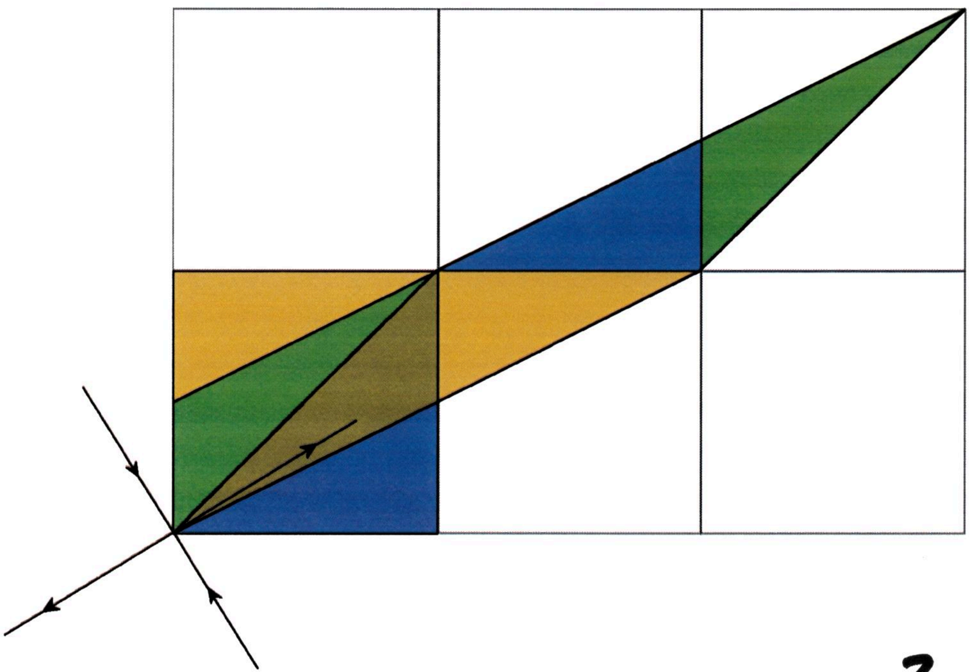
Lecture I



$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$$

cat map

eigenvalues $\lambda = \dfrac{3+\sqrt{5}}{2} > 1.$

$\lambda^{-1} < 1.$

Arnold's Cat Map $A : \mathbb{T}^2 \circlearrowleft$

is structurally stable:

every $f \sim A$ is topologically conjugate to A $\left[\begin{array}{l} f = h^{-1} \circ A \circ h \\ \text{for a homeo } h \end{array} \right]$

That's topology.

What about measure theory?

A is ergodic (w.r.t. Lebesgue)

Smale to Anosov:

Is every $f \sim A$ also ergodic?

Birkhoff Ergodic Theorem (1931)

Let $f: M \circlearrowleft$ be a diffeomorphism and μ an inv't meas. ($\mu(M)=1$)

For any integrable $\phi: M \to \mathbb{R}$ the limits

$$\phi^+(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi f^k(x)$$

and

$$\phi^-(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi f^{-k}(x)$$

exist and are equal for μ-a.e. x

"Def'n" (f, μ) is ergodic if ϕ^+ and ϕ^- are constant μ-a.e. for every continuous $\phi: M \to \mathbb{R}$.

