Models of chaos in dimensions 2 and 3.

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Joint w/ C. Bonatti, A. Gogolev, R. Potrie

Consider a flow $\varphi_t : M \times \mathbb{R} \to M$.

What is the long term behaviour?

Define the <u>ω-limit set</u>

$$\omega(x) := \lim_{T \to \infty} \overline{\{ \varphi^t(x) : t > T \}}$$

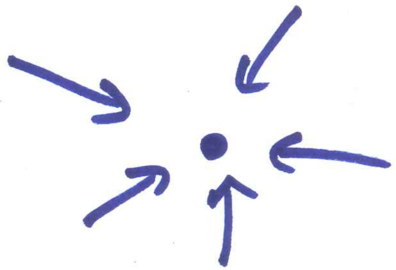
$$\left(y \in \omega(x) \iff \exists\, t_k \to \infty \text{ s.t. } \varphi^{t_k}(x) \to y \right)$$

What is $\omega(x)$?

What are the dynamics on $\omega(x)$?

Examples a sink $s \in M$

$$\omega(x) = \{s\}$$



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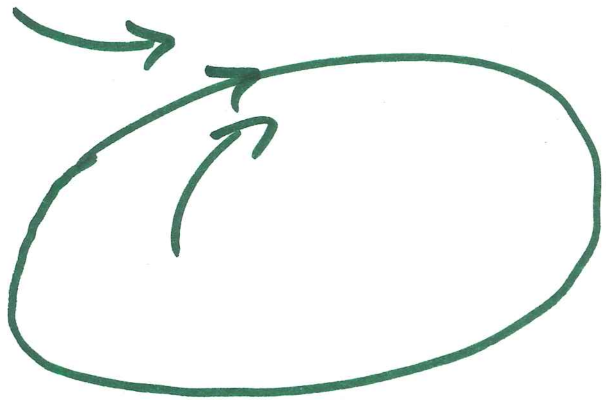
a periodic orbit



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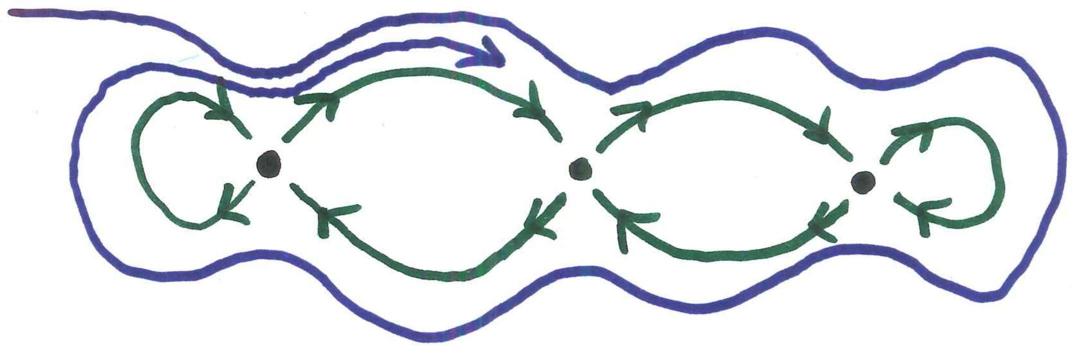
a "graphic"

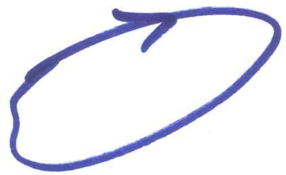


Thm (Poincaré – Bendixson 1901)

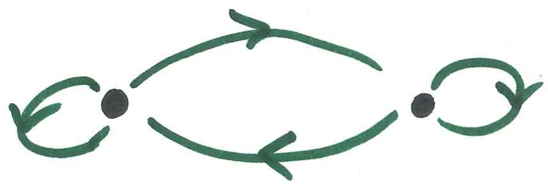
For a flow on $\mathbb{R}^2$ or S^2 and any point x,

