Models of Chaos

in dimensions 2 and 3

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Consider a flow $\varphi_t : M \times \mathbb{R} \to M$

What is the long-term behaviour?

Define the <u>ω-limit set</u>

$$\omega(x) := \lim_{T \to \infty} \overline{\{\varphi^t(x) : t > T\}}$$

$\left(y \in \omega(x) \iff \exists\, t_k \to \infty \text{ such that } \varphi^{t_k}(x) \to y \right)$

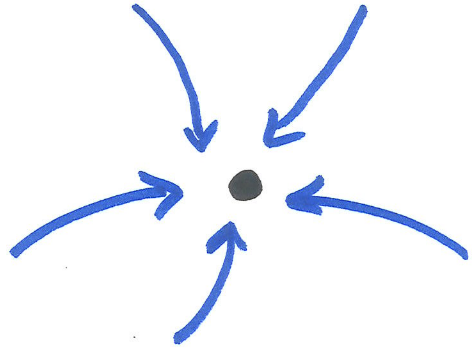
What is $\omega(x)$?

What is the dynamics on $\omega(x)$?

Examples

a sink $s \in M$

$\omega(x) = s$



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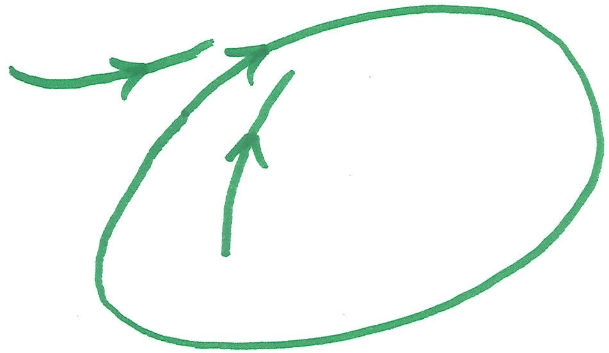
a periodic orbit



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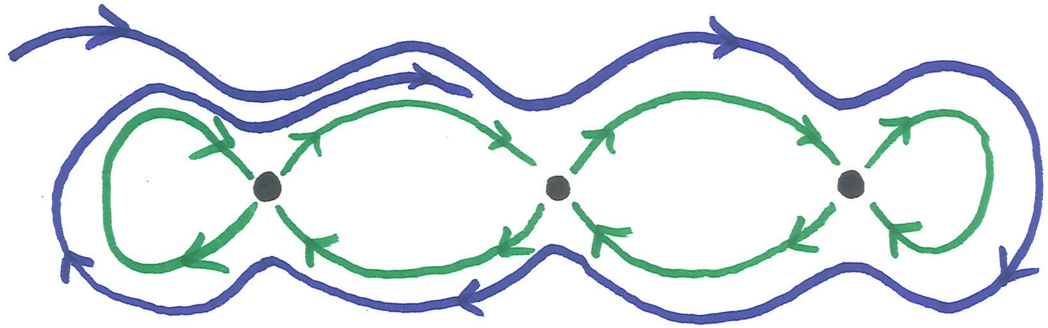
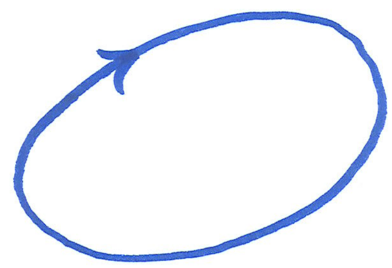
a "graphic"

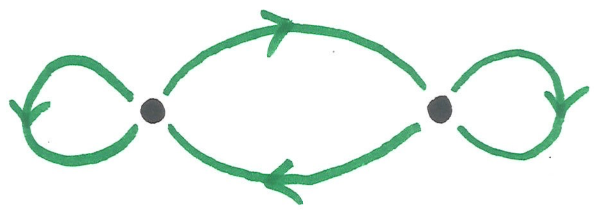


Thm (Poincaré - Bendixson 1901)

For a flow* on $\mathbb{R}^2$ or S^2 and any point x, $\omega(x)$ is

1) a point, 

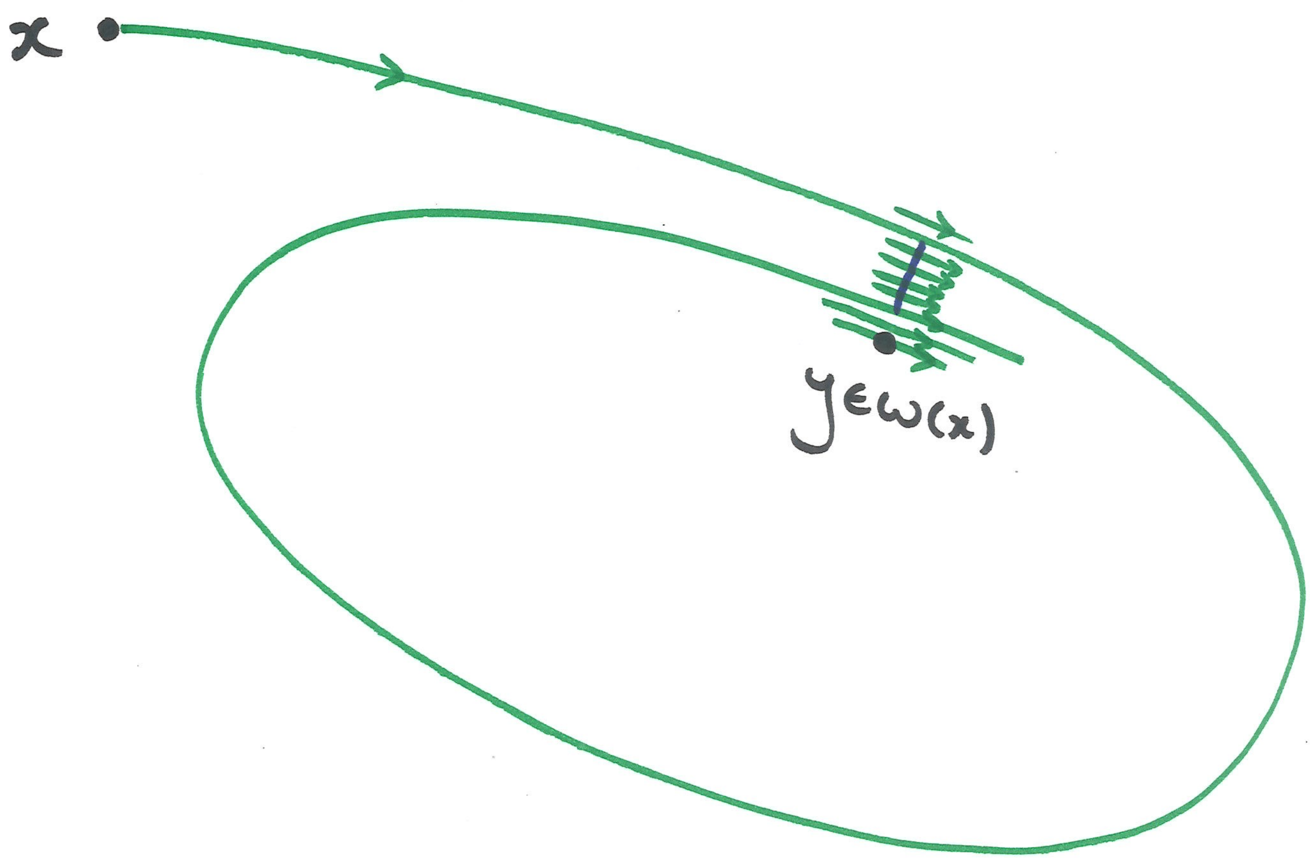
2) a periodic orbit, or 

3) a "graphic". 

* with isolated fixed points

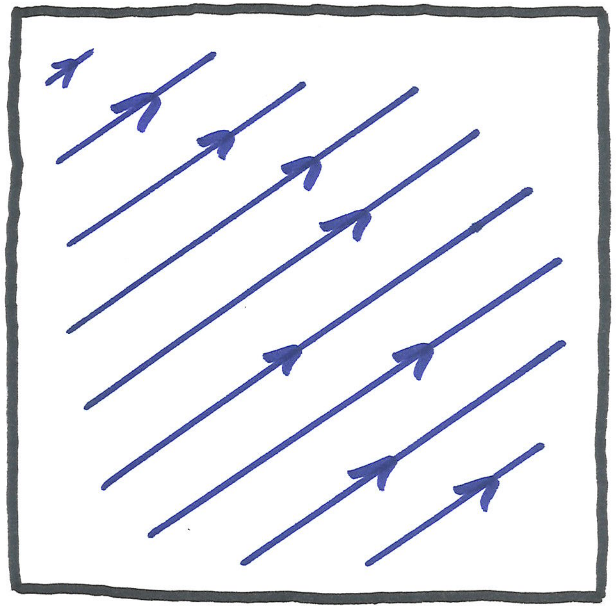
Key idea of proof:

Jordan curves and trapping regions



The result does not hold on other surfaces.