$\cdot\, fx$

$\cdot\, f^2x$

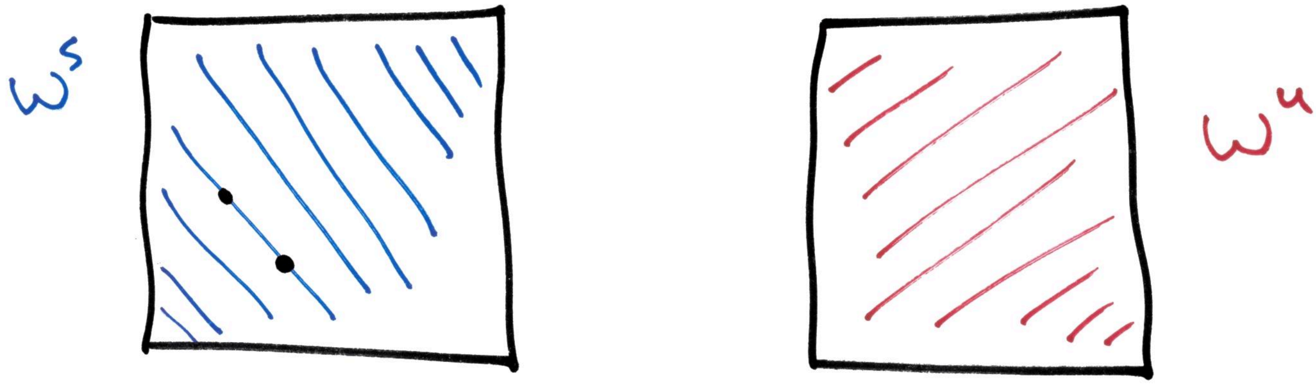
$\overset{\bullet}{x}$

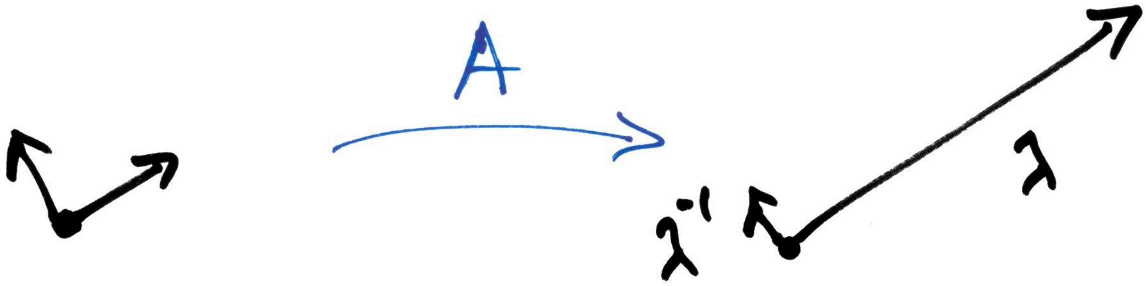
$\cdot\, f^3x$

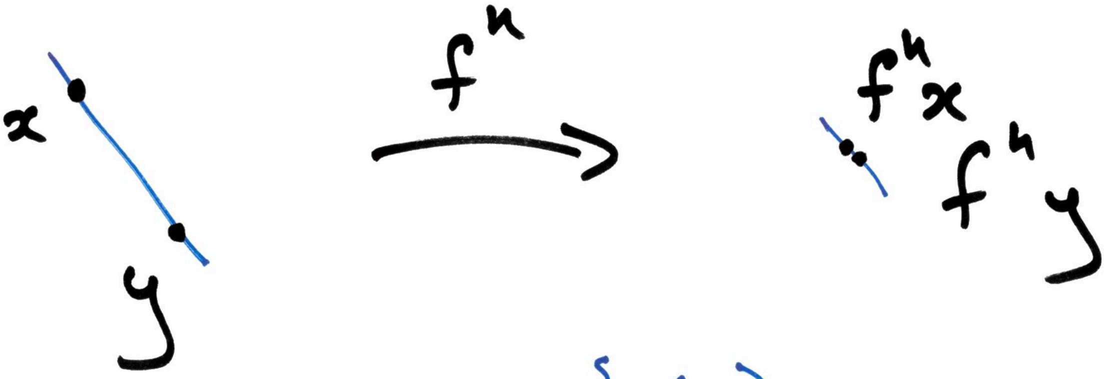
$$\frac{1}{4}\sum_{k=0}^{3}\phi f^{k}(x)$$

Cat map $A : \mathbb{T}^2 \circlearrowleft$

eigenvalues $\lambda > 1 > \lambda^{-1}$







$x \in W^s_{loc}(y)$

$\Longrightarrow \quad d(f^n x, f^n y) \leq \lambda^{-n} d(x,y).$

$\phi : M \to \mathbb{R}$ unif continuous

$$|\phi f^n(x) - \phi f^n(y)| \to 0$$

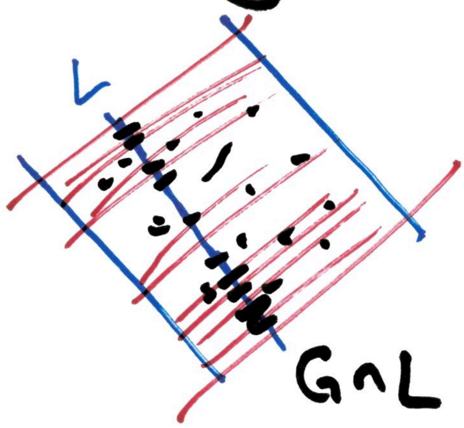
$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \phi f^k(x) - \frac{1}{n} \sum_{k=0}^{n-1} \phi f^k(y) \right| \to 0.$$

$$\phi^+(x) = \phi^+(y).$$

ϕ^+ is constant on stable leaves

ϕ^- is constant on unstable leaves.

