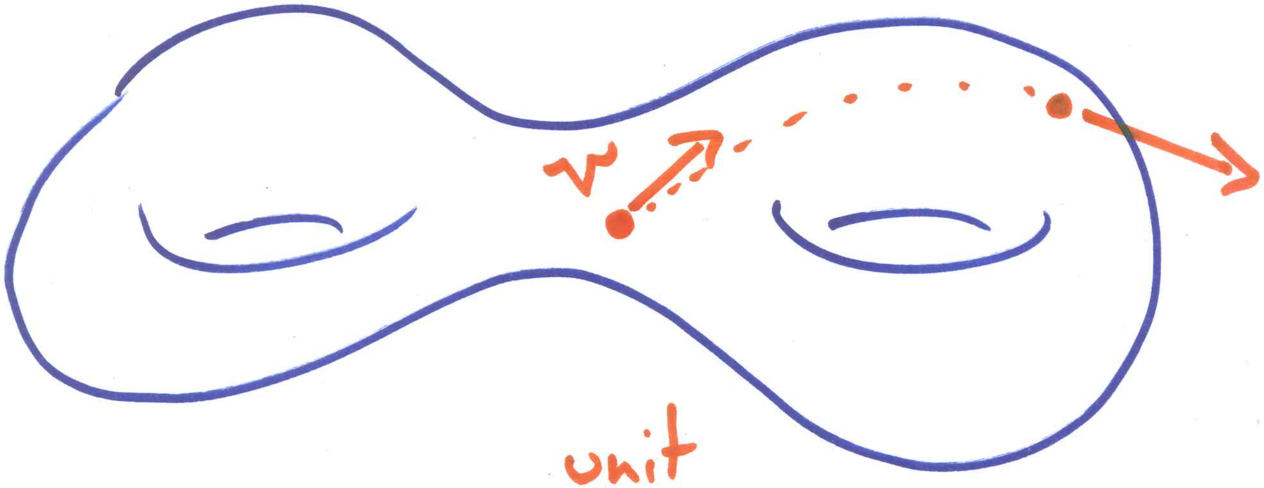
Stable Ergodicity

Andy Hammerlindl
Monash University

Hopf (1939):

The geodesic flow on a negatively curved manifold is ergodic.

S



$\varphi_t(v)$

unit vector

$$M = \underbrace{T^1 S}_{\text{unit vectors}}$$

$$\varphi_t : M \longrightarrow M \quad \text{flow}$$

The Hopf Argument for the cat map

$A : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Say $\phi : \mathbb{T}^2 \to \mathbb{R}$ is (uniformly) continuous.

We want to show the forward Birkhoff average

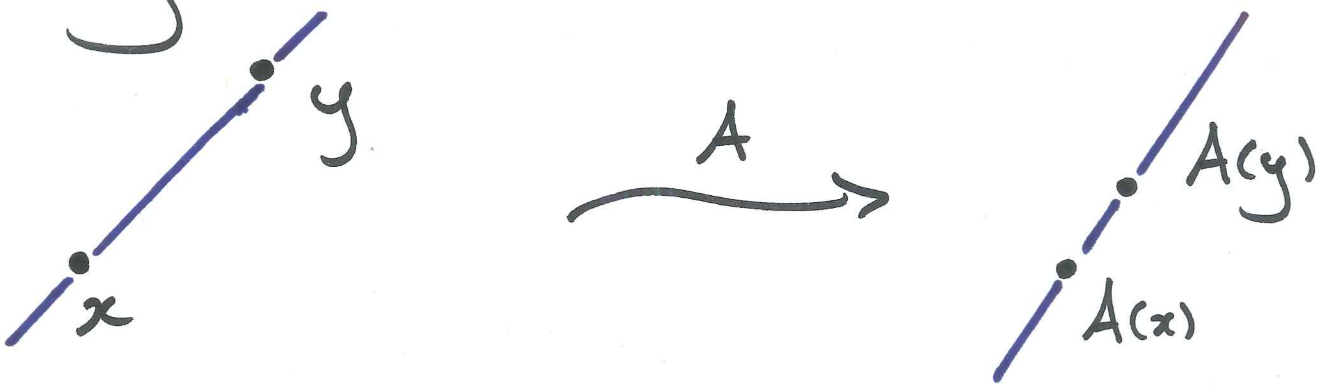
$$\phi^+(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \phi(A^k(x))$$

is constant a.e.