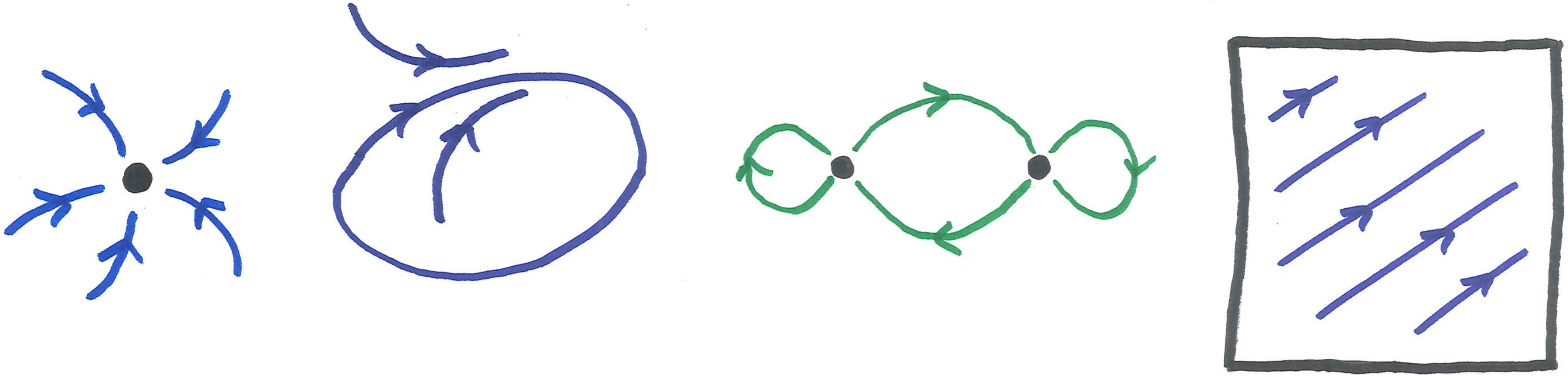
<u>Ex</u> Irrational slope flow on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$



For any $x \in \mathbb{T}^2$,

$$\omega(x) = \mathbb{T}^2.$$

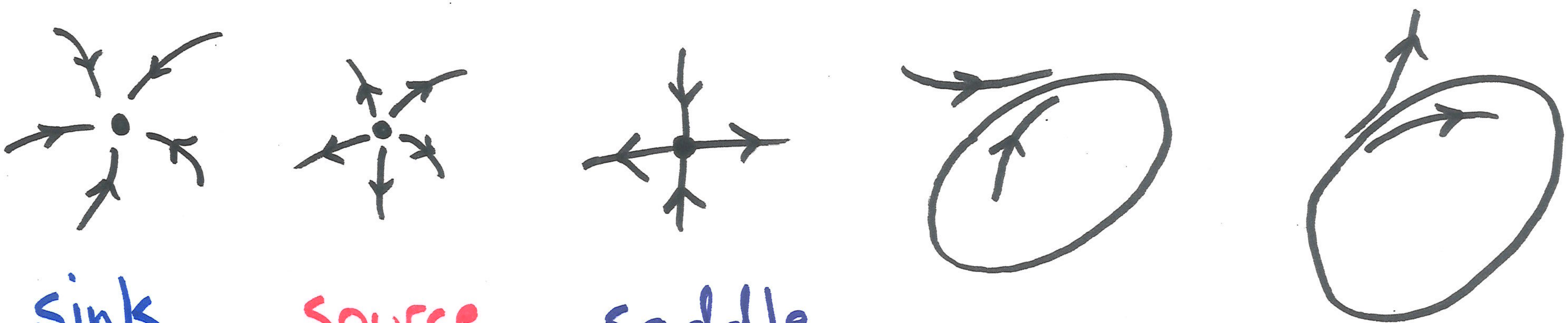
Examples



Which of these is robust under perturbation?

Thm (Peixoto 1962)

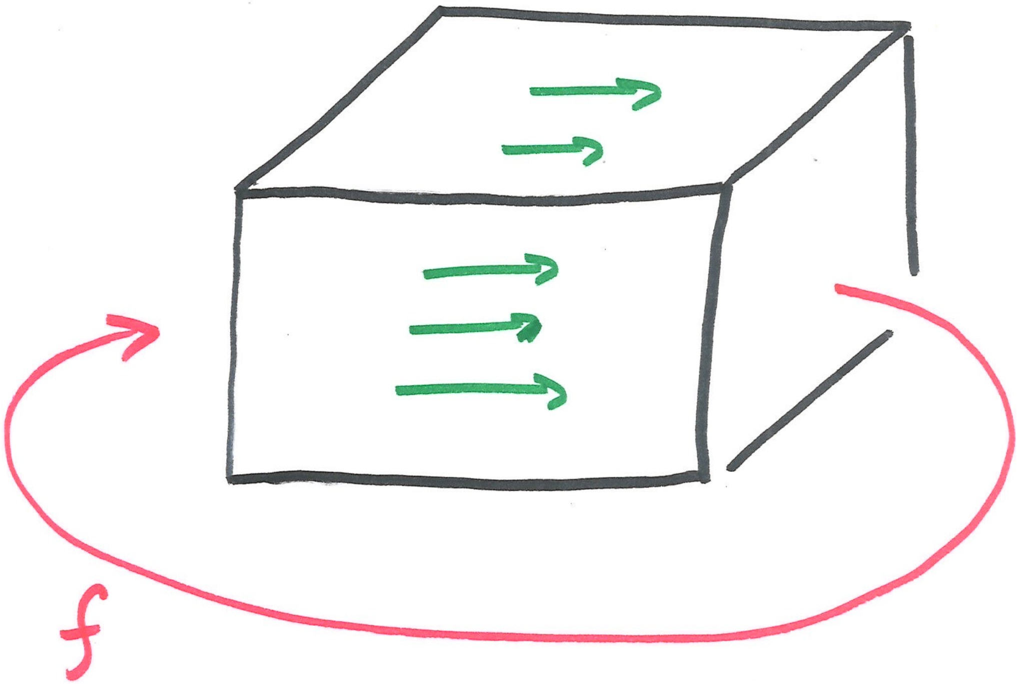
Let φ^t be a flow on a compact surface.
After a C^r-small perturbation,
every ω-limit set is of the form:



sink   source   saddle   attracting / repelling periodic orbit

What about flows in dim 3?

First, consider diffeo(morphism)s in dim 2.



