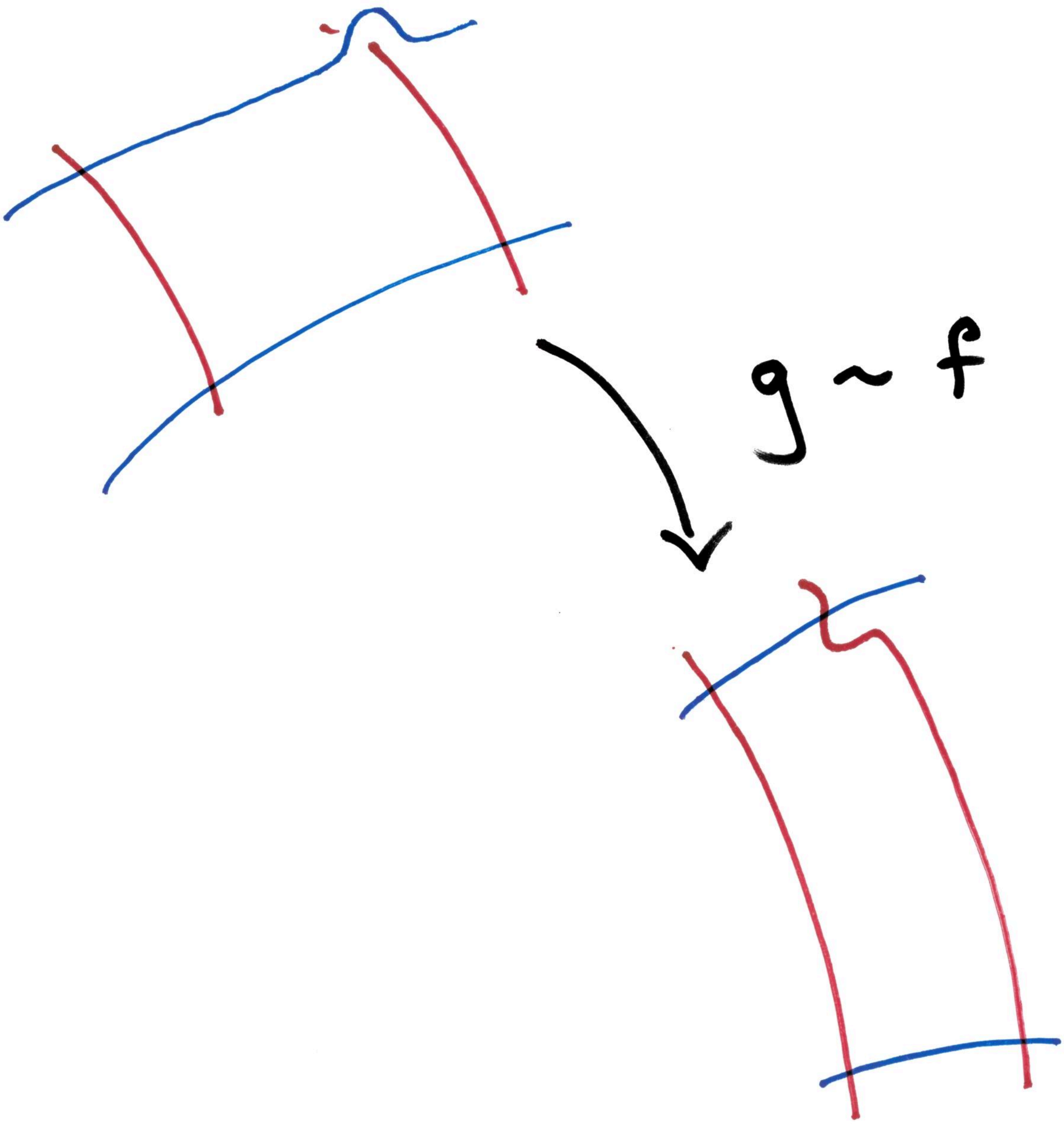


stable ~~manifold~~ leaf
determined by the future

⇝ changed

unstable leaf
determined by the past

⇝ unchanged.