$\omega(x)$ is

1) a point

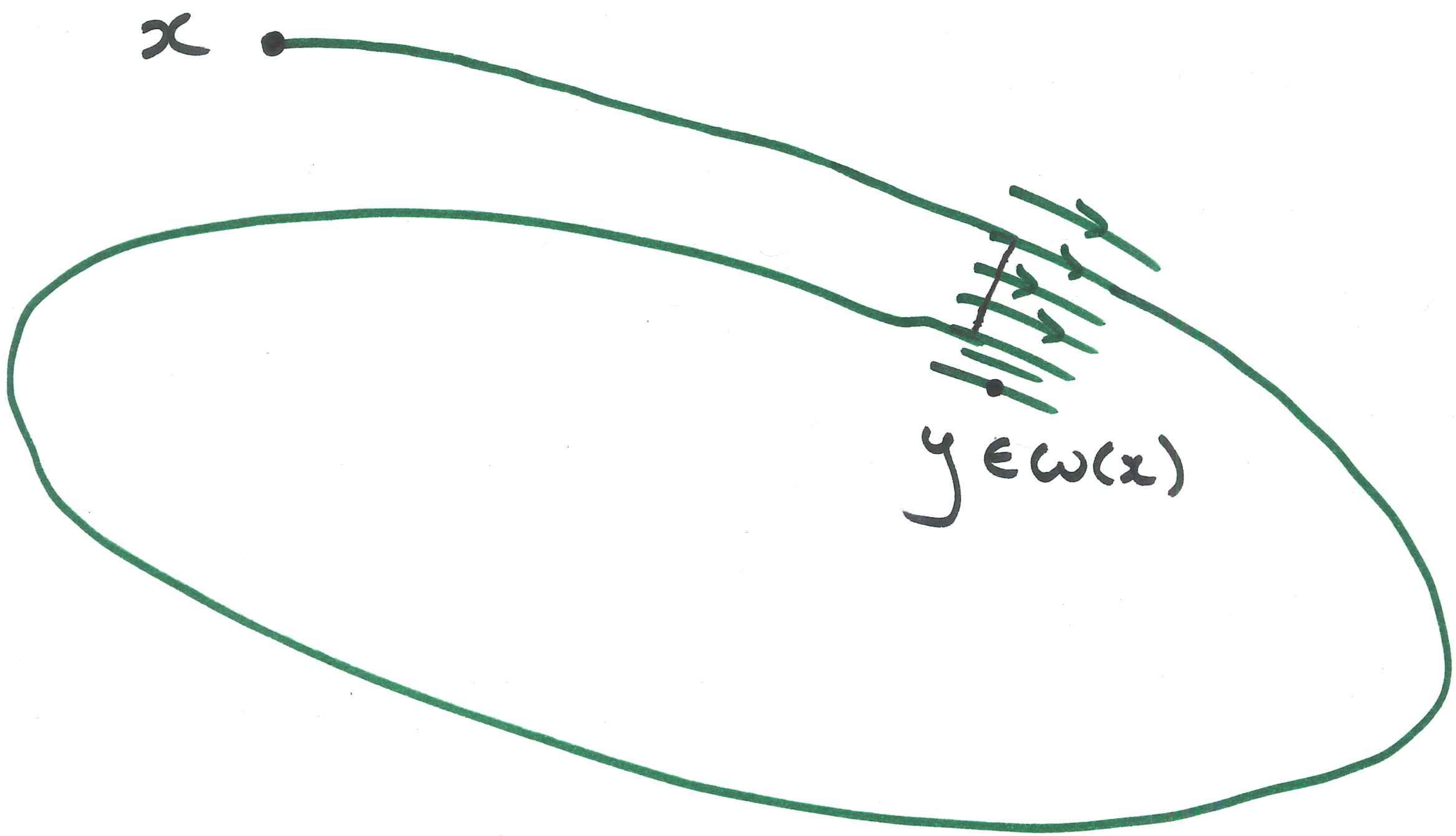
2) a periodic orbit, 

or

3) a "graphic". 