Rectangle R.



$\phi : M \to \mathbb{R}$ continuous

a "good" set G

$x \in G$

if $\phi^+(x) = \phi^-(x)$ exist.

G has full meas in R.

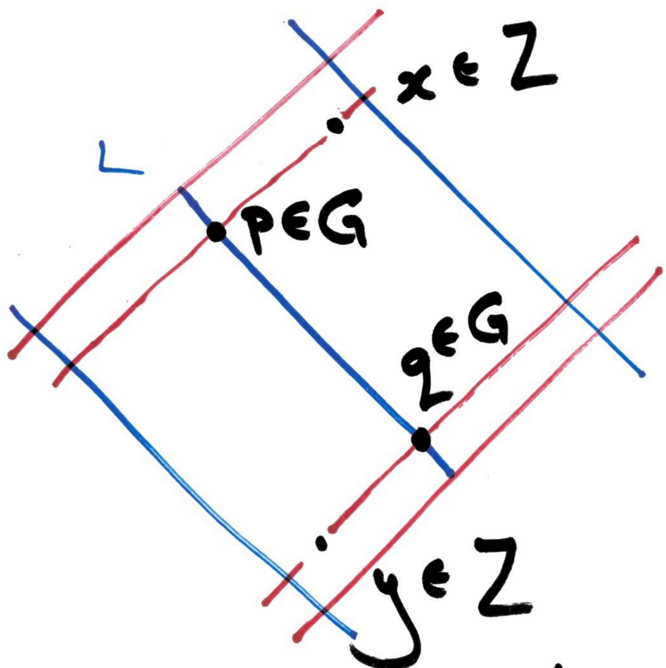
Fubini $\Rightarrow \exists L$ stable segment

s.t. $G \cap L$ in a set of full (1-dim'l) measure.

Define $Z = \bigcup_{z \in G \cap L} W^u(z)$.

Z has full measure in R.

Claim if $x, y \in Z$ then $\phi^-(x) = \phi^-(y)$



$x \in W^u(p)$.

$p \in W^s(q)$.

$q \in W^u(y)$

$$\phi^-(x) = \phi^-(p) = \phi^+(p) = \phi^+(q) = \phi^-(q) = \phi^-(y$$

ϕ^- is constant a.e. on R.

ϕ^- is constant a.e. on M

$\overset{\shortparallel}{\phi^+}$

Conclude, A is ergodic.

What happens when we perturb?

A is an Anosov diffeo

$\Longrightarrow$ all $f \sim A$ are also Anosov.

<u>Def'n</u> A diffeo $f : M \circlearrowleft$ is <u>Anosov</u>

if $\exists \lambda > 1$ and a Tf-inv't splitting

$$TM = E^u \oplus E^s$$

(that is, $T_x M = E^u_x \oplus E^s_x$ for all $x \in M$)

s.t.