$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ has eigs $\lambda^{-1} < 1 < \lambda$

and a splitting $TM = E^s \oplus E^u$ given by the eigenspaces.

If x and y are on the same stable leaf,



then

$$d(A^k x, A^k y) \to 0 \Rightarrow |\phi(A^k x) - \phi(A^k y)| \to 0 \Rightarrow \phi^+(x) = \phi^+(y)$$

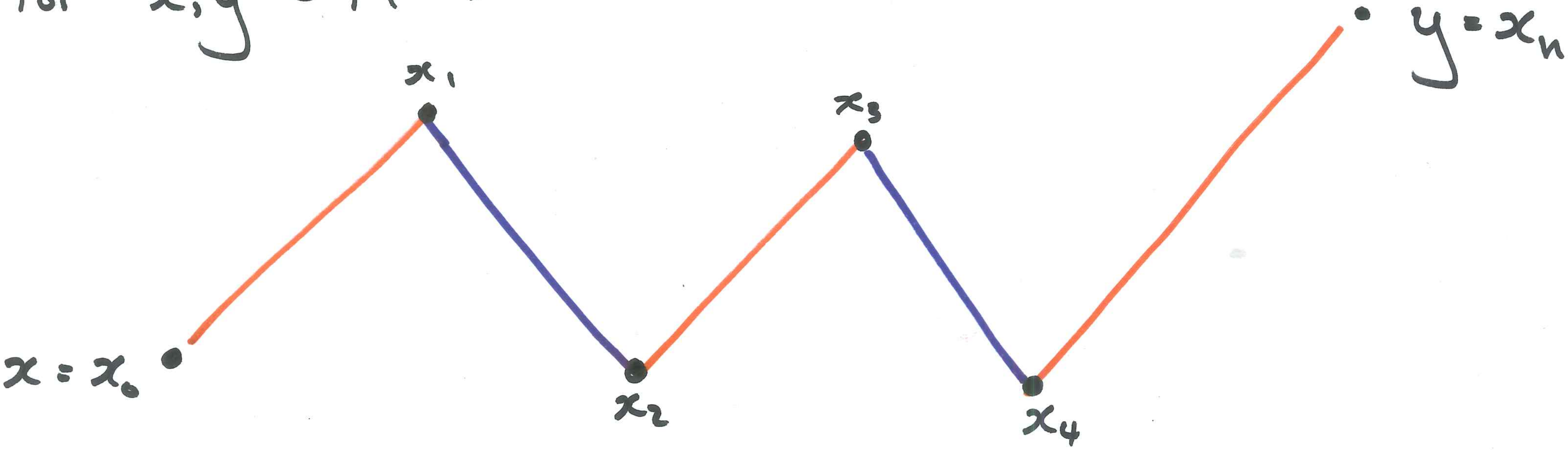
ϕ^+ is constant on stable leaves

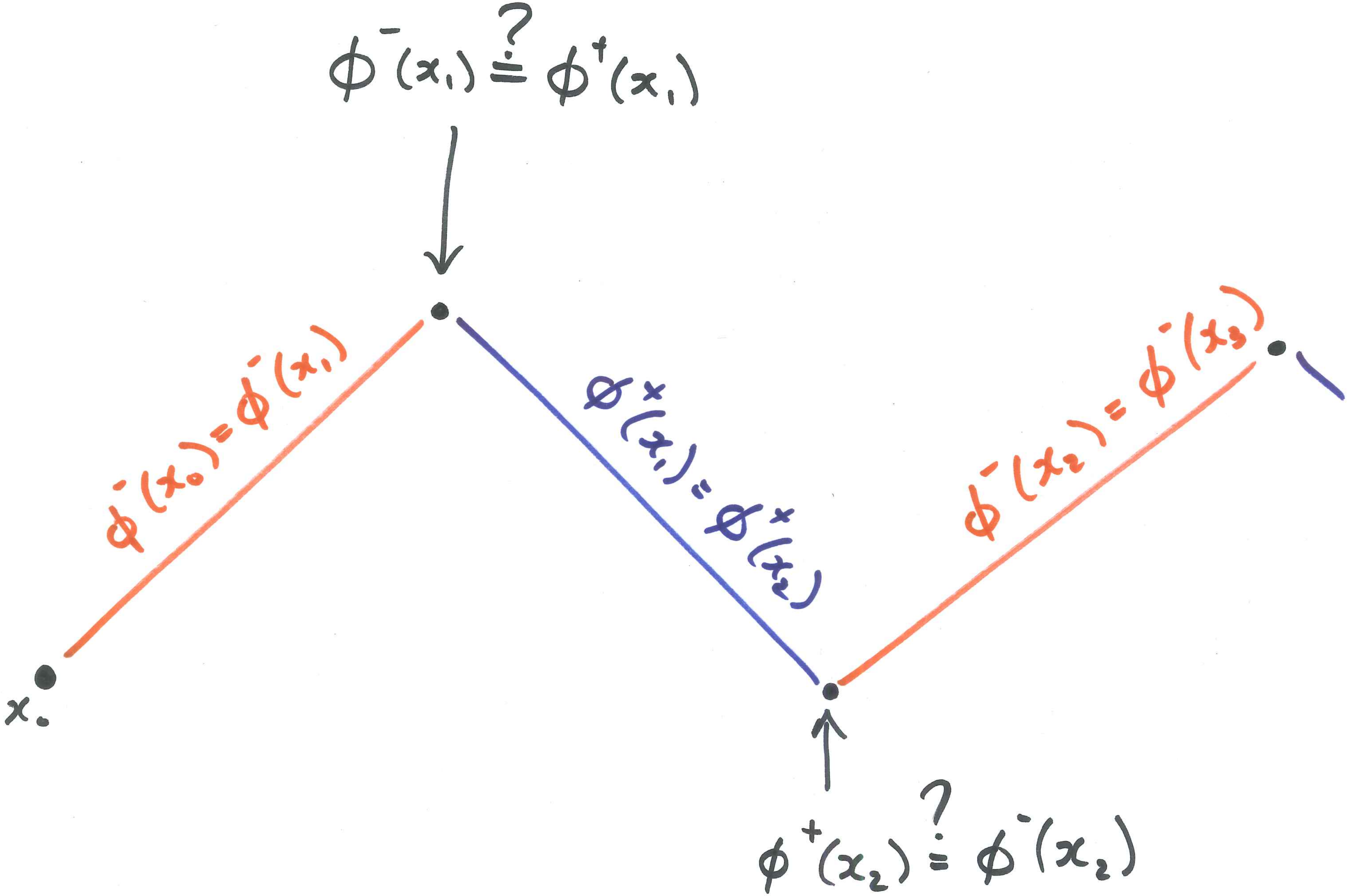
ϕ^+ is constant on unstable leaves

and $\phi^+ = \phi^-$ almost everywhere.

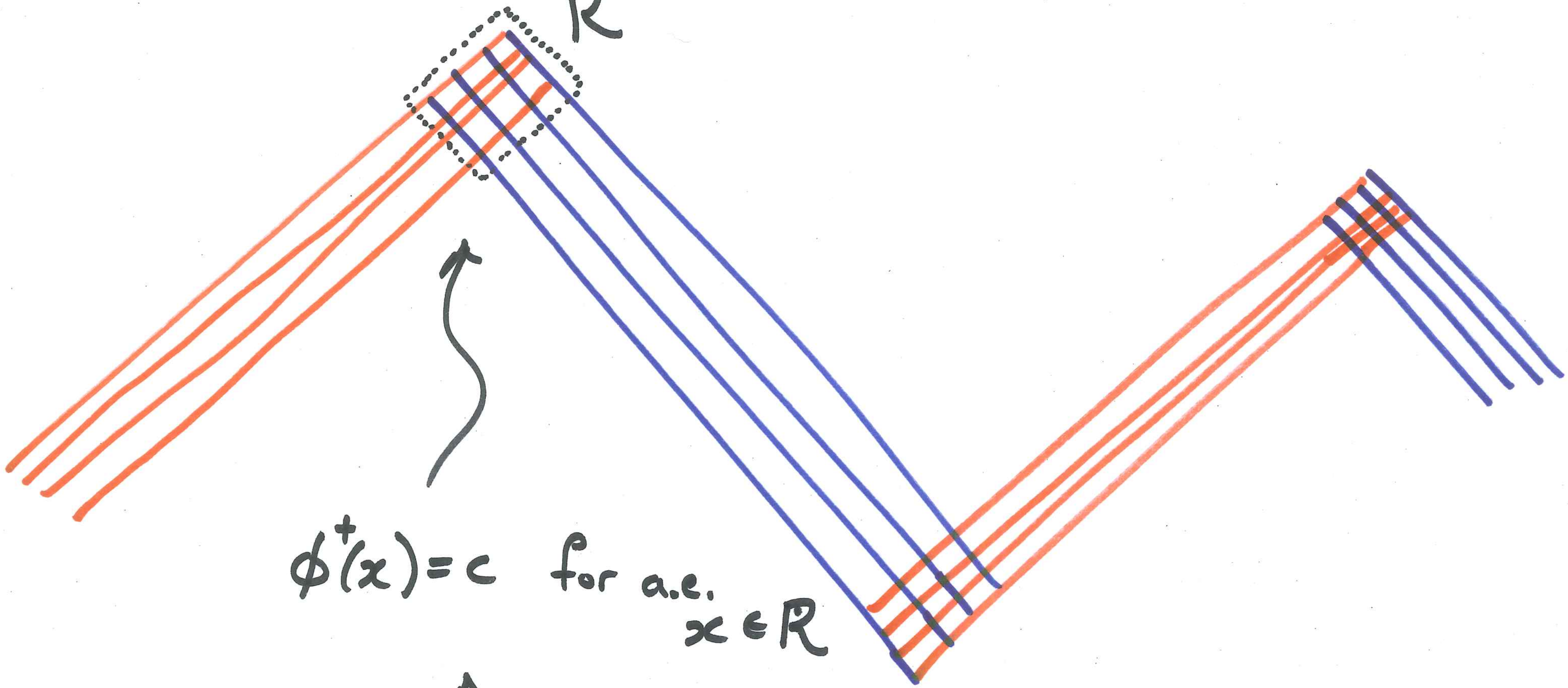
First, a non-proof that ϕ^+ is constant a.e.