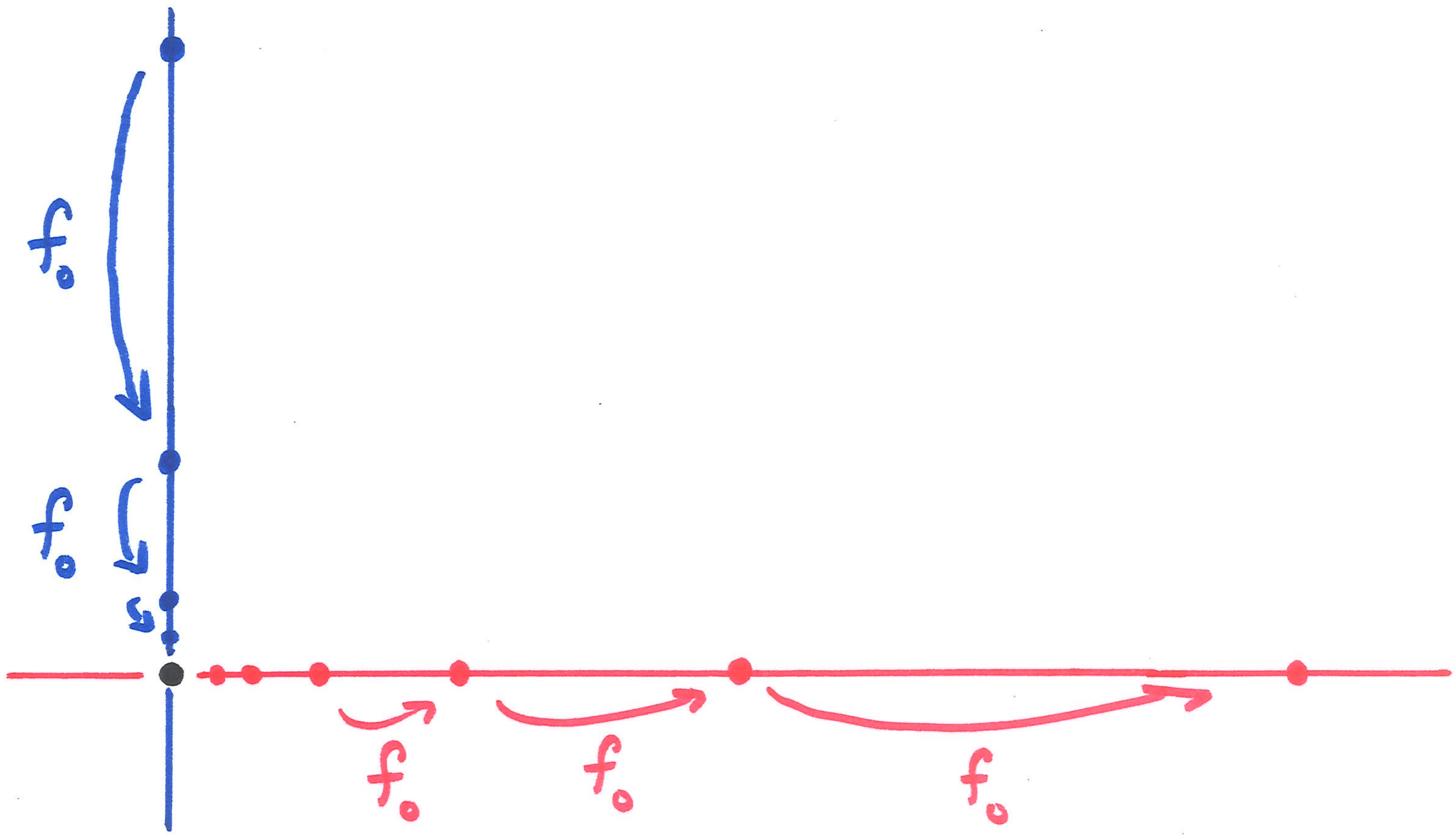
$f: M^2 \to M^2$

can be suspended to a flow

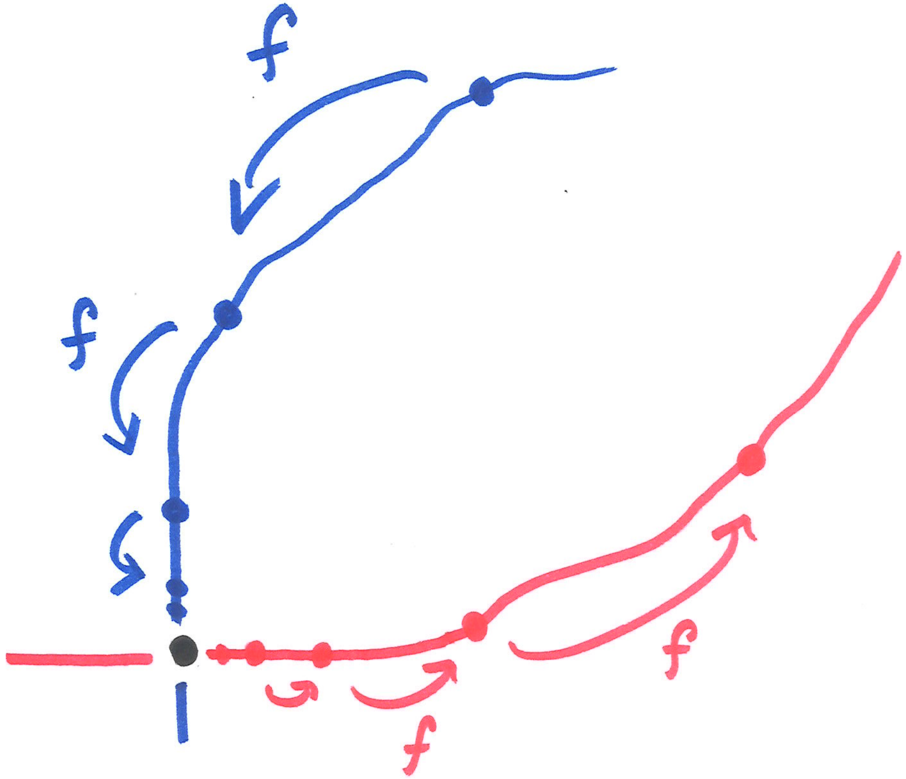
φ^t in dim 3

For diffeos in dim 2, chaos is possible.

__Ex__ In $\mathbb{R}^2$, $f_0(x,y) = (2x, \frac{1}{3}y)$



Deform f_0 away from the origin to get f



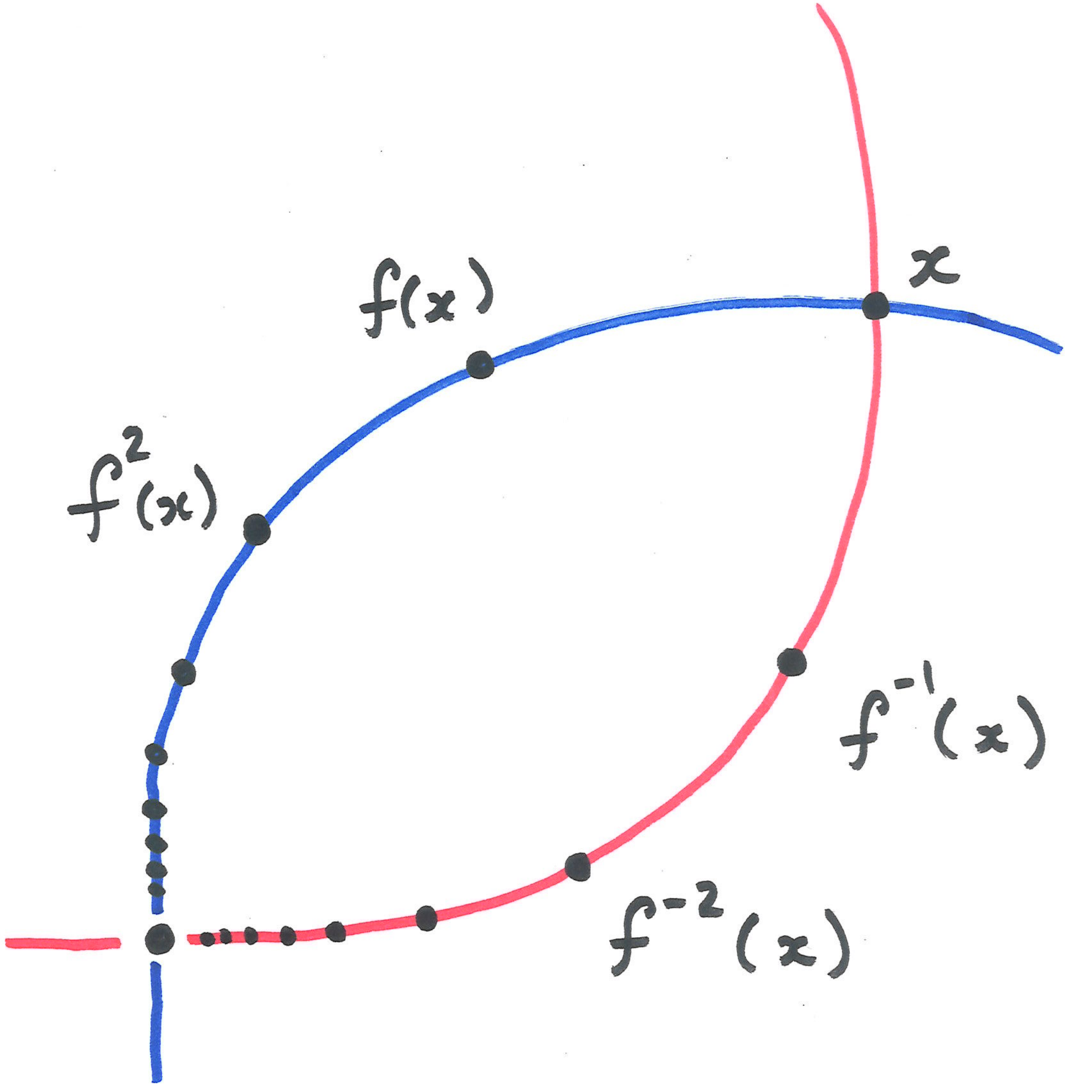
We still have invariant curves.

Don't cross the streams.

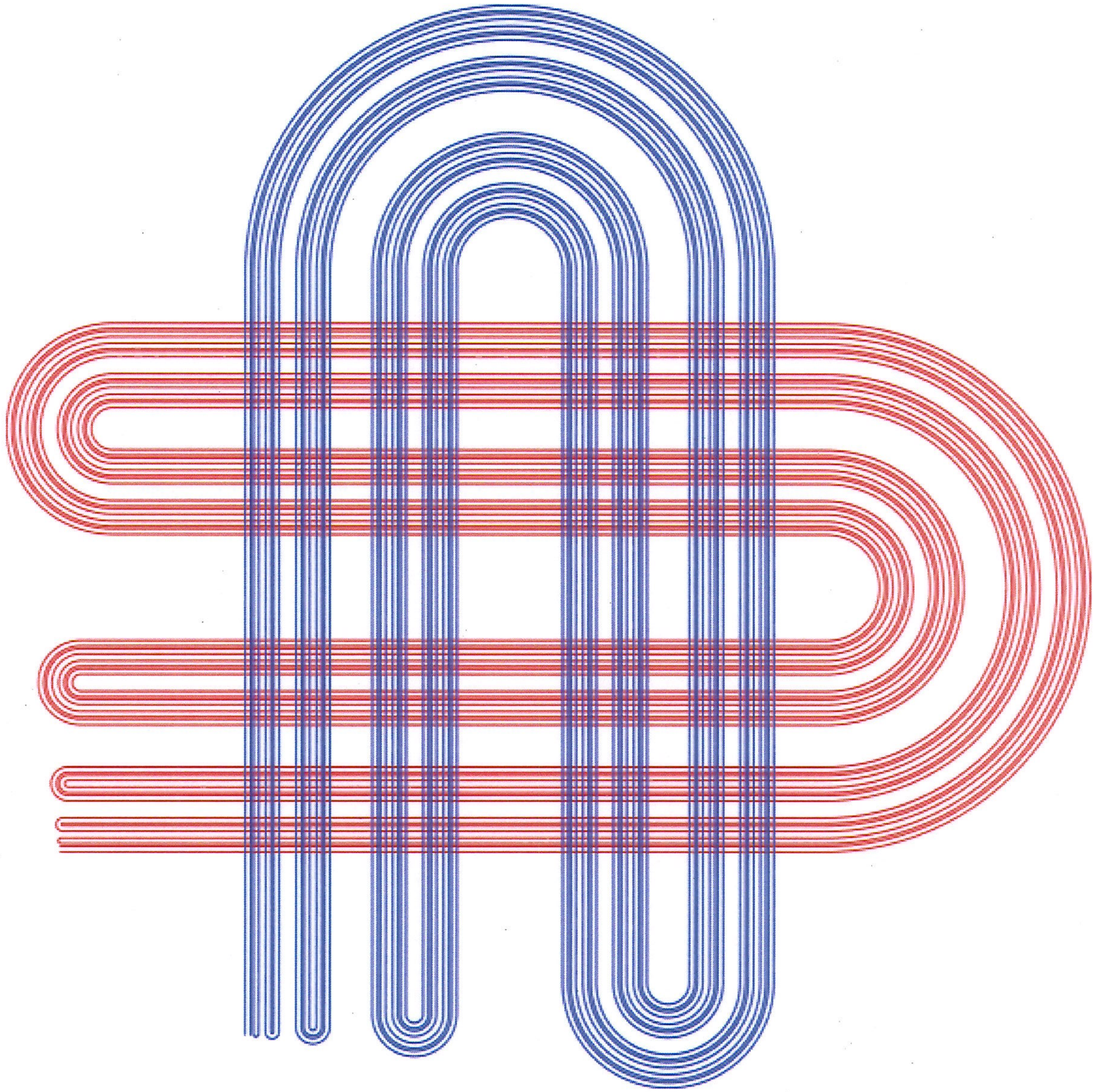
Why?