$$\| Df\, v \| > \lambda \| v \| \quad \text{for } v \in E^u$$

and $$\| Df\, v \| < \frac{1}{\lambda} \| v \| \quad \text{for } v \in E^s.$$

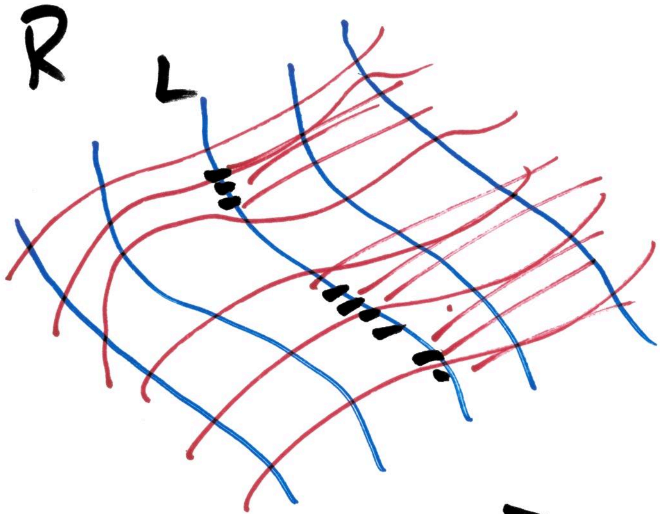
A

f

ϕ^+ is const on stable leaves

ϕ^- is const on unstable leaves

R L



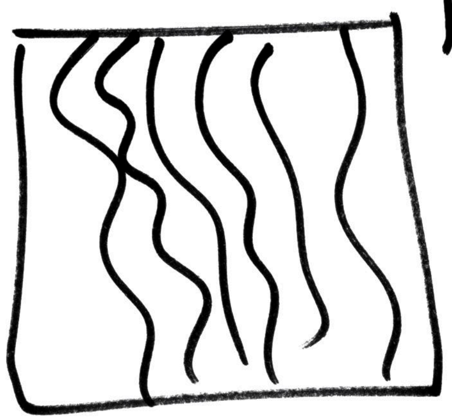
"good" set G_f of full meas

Does G intersect a stable segment in full meas?
1 dim'l.

Given $G \cap L$ of full meas.

Does $\bigcup_{z \in G \cap L} W^u(z)$ have full meas?

There is a C^0 foln



R of the square

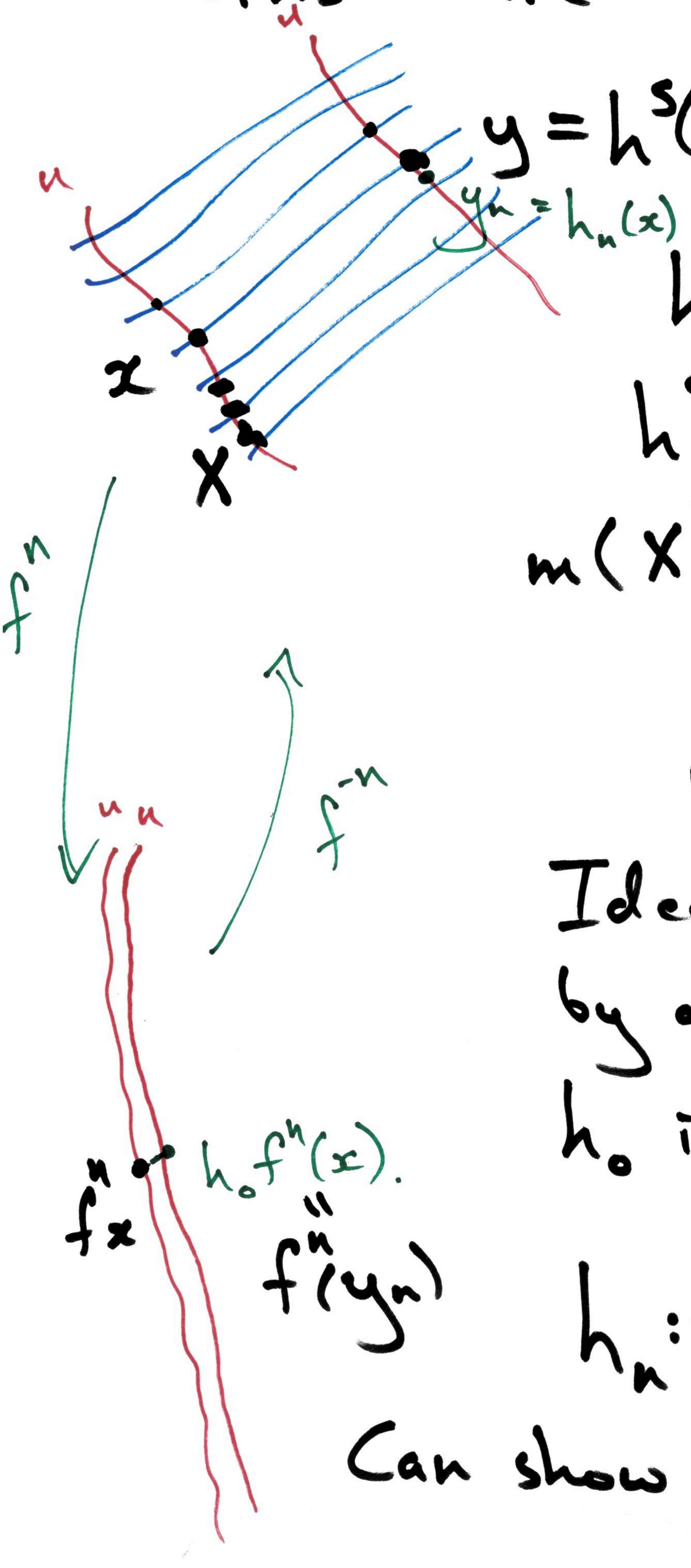
Each leaf is C^∞

And there is a full meas subset $X \subset R$ such each leaf intersects X in <u>one point</u>.