For $x, y \in M = \mathbb{T}^2$




$\phi^-(x_1) \overset{?}{=} \phi^+(x_1)$
$\phi^-(x_0) = \phi^-(x_1)$
$\phi^+(x_1) = \phi^+(x_2)$
$\phi^-(x_2) = \phi^-(x_3)$
x_0
$\phi^+(x_2) \overset{?}{=} \phi^-(x_2)$

Use Fubini's Theorem

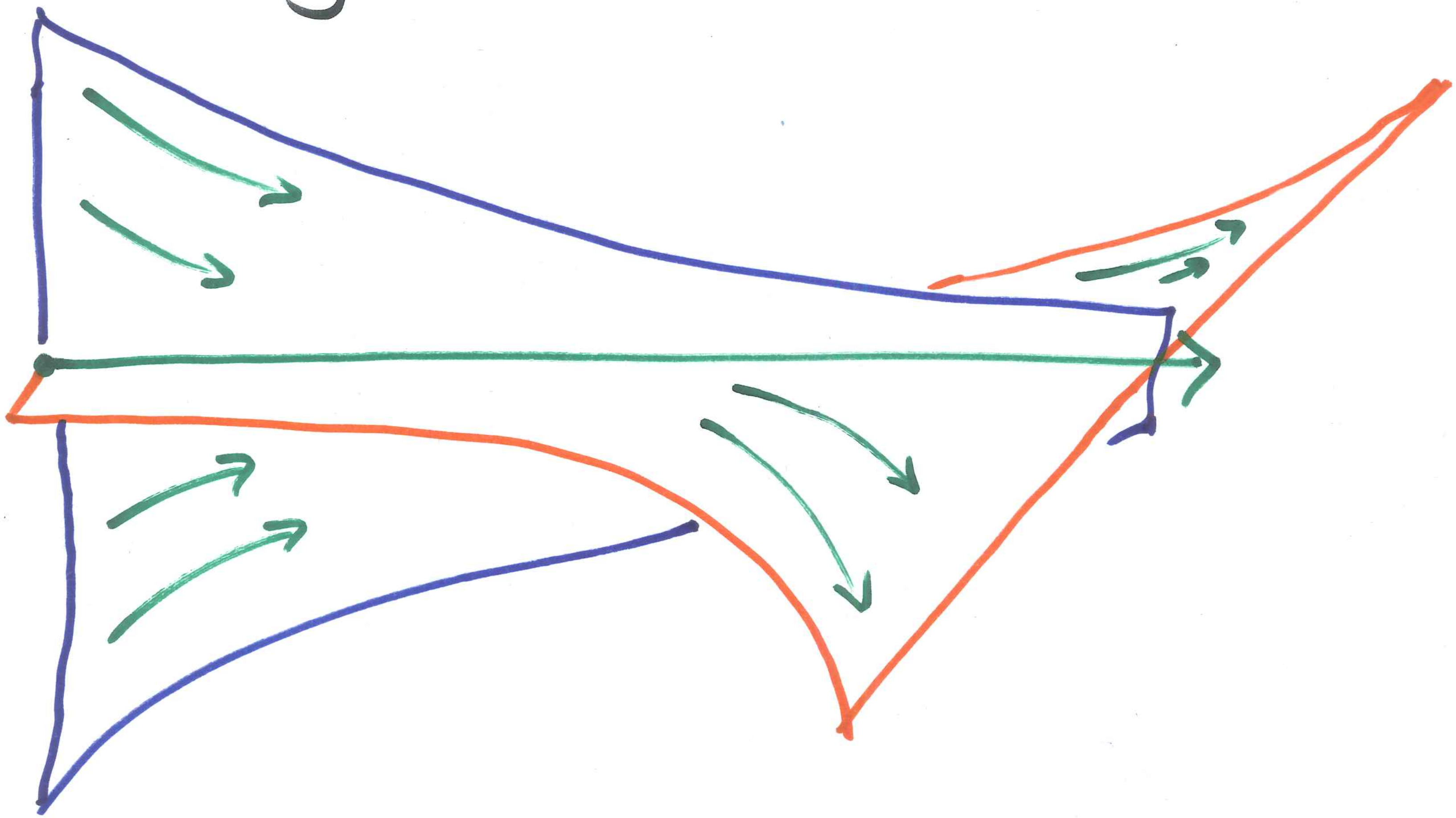


$\phi^{+}(x) = c$ for a.e. $x \in R$

$\Updownarrow$

$\phi^{-}(x) = c$ for a.e. $x \in R$

A similar argument works for geodesic flows.



Smale (1961):

Is the cat map

<u>stably</u> ergodic?

Anosov (1962, 1963): Yes!