It would be bad.

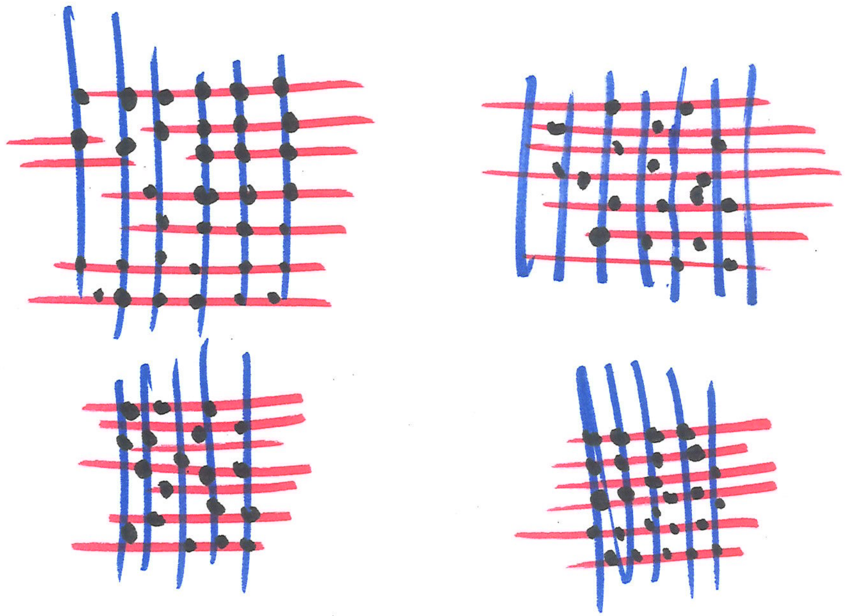
– Ghostbusters (1984)



A simpler example: Smale's horseshoe



Take the closure Λ of all the intersections in the picture.



This is an example of a <u>uniformly hyperbolic</u> set.

For a diffeo $f: M \to M$,

a closed, invariant subset Λ is

<u>uniformly</u> <u>hyperbolic</u>

if at each $x \in \Lambda$, $T_x M$ splits as

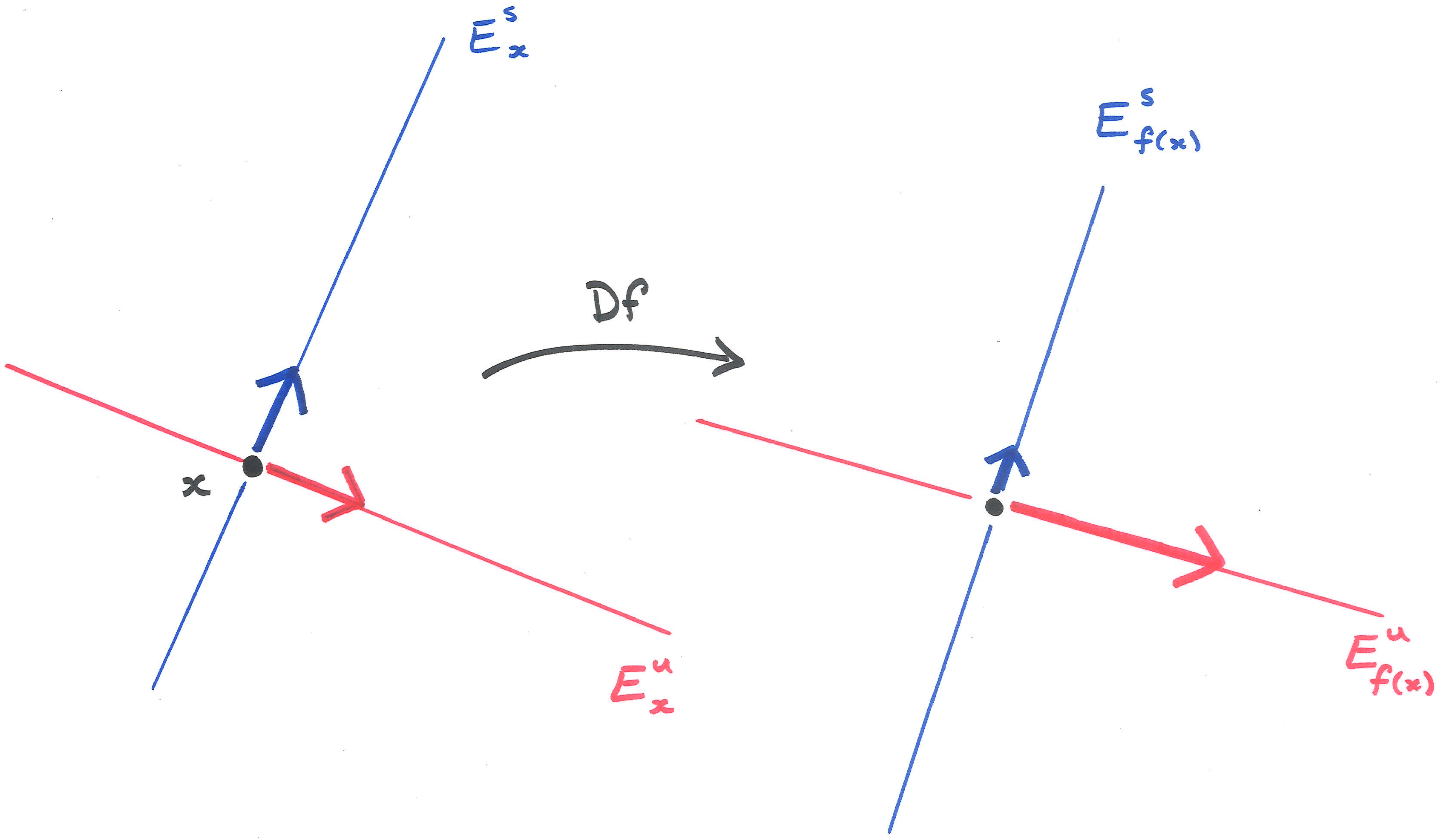
$$T_x M = E_x^u \oplus E_x^s$$

unstable

expanded by Df

stable

contracted by Df

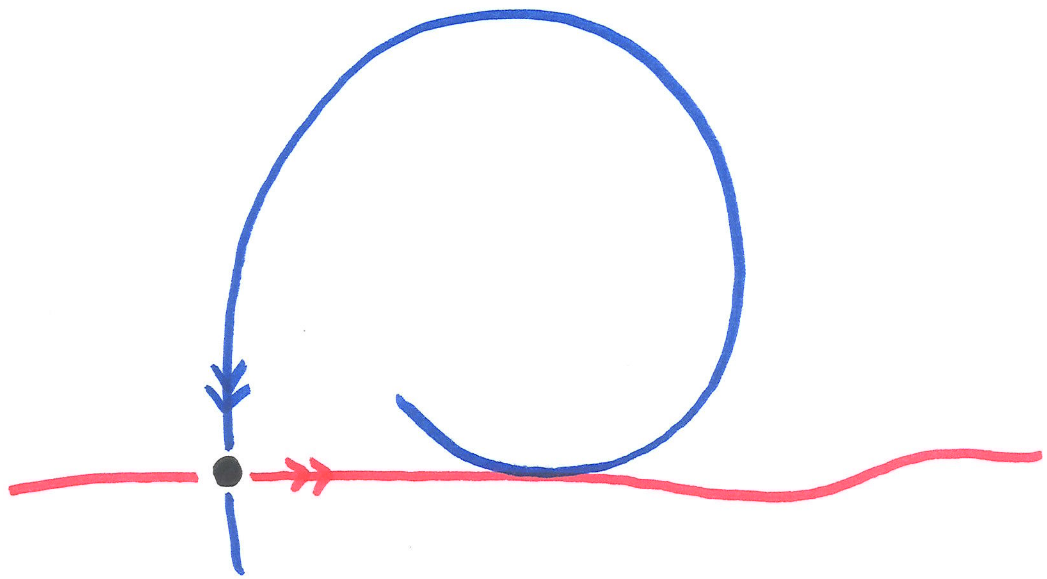

E^s_x
x
E^u_x
Df
$E^s_{f(x)}$
$E^u_{f(x)}$

For diffeos in dim 2:

After perturbation, is every $\omega(x)$ uniformly hyperbolic?