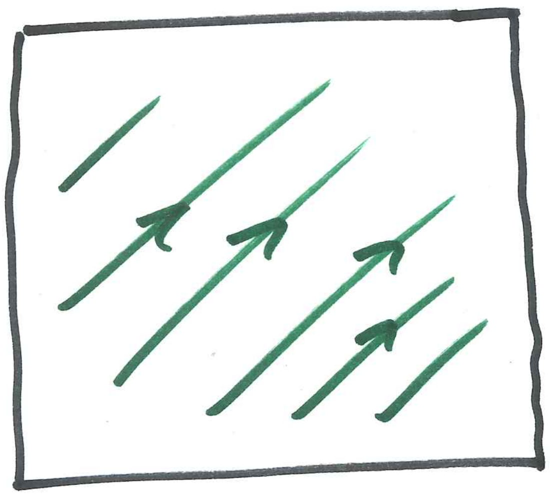
Key Idea of Proof

Jordan curves and trapping regions



The result does not hold on other surfaces.

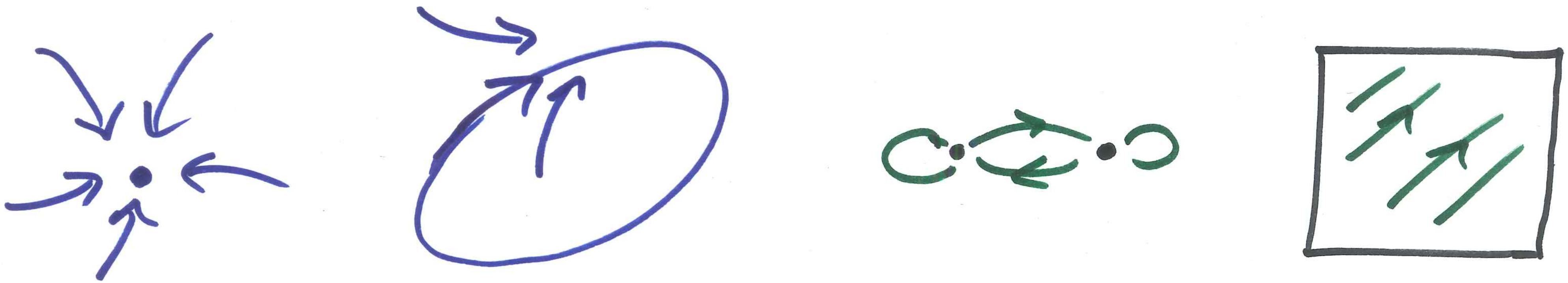
<u>Ex</u> Irrational slope flow on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$