Def A diffeo $f : M \to M$ is **Anosov**

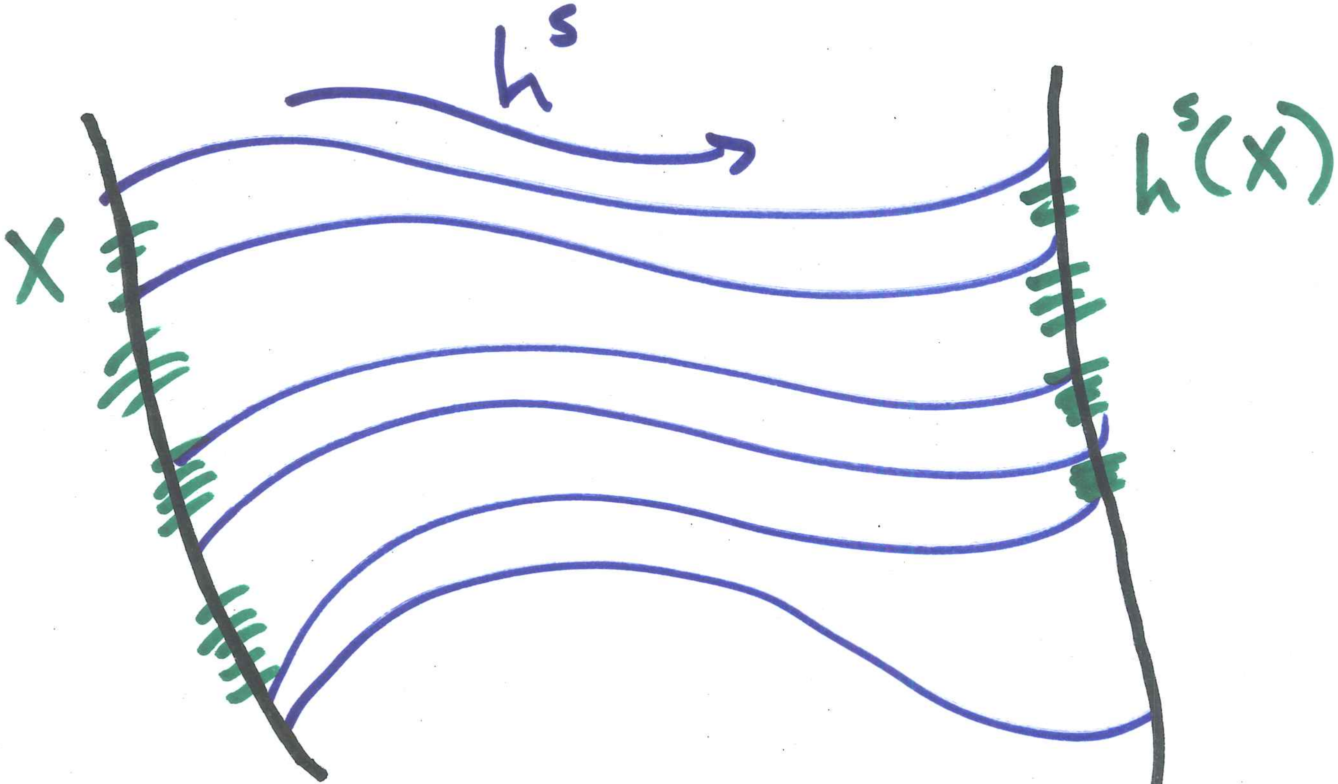
if $\exists$ an inv't splitting

$$TM = \underbrace{E^s}_{\text{contracting}} \oplus \underbrace{E^u}_{\text{expanding}}$$