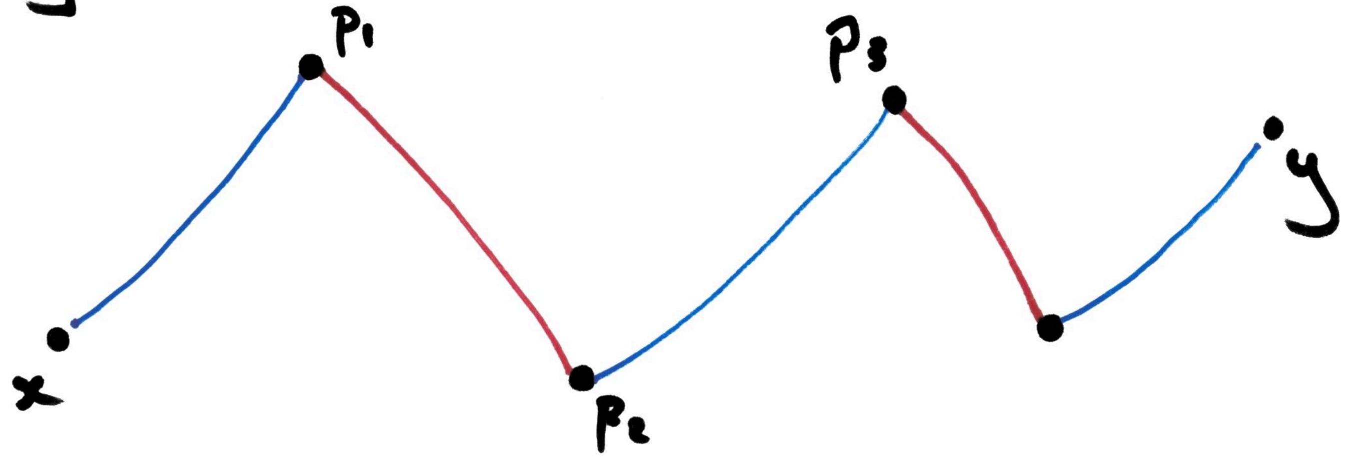


Not quite that simple as orbits can go through the perturbed area multiple times.

"Proof" on Conj 3

For any $x, y \in M$

$\exists$



$$\phi^+(x) \overset{\checkmark}{=} \phi^+(p_1) = \phi^-(p_1) \overset{\checkmark}{=} \phi^-(p_2) = \phi^+(p$$

$\vdots$

Don't know if this holds

BET holds almost everywhere.

Lebesgue density pts.

Idea: Using Leb. density pts.

$A \subset \mathbb{R}$ meas

$x \in \mathbb{R}$ is a <u>Leb</u> <u>density</u> <u>pt</u>

if $$\lim_{\varepsilon \to 0} \frac{m(A \cap [x-\varepsilon, x+\varepsilon])}{m([x-\varepsilon, x+\varepsilon])} =$$

<u>Thm</u>

If $\hat{A}$ is the set of Leb density points of A

then

A equals $\hat{A}$ mod zero.

In a manifold M, define Leb density points using balls:

$$\lim_{r \to 0} \frac{m(A \cap B_r(x))}{m(B_r(x))} = 1$$

Thm still holds!

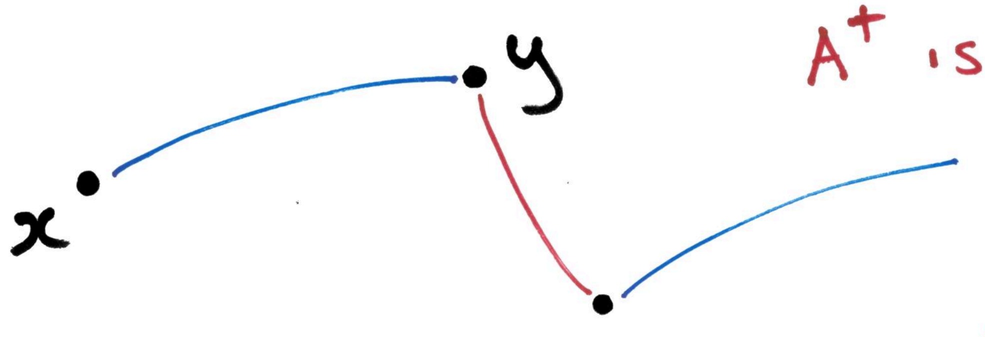
Can show: $\phi : M \to \mathbb{R}$ continuous

$c \in \mathbb{R}$

$A^+ := \{ x \in M : \phi^+(x) > c \}$

then A^+ is s-saturated

A^+ is essential u-saturated



$A^+ \equiv A^-$ mod zero.