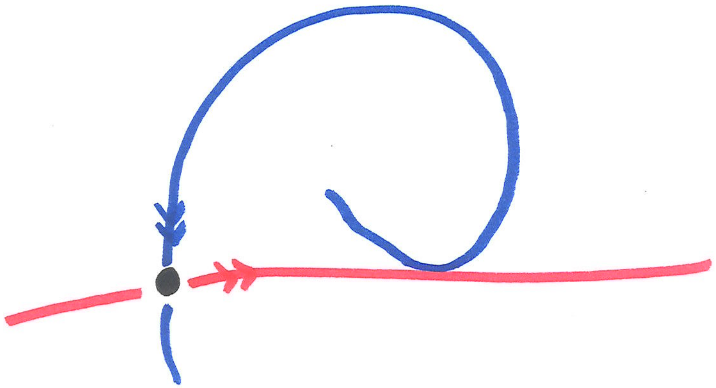
A possible obstruction:

a <u>homoclinic tangency</u>



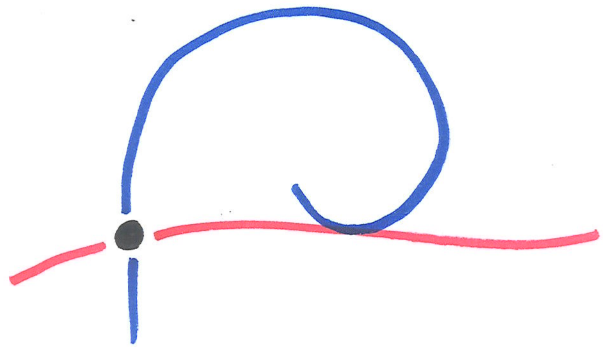
Newhouse (1970):

$\exists$ a surface diffeo f so that every g C^2-close to f has a tangency.



Open question for surface diffeos:

can a C^1-perturbation remove all tangencies?



Say a system is "far from tangencies" if $\exists$ a C^1-nbhd with no tangencies.

<u>Thm</u> (Pujals - Sambarino 2000)

"Far from tangencies,"

a surface diffeo may be

C^1-perturbed

so that every $\omega(x)$ is

uniformly hyperbolic.

For a diffeo $\omega(x) = \lim_{N\to\infty} \overline{\{ f^n(x) : n > N\}}$

A system is <u>transitive</u> if there is a dense orbit, i.e. there is $x \in M$ such that

$$\omega(x) = M$$

Are there robustly transitive diffeos?

Ex the "cat map"

$$A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x+y\\x+y\end{pmatrix}$$

on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

Here, the entire manifold $M = \mathbb{T}^2$ is a uniformly hyperbolic set.

In such a case, the dynamical system is called **Anosov**.

Anosov: $\exists$ global splitting $TM = E^u \oplus E^s$

<u>Thm</u> (Mañé 1982) For a diffeo of a cmpt surface:

C^1-robustly transitive $\Longleftrightarrow$ Anosov

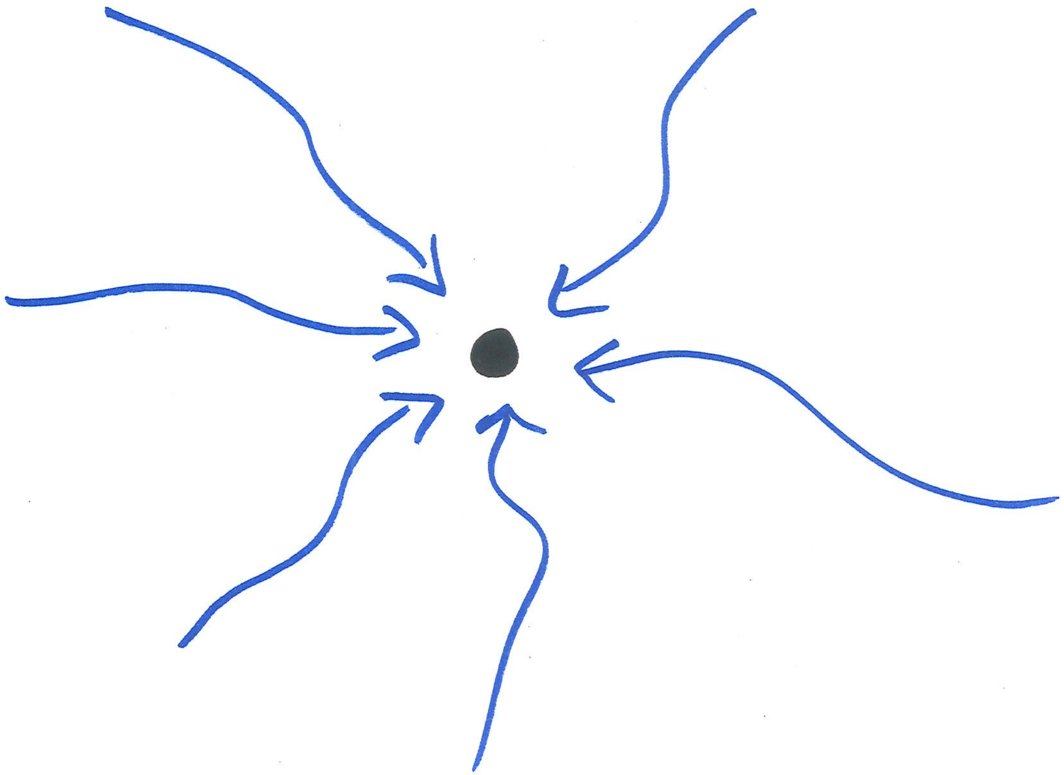
Note: Anosov $\Rightarrow$ line field on M

$\Rightarrow M = \mathbb{T}^2$

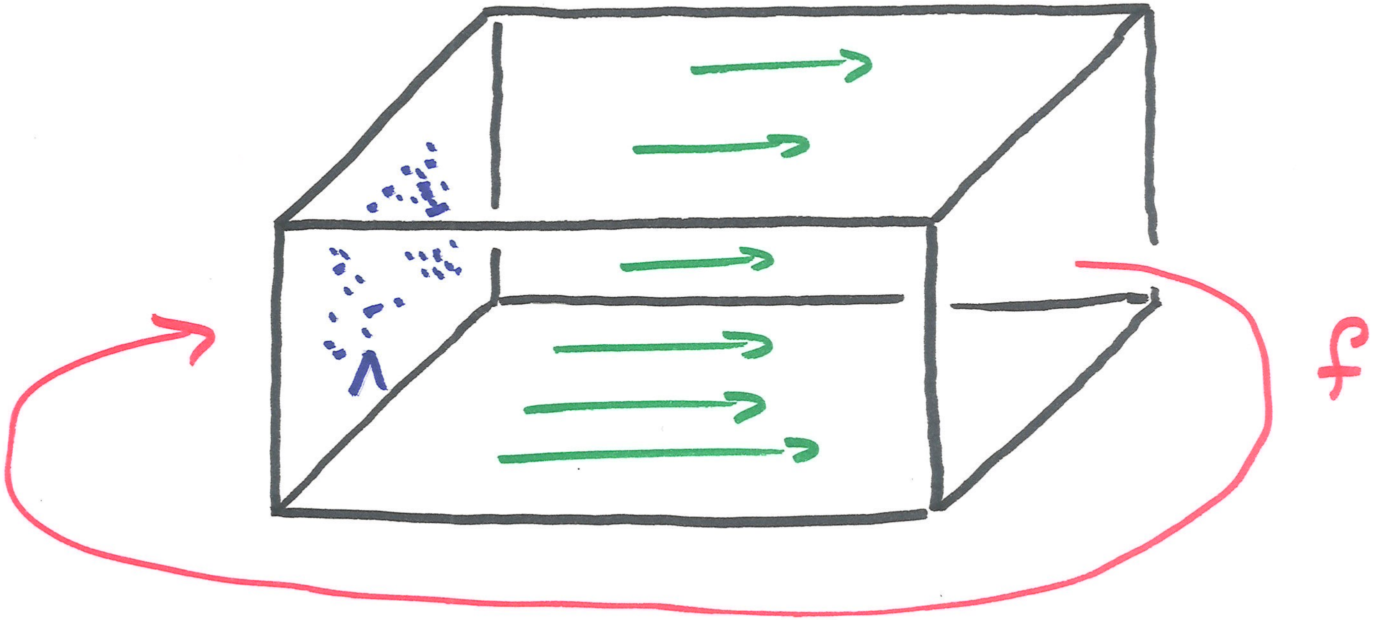
flows in dim 3

Examples

$$\omega(x) = \{\text{point}\}$$

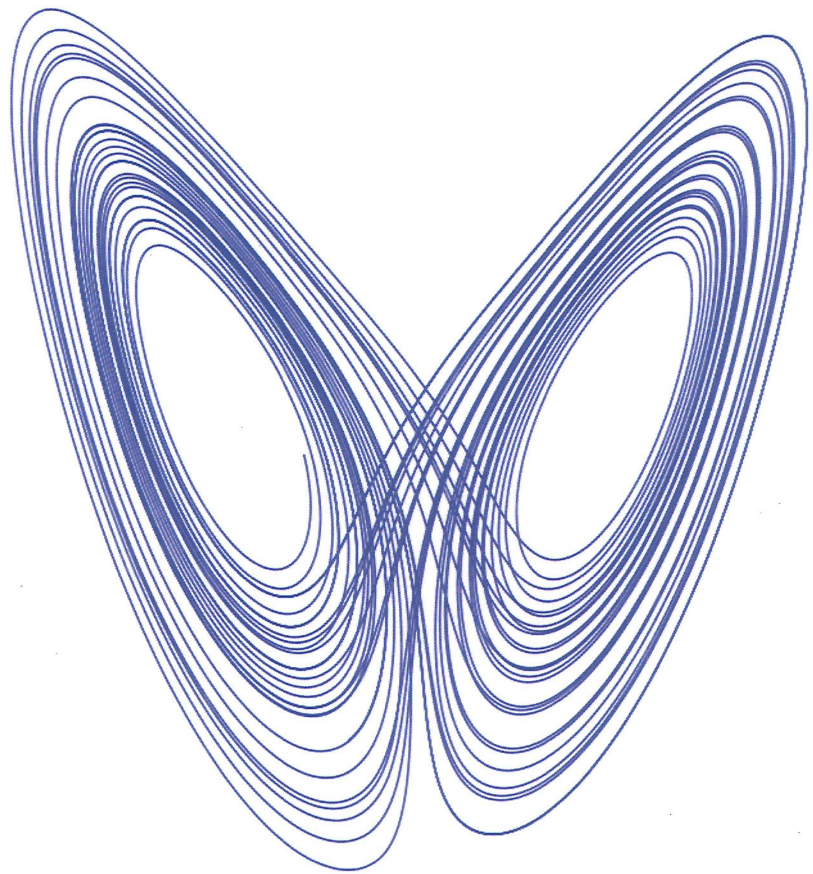


Example Take a uniformly hyperbolic set Λ for a surface diffeo f and suspend it.