$\exists\ \sigma < 1$ s.t. $\|Df v\| \leq \sigma \|v\|$ for $v \in E^s$

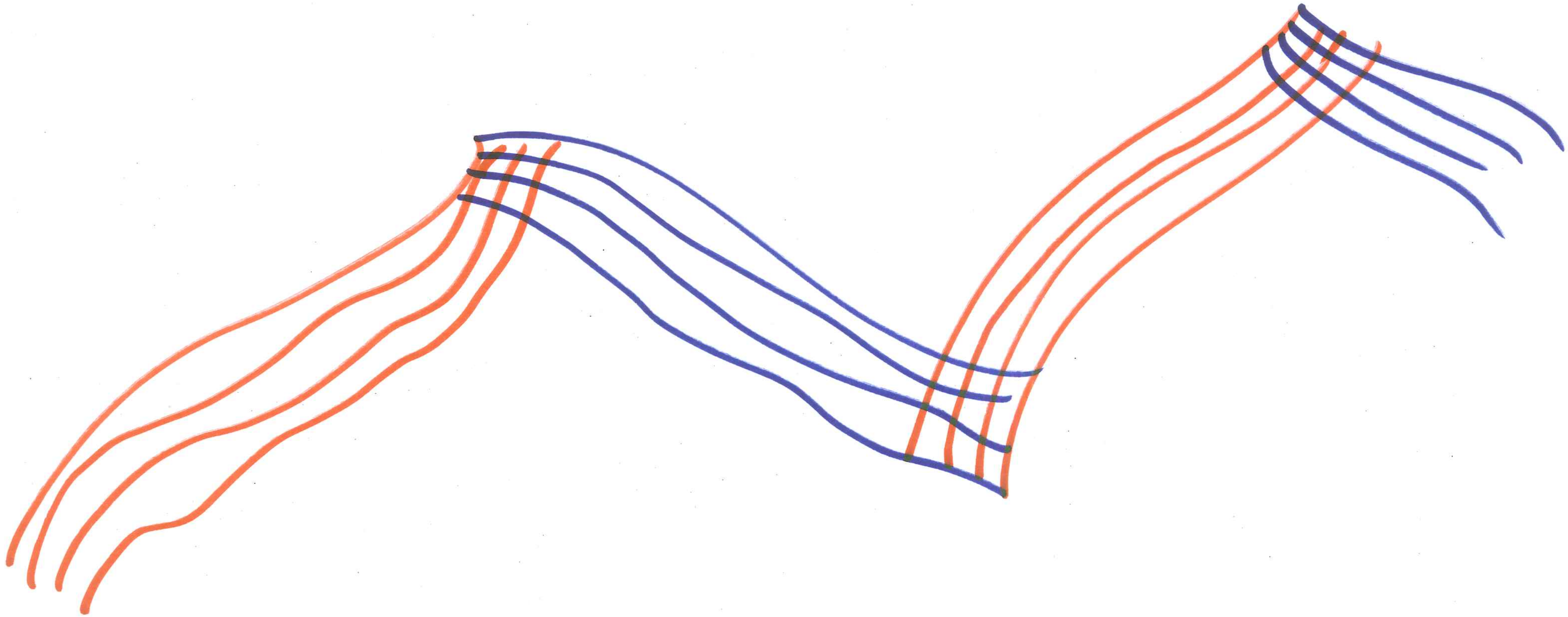
$\|Df^{-1} v\| \leq \sigma \|v\|$ for $v \in E^u$.