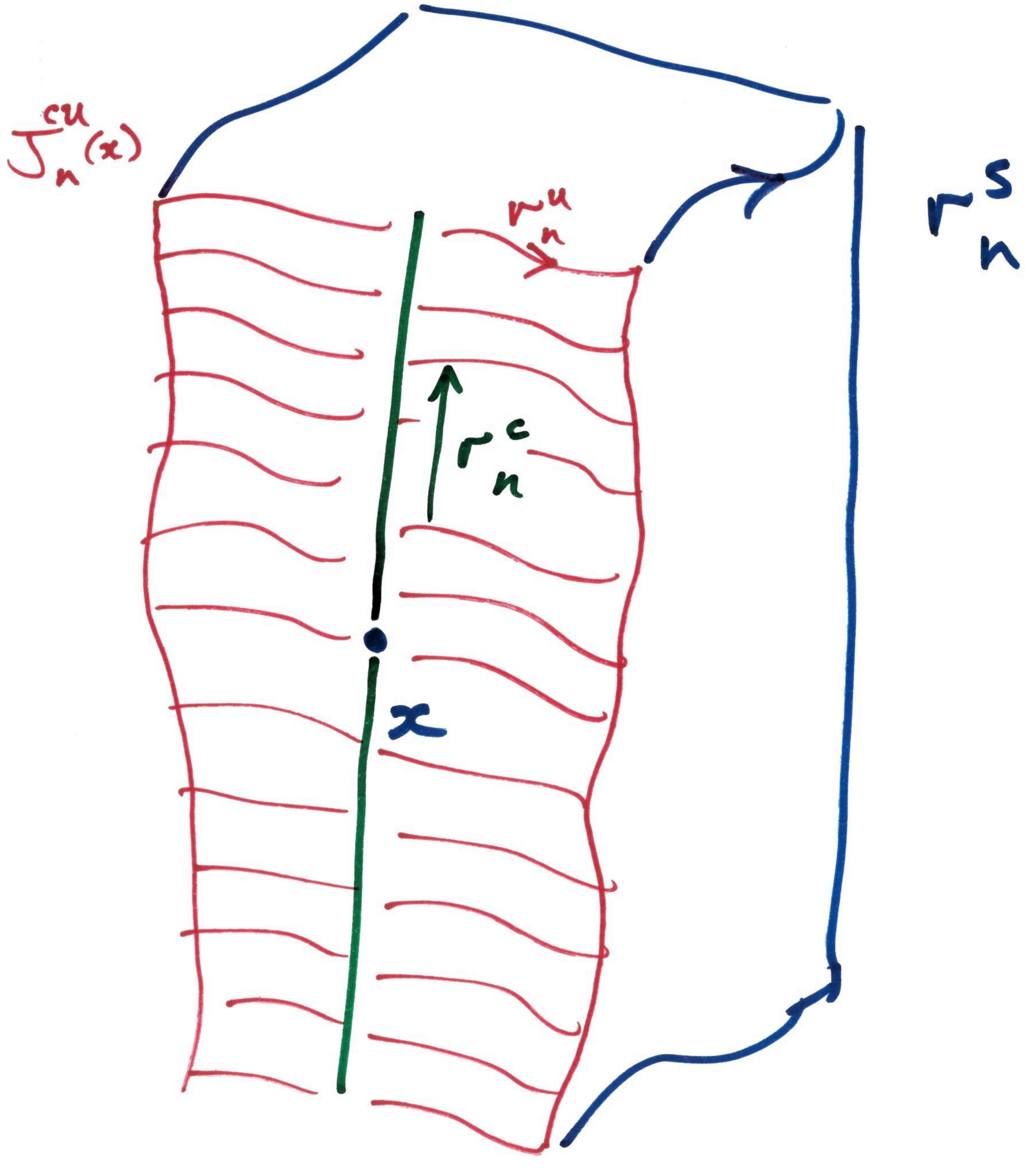
x is a Leb density pt of A^+

iff y ________ , ________

________ , ________

Julienne

$J_n(x)$



$r_n^* \to 0$ as $n \to \infty$

$J_n(x)$ nest down to x.