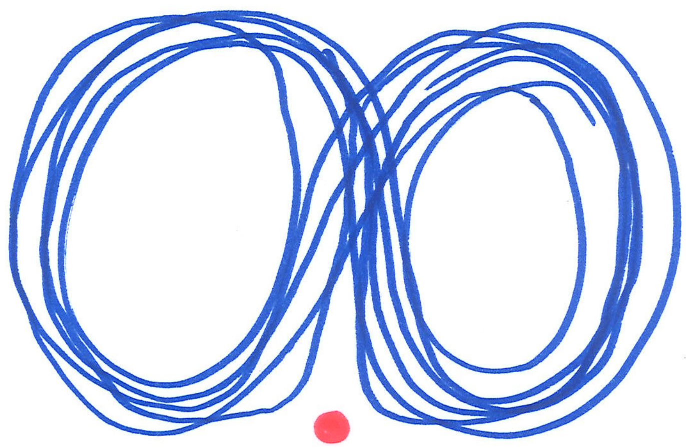


This is robust under perturbation.

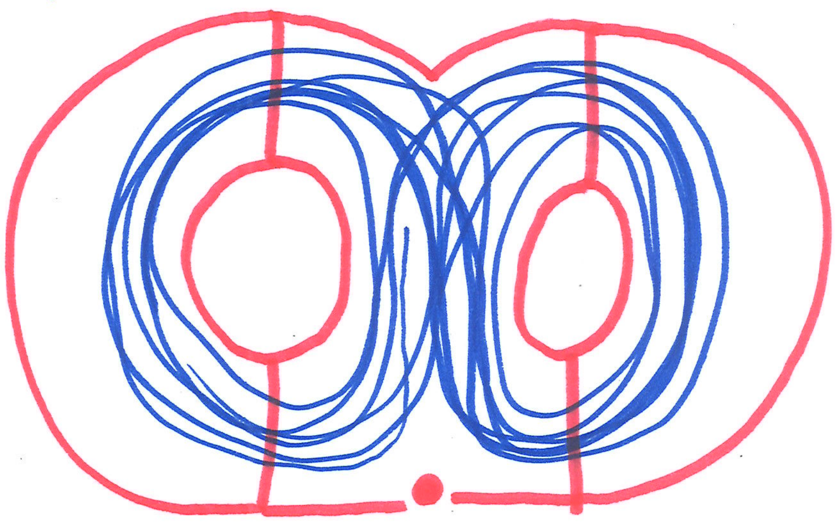
Example The Lorenz Attractor



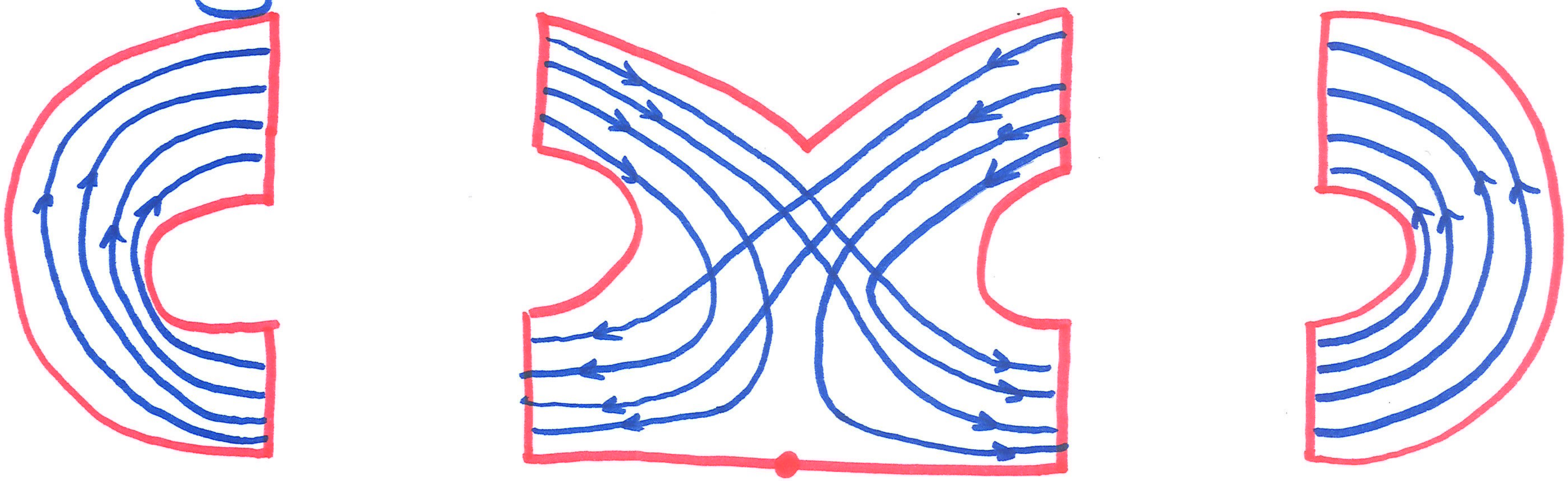
Split the Lorenz attractor



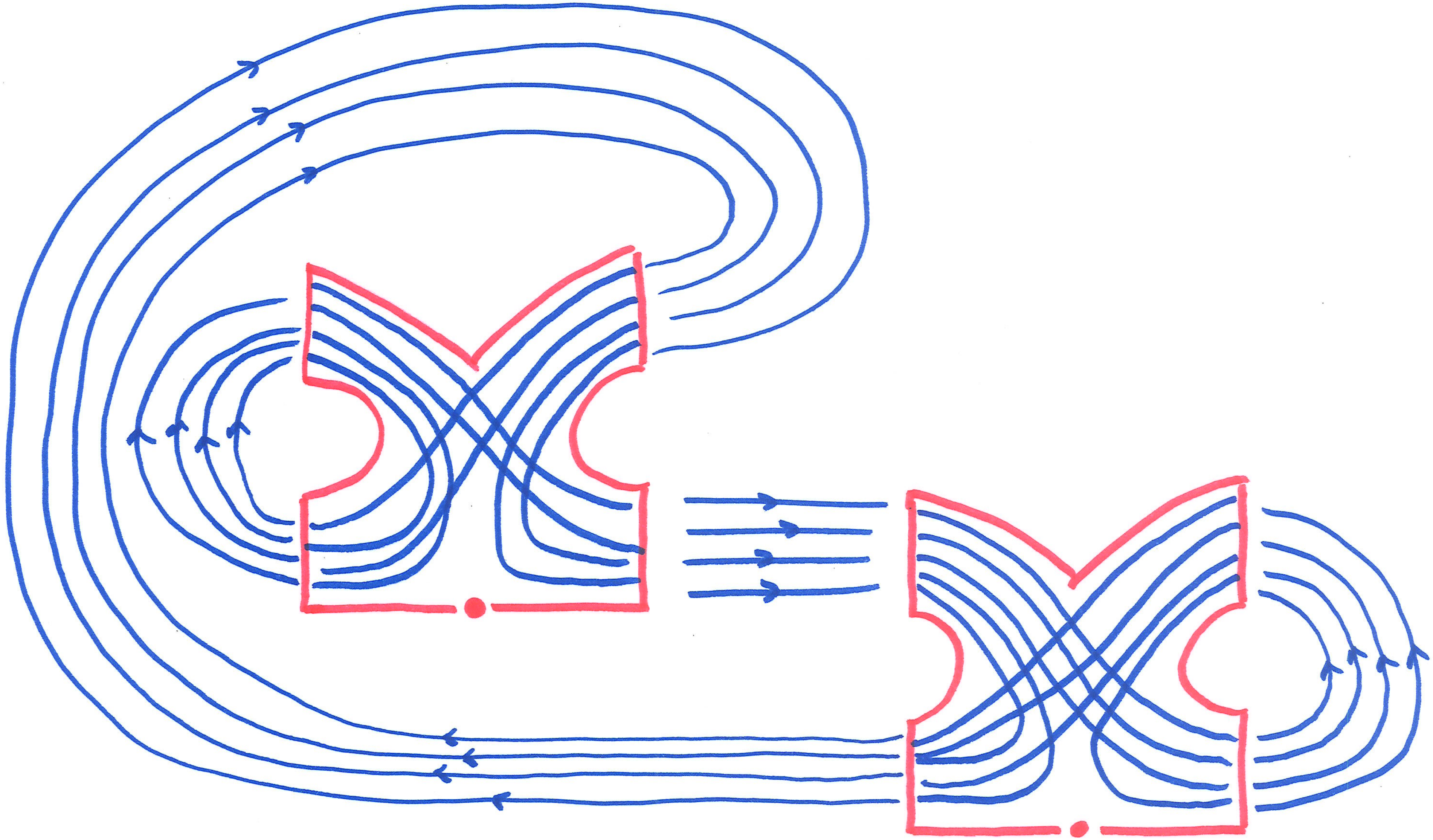
into three pieces



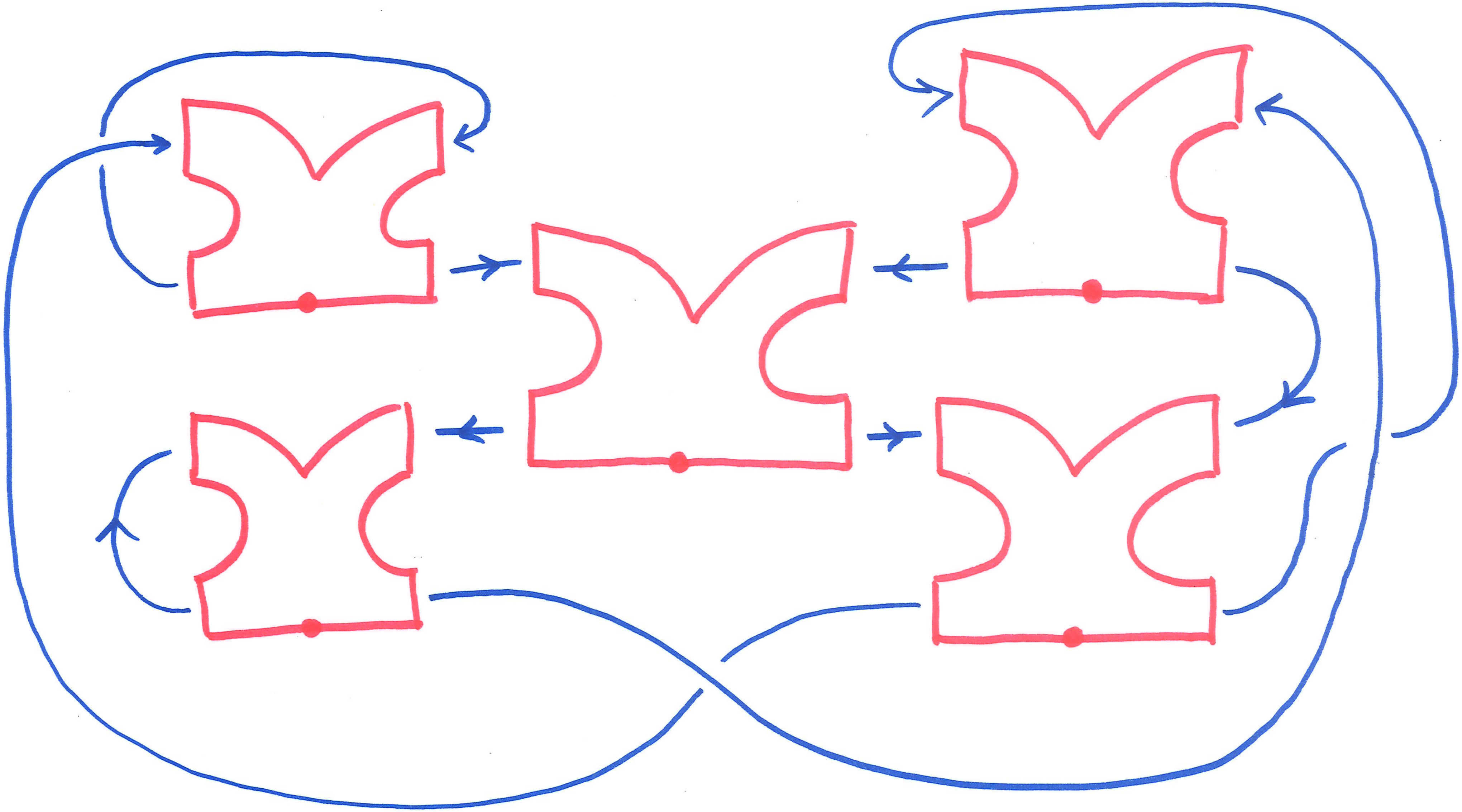
Only the middle piece is chaotic.



Multiple copies of this middle piece may be connected together.



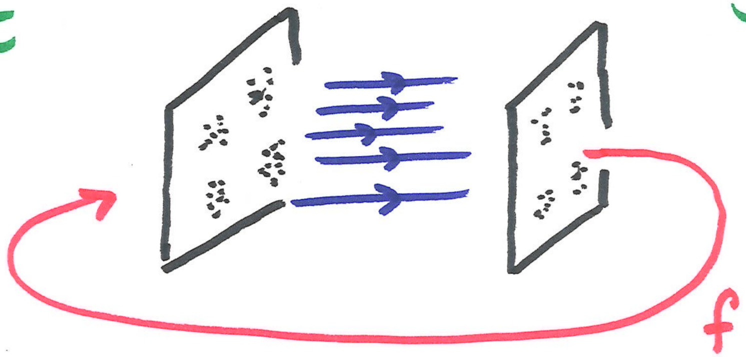
These are examples of singular hyperbolic sets.



Thm (2015) "Far from tangencies" a flow φ^t on a 3-mfld M may be C^1-perturbed so that every ω-limit set $\omega(x)$ is one of:

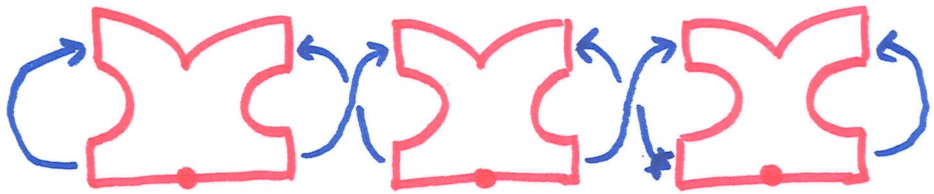
1) $\omega(x)$ is a single point •

2) $\omega(x)$ is the suspension of a 2-dim'l uniformly hyperbolic set



3) $\omega(x)$ is singular hyperbolic

"Lorenz-like"



4) $\omega(x)=M$ and φ^t is a robustly transitive Anosov flow