Thm (Anosov) For a C^2 Anosov diffeo or flow, the stable and unstable foliations have absolutely continuous holonomies.



X has meas zero $\Longleftrightarrow$ $h^s(X)$ has meas zero

<u>Thm</u> (Anosov) A volume preserving C^2 Anosov system is ergodic.



Fact: Every vol. pres. C^2 diffeo which is C^1-close to the cat map is ergodic.

OPEN QUESTION: If a vol. pres. $\underline{\underline{C^1}}$ diffeo is C^1-close to the cat map, is it ergodic?

For the rest of the talk,

a vol. pres. C^2 diffeo is

stably ergodic

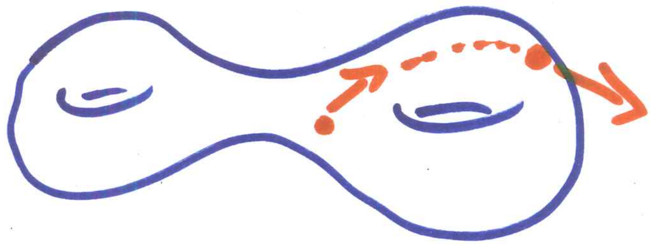
if every vol. pres. C^2 diffeo in a C^1-neighbourhood is ergodic.

From 1963 to 1994,

Anosov systems were the only known stably ergodic systems.

Grayson-Pugh-Shub 1994:

Let $\varphi_t : M \to M$ be the geodesic flow of a surface of curvature $\kappa = -1$



The time-1 map $f = \varphi_1 : M \to M$ is stably ergodic <u>as a diffeo</u>.

Two key concepts in the proof:

1) A diffeo $f: M \to M$ is partially hyperbolic if there is an invariant splitting

$$TM = \underbrace{E^s}_{\text{strongly contracting}} \oplus \underbrace{E^c}_{\text{no strong contraction or expansion}} \oplus \underbrace{E^u}_{\text{strongly expanding}}$$

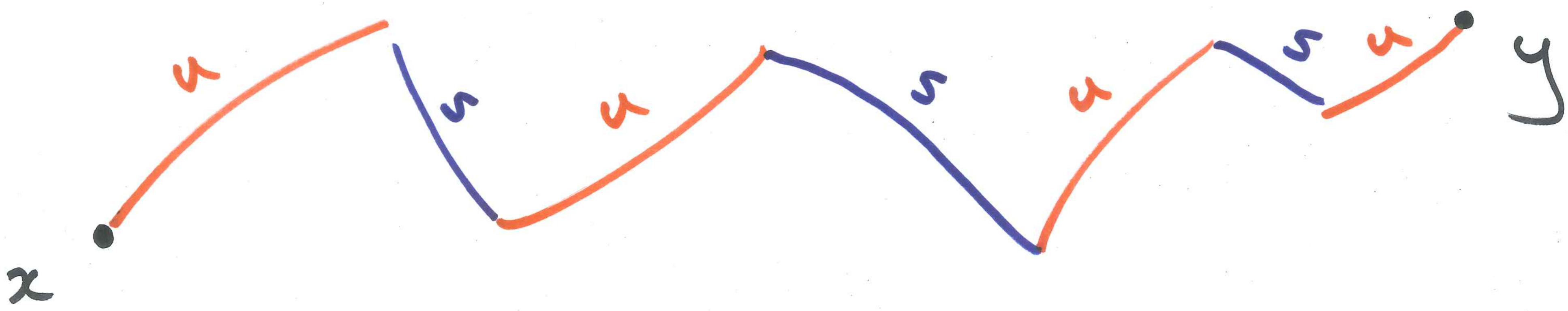
Example: the time-1 map of an Anosov flow

Two key concepts in the proof:

2) A partially hyperbolic diffeo is <u>accessible</u>

if for any $x, y \in M$

there is a path



Grayson-Pugh-Shub showed that any $f: M \to M$ near the time-1 map is both partially hyperbolic and accessible.

Then did lots of analysis to show f is ergodic.

This work lead to the Pugh-Shub conjectures.