Lebesgue density point:

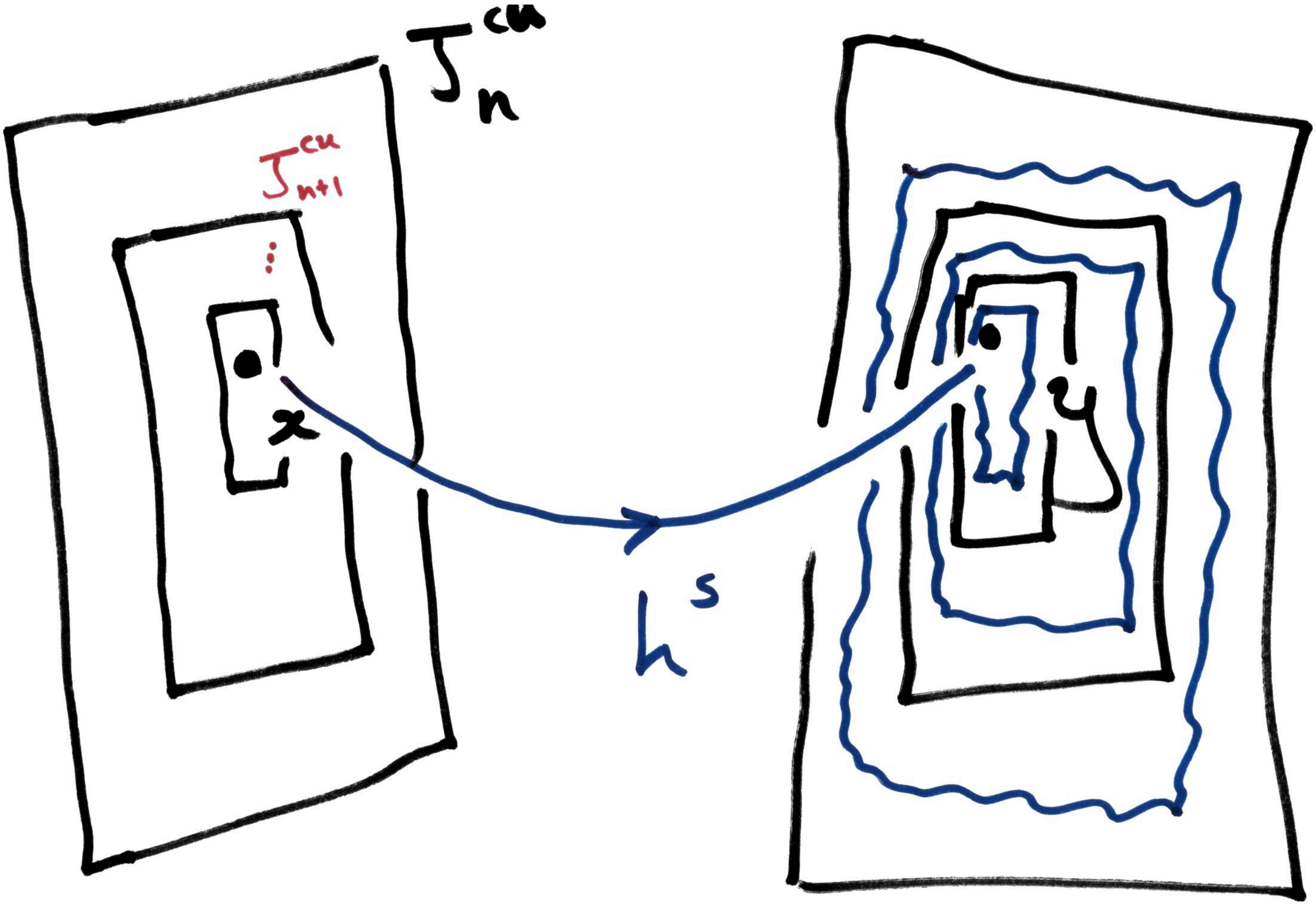
$$\lim_{n\to\infty} \frac{m(X \cap B_n(x))}{m(B_n(x))} = 1$$

equiv (in this setting)
to a
Julienne density point

$$\lim_{n\to\infty} \frac{m(X \cap J_n(x))}{m(J_n(x))} = 1$$

If X is s-saturated, then also equiv to

$$\lim_{n\to\infty} \frac{m(X \cap J_n^{cu}(x))}{m(J_n^{cu}(x))} = 1$$



Sequences

$$h^s(\mathcal{J}_n^{cu}(x)) \quad \text{and} \quad \mathcal{J}_n^{cu}(y)$$

are "internested".