Theorem Credits

Doering (1985)

Morales, Pacifico, Pujals (1999, 2004)

Arroyo, F. Rodriguez Hertz (2003)

Araújo, Pacifico, Pujals, Viana (2009)

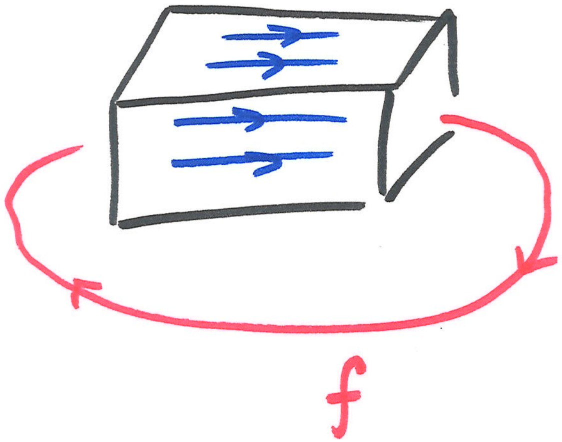
Crovisier, D. Yang (2015)

Also Tucker (2002) and others...

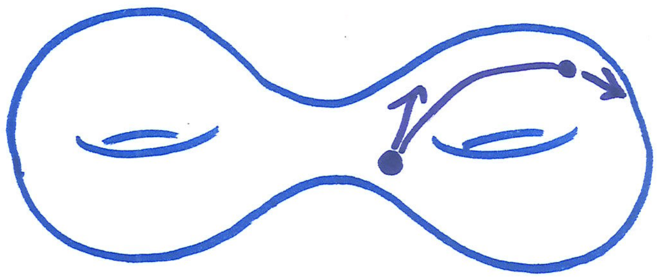
Thm (Doering 1985) Any robustly transitive flow on a 3-manifold is Anosov.

Main examples of Anosov flows:

1) Suspensions of Anosov diffeos



2) Geodesic flows (negative curvature)



3) Surgery on these examples

There are no Anosov flows on S^3, $\mathbb{T}^3$, most Seifert fiber spaces, the Weeks manifold, reducible manifolds, ...

For flows in dim 3:

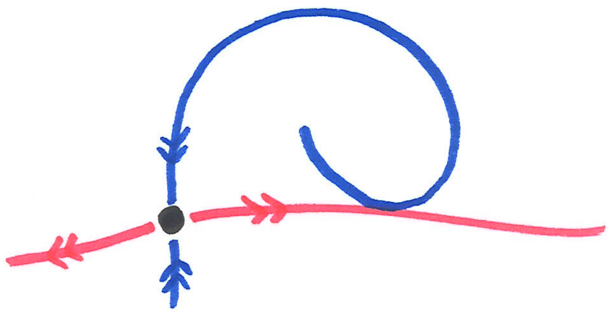
Robust Transitivity $\Longrightarrow$ Anosov $\Longrightarrow$ Restrictions on the geometry

The Great Unknown

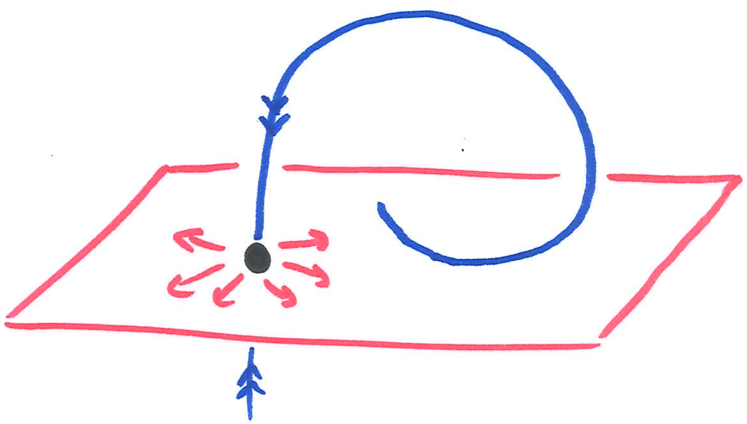
Diffeos in Dimension 3

Are tangencies robust in the C^1-topology?