Pugh-Shub Conjectures

In the space of vol. pres. partially hyperbolic C^r diffeos $(r \geq 2)$:

1) ergodicity is C^r-open and C^r-dense

2) accessibility is C^r-open and C^r-dense

3) accessibility implies ergodicity

For C^2 vol. pres. partially hyperbolic systems:

Rodriguez Hertz – Rodriguez Hertz – Ures 2008:

if $\dim E^c = 1$, then ergodicity is C^1-open and C^r-dense $(r \geq 2)$

Burns-Wilkinson 2010:

accessibility & center bunching $\Longrightarrow$ ergodicity

Avila-Crovisier-Wilkinson, preprint in 2014:

ergodicity is C^1-open and C^1-dense

Converse Results (based on work of Arbieto-Mathews)

For diffeos:

In dim 2 stably ergodic $\Longleftrightarrow$ Anosov (Mañé-Bochi)

In dim 3 stably ergodic $\Longrightarrow$ (weakly) partially hyperbolic (Díaz-Pujals-Ures)

In dim ≥ 4 stably ergodic $\Longrightarrow$ volume partially hyperbolic (Bonatti-Díaz-Pujals)

Questions:

What types of partially hyperbolic systems exist?

What are the non-ergodic ones?

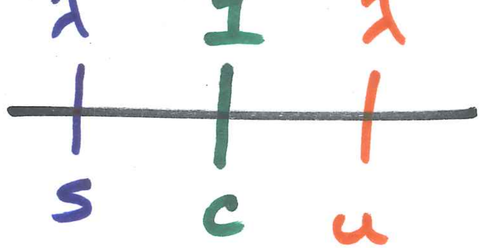
What are their ergodic decompositions?

Consider linear maps on the 3-torus $\mathbb{T}^3$

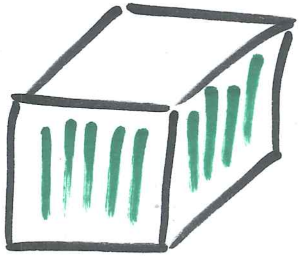
Example:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

eigs



$A \times \mathrm{id}$ on $\mathbb{T}^2 \times S^1$

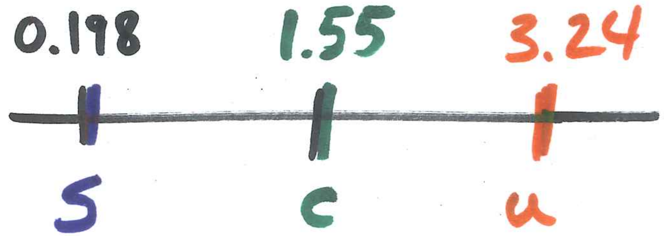


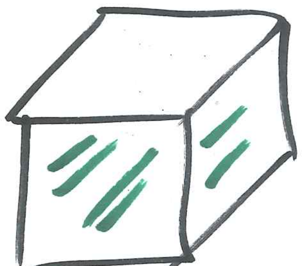
$\uparrow E^c$

Example:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

eigs





$\nearrow E^c$ has irrat slope

<u>Thm</u> (H, 2009)

Every partially hyperbolic diffeo on $\mathbb{T}^3$ is leaf conjugate to a linear map.

(After a C^0 change of coordinates on $\mathbb{T}^3$, f is given by a ~~[illegible]~~ linear map composed with some sliding along the E^c direction.)

Thm (H - Ures, 2013)

If $f : \mathbb{T}^3 \circlearrowleft$ is vol pres, partially hyperbolic and homotopic to an Anosov diffeo A, then either

1) f is ergodic, or

2) f is non-ergodic and topologically conjugate to A ($\exists h$ so that $f = h^{-1} A h$)

We don't know if case 2 exists.

Thm (H-Potrie, 2014)

Every partially hyperbolic diffeo on a 3-mfld with solvable fundamental group is leaf conjugate to

1) an Anosov diffeo on $\mathbb{T}^3$,

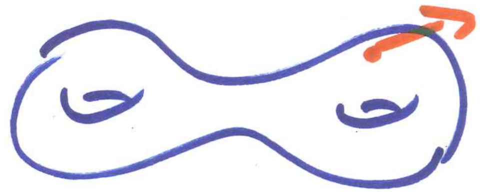
2) a skew product, or

3) the time-1 map of an Anosov flow.

This allowed me to give a classification of all of the accessibility classes of these diffeos.

Except for the open question on $\mathbb{T}^3$, these accessibility classes coincide with the ergodic decomposition of the system.

In recent work with
Bonatti, Gogolev, & Potrie,
we've found new stably ergodic
partially hyperbolic systems on unit
tangent bundles



which are not homotopic to the
time-1 maps of Anosov flows.