$$\Rightarrow \lim \frac{m(X \cap \mathcal{J}_n^{cu}(x))}{m(\mathcal{J}_n^{cu}(x))} = 1 \Leftrightarrow \lim \frac{m(X \cap \mathcal{J}_n^{cu}(y))}{m(\mathcal{J}_n^{cu}(y))}$$

$$\Rightarrow \lim_{n\to\infty} \frac{m\left(X \cap J_n^{cu}(x)\right)}{m\left(J_n^{cu}(x)\right)} = 1$$

if and only if

$$\lim_{n\to\infty} \frac{m\left(X \cap J_n^{cu}(y)\right)}{m\left(J_n^{cu}(y)\right)} = 1$$

To prove erg

ETS

for every cts $\phi : M \to \mathbb{R}$

and $c \in \mathbb{R}$

the set

$$\left\{ \cancel{\phi^+} x \ : \ \phi^+(x) > c \right\}$$

has either full

meas or zero

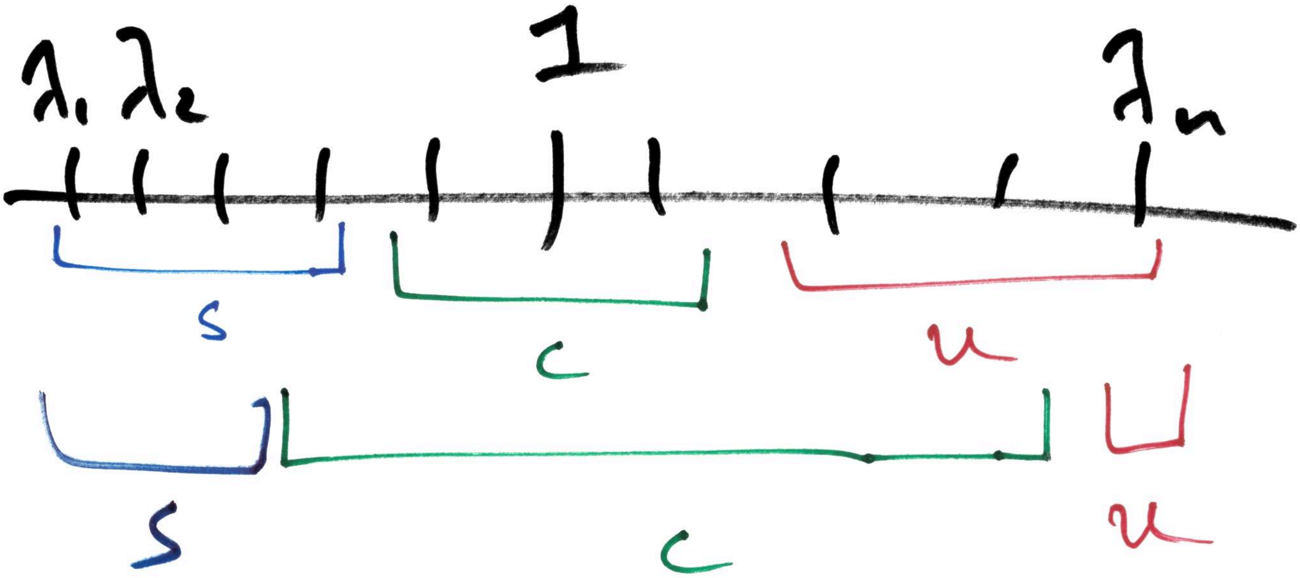
meas.

$f \sim A$ $n \times n$ matrix

integer vals

$\det A = 1$

$A : \mathbb{R}^n / \mathbb{Z}^n \circlearrowleft$



Conj [Pujals]

Every PH diffeo. in dim 3

is "flow-like"

skew-product over $A : \mathbb{T}^2 \circlearrowleft$

or

Derived-from-Anosov on $\mathbb{T}^3$.

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ on } \mathbb{T}^3.$$

p.h.

Leb meas.

not ergodic.

$A \times id.$