Milnor Math. Intelligencer

Katok.

Need to show W^s and W^u foli'ns are absolutely continuous



stable holonomy

$h^s : W^u_{loc}(x) \to W^u_{loc}(y)$

h^s is abs cts.

$m(X) = 0 \iff m(h^s(X)) = 0$

m is Lebesgue.

Idea: approx W^s by a smooth foln W_0

h_0 is holonomy by W_0

$h_n := f^{-n} \circ h_0 \circ f^n$

Can show $h_n \rightrightarrows h^s$.

$$h_n = f^{-n} \circ h_0 \circ f^n$$

Let Jh_n be the Radon-Nikodym derivative of h_n

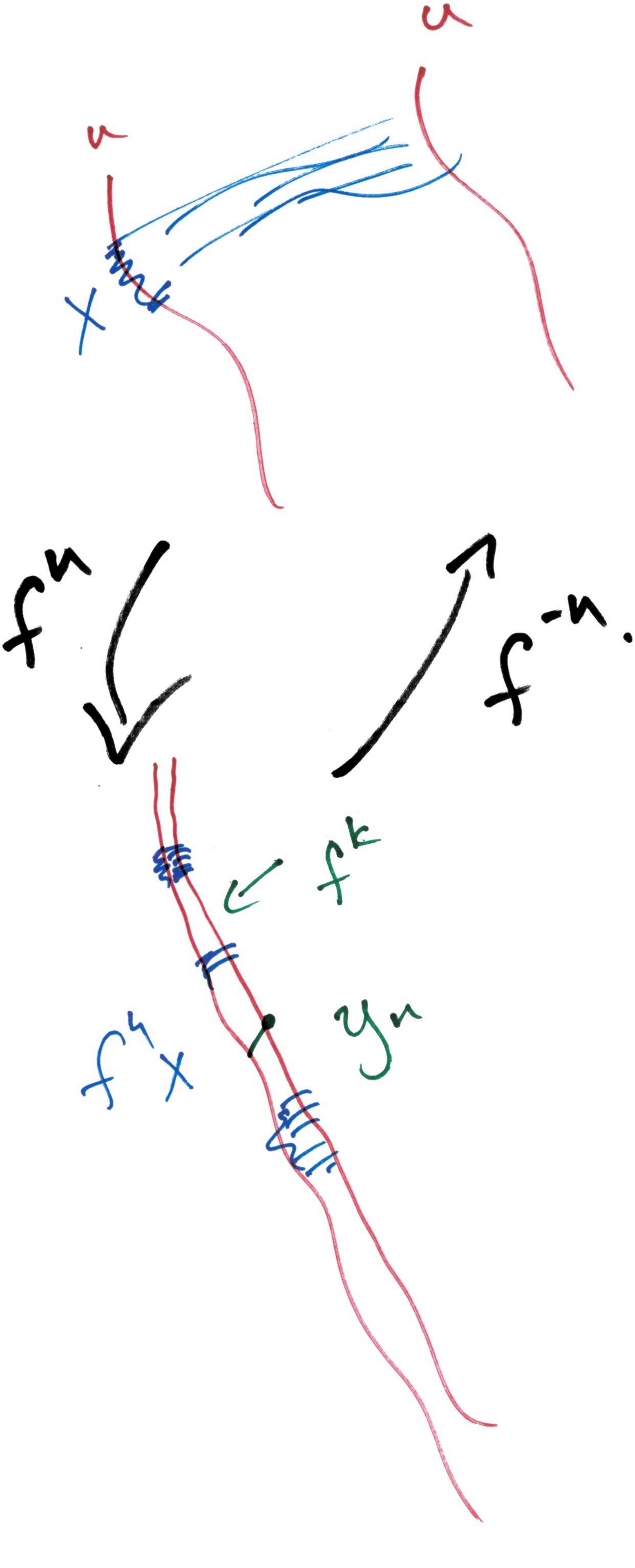
If $h_n \rightrightarrows h^s$

$\lim_{n\to\infty} Jh_n$ exists and conv uniformly

then

Jh^s exists.