Open question for diffeos in dim 2
and
flows in dim 3.



For diffeos in dim 3: there are known examples of C^1-robust tangencies.



Recall: A system is <u>transitive</u> if it has a dense orbit.

For diffeos in dim 2:

robustly transitive $\Leftrightarrow$ Anosov $\Rightarrow$ $M = \mathbb{T}^2$

For flows in dim 3:

robustly transitive $\Rightarrow$ Anosov $\Rightarrow$ restrictions on M

For diffeos in dim 3:

robustly transitive $\overset{?}{\Rightarrow}$ any restrictions on M?

For diffeos in dim 3:

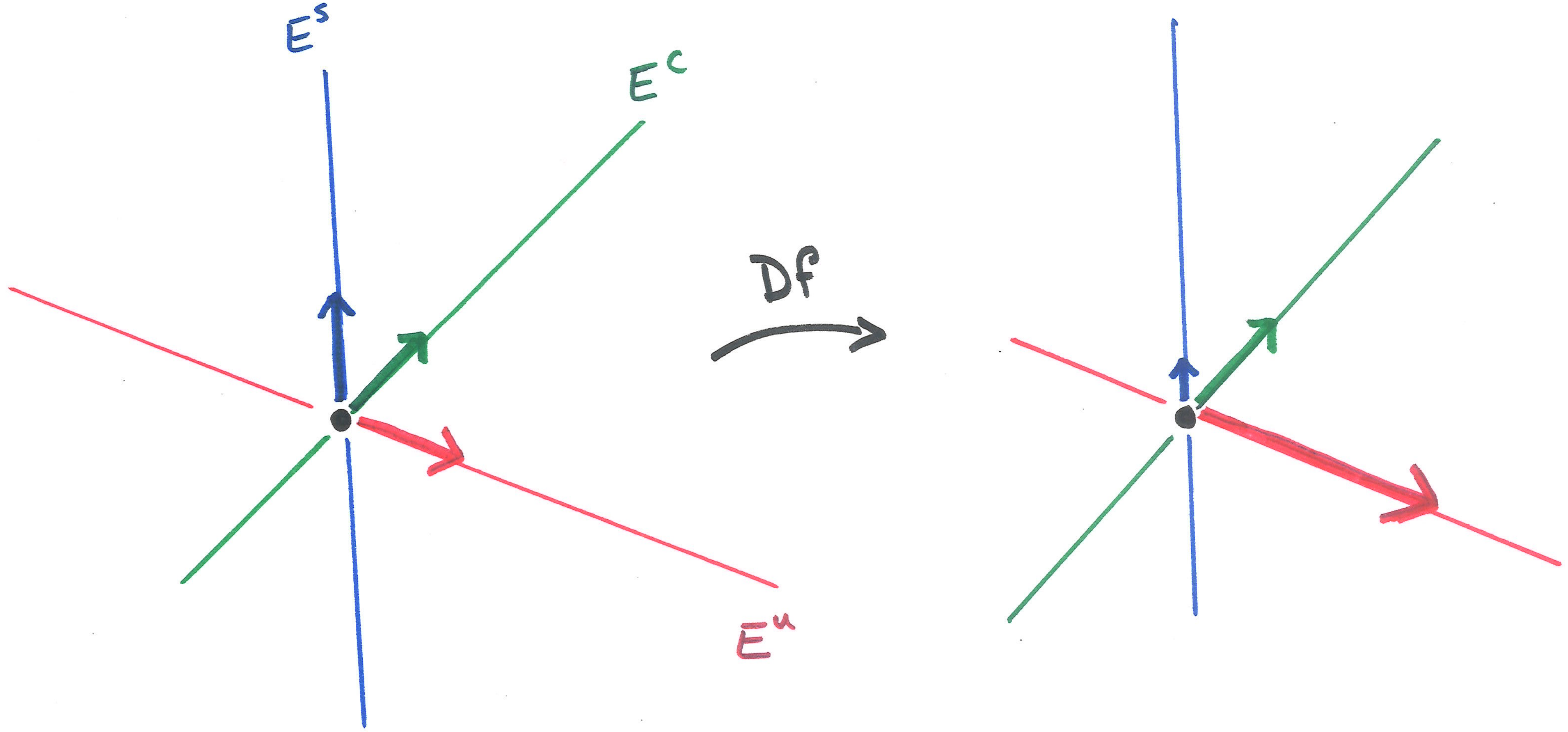
$$\text{robustly transitive} \overset{\text{Mañé 1978}}{\not\Rightarrow} \text{Anosov}$$

$$\text{robustly transitive}$$
$$\text{Díaz-Pujals-Ures 1999} \Downarrow$$

f is either weakly partially hyperbolic

or

strongly partially hyperbolic



E^s
E^c
E^u
Df

Def A diffeo $f : M^3 \to M^3$ is (strongly) partially hyperbolic

if there is a Df-invariant splitting

$$TM = E^u \oplus E^c \oplus E^s$$

E^u: expanded by Df

E^c: no strong expansion or contraction

E^s: contracted by Df

Classic Examples of partial hyperbolicity in dim 3

1) linear maps on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$
with eigenvalues $\lambda^s < \lambda^c < \lambda^u$

The eigenspaces give the splitting $E^s \oplus E^c \oplus E^u$

2) algebraic maps on nilmanifolds

3) the time-one map φ^1 of
an Anosov flow φ^t

Thm (H - Potrie 2015)

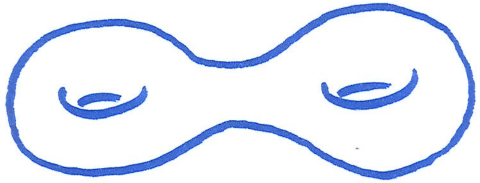
If M is a 3-mfld with solvable fundamental group and $f: M \to M$ is partially hyperbolic, the f is "leaf conjugate" to

1) a linear map on $\mathbb{T}^3$

2) an algebraic map on a nilmanifold, or

3) the time-one map of an Anosov flow.

Surface diffeo $\sigma : S \to S$ (genus ≥ 2)



Derivative $D\sigma : TS \to TS$

Normalize $D\sigma : T^1S \to \underline{T^1S}$ unit tangent bundle

<u>Thm</u> (Bonatti-Gogolev-H-Potrie 2020)

For any surface diffeo $\sigma : S \to S$ there is a robustly transitive, partially hyperbolic diffeo $g : T^1S \to T^1S$ isotopic to $D\sigma$.

Instead of studying diffeomorphisms, $M \underset{f^{-1}}{\overset{f}{\rightleftarrows}} M$

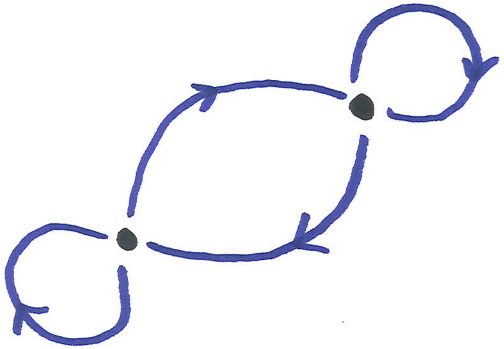
we can look at endomorphisms:

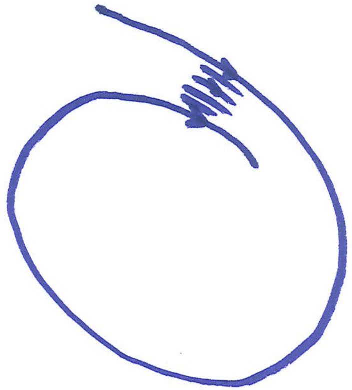
non-invertible covering maps $M \xrightarrow{f} M$

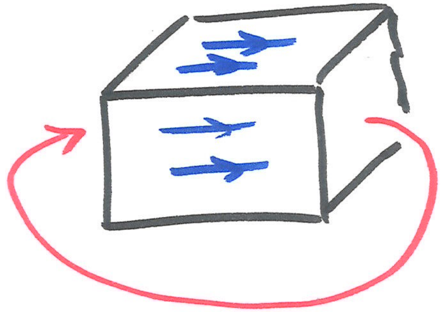
Joint with Layne Hall (2021, 2022),

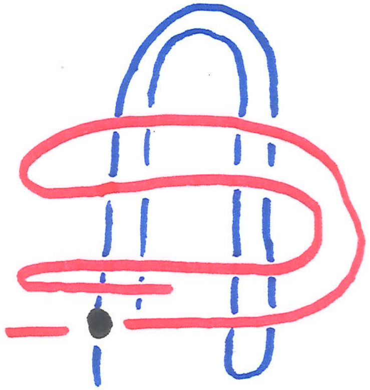
we completed a classification of

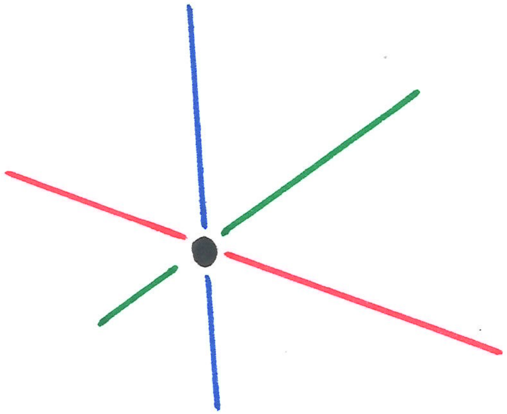
partially hyperbolic surface endomorphisms.











Thank you!