and h^s abs continuous.



u
u
x
f^n
f^{-n}.
f^k
y_n
$f^n x$

$$h_n = f^{-n} \circ h_0 \circ f^n$$

$$Jh_n(x) = \left[Jf^n(y_n)\right]^{-1} \cdot Jh_0(f^n x) \cdot Jf^n(x$$

$$= \underbrace{Jh_0(f^n x)}_{\text{tends to one.}} \cdot \prod_{k=0}^{n-1} \frac{Jf(f^k x)}{Jf(f^k y_n)}$$

$$y_n = h_n(x) \xrightarrow{n\to\infty} y.$$

Jf is the $^{R\text{-}N}$ deriv of $Df|_{E^u}$.

Now $y \in W^s_{loc}(x)$

$$\Rightarrow \quad d(f^k x, f^k y) < \lambda^{-k}$$

$k > 0$

$\lambda > 1.$

Can show

$$d(f^k x, f^k y_n) < \lambda^{-k}$$

indep of n.

$f \in C^2$.

Jf is Lipschitz

$\log Jf$ is Lipschitz.

$$\log \prod_{k=0}^{n-1} \frac{Jf(f^k x)}{Jf(f^k y_n)}$$

$$= \sum_{k=0}^{n-1} \left[\log Jf(f^k x) - \log Jf(f^k y_n) \right]$$

$$\leq \sum_{k=0}^{n-1} L \lambda^{-k} < \sum_{k=0}^{\infty} L \lambda^{-k} < \infty$$

$$\lim Jh_n = \prod_{k=0}^{\infty} \frac{Jf(f^k x)}{Jf(f^k y)}$$

$y = h^s(x)$.

Can show

$$\int h_n \rightrightarrows \int h^s.$$

so h^s is abs cts.

W^s is abs cts — W^u ~~at~~ as well.

"Fubini" holds

and f is ergodic.

I'm lying slightly.

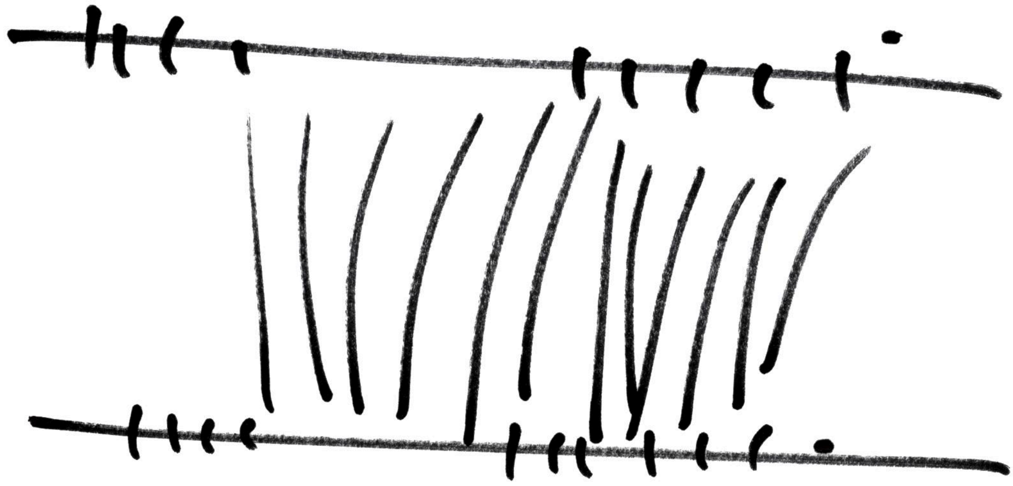
Jf is deriv of $Df|_{E^u}$

and E^u is Hölder cts.

So $\log Jf$ is Hölder

and

$$\sum L(\lambda^{-k})^{\theta} < \infty.$$



Cantor set

$m(X_0) = 0$

Cantor set

$m(X_1) = 0.1.$

Anosov:

If $f \in C^2$, meas. pres. and Anosov, then f is ergodic.

Robinson - Young (1980)

There is $f \in C^1$ arbitrarily C^1-close to the cat map s.t.

W^s_f and W^u_f are

NOT absolutely continuous

Open question:

Is every $f \in C^1$ which is C^1 close to the cat map ergodic?