For any $x \in \mathbb{T}^2$

$$\omega(x) = \mathbb{T}^2.$$

Examples



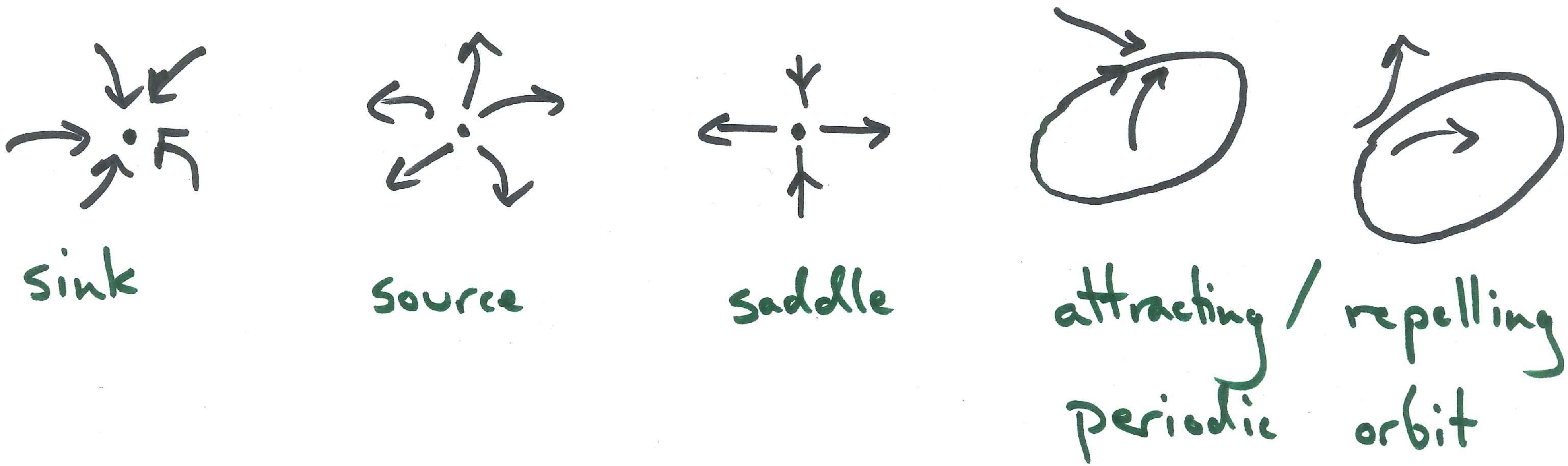
Which of these is robust under perturbation?

Thm (Peixoto 1962)

Let φ^t be a flow on a compact surface.

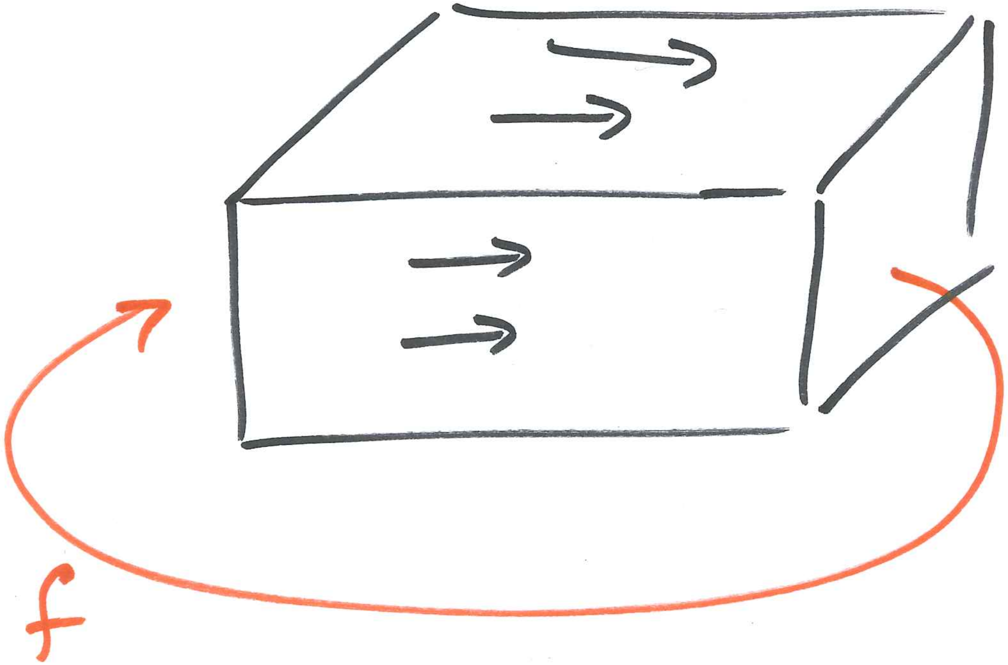
After a C^r-small perturbation,

every ω-limit set is of the form:



What about flows in dim 3?

First, consider diffeo(morphism)s in dim 2.



$f: M^2 \to M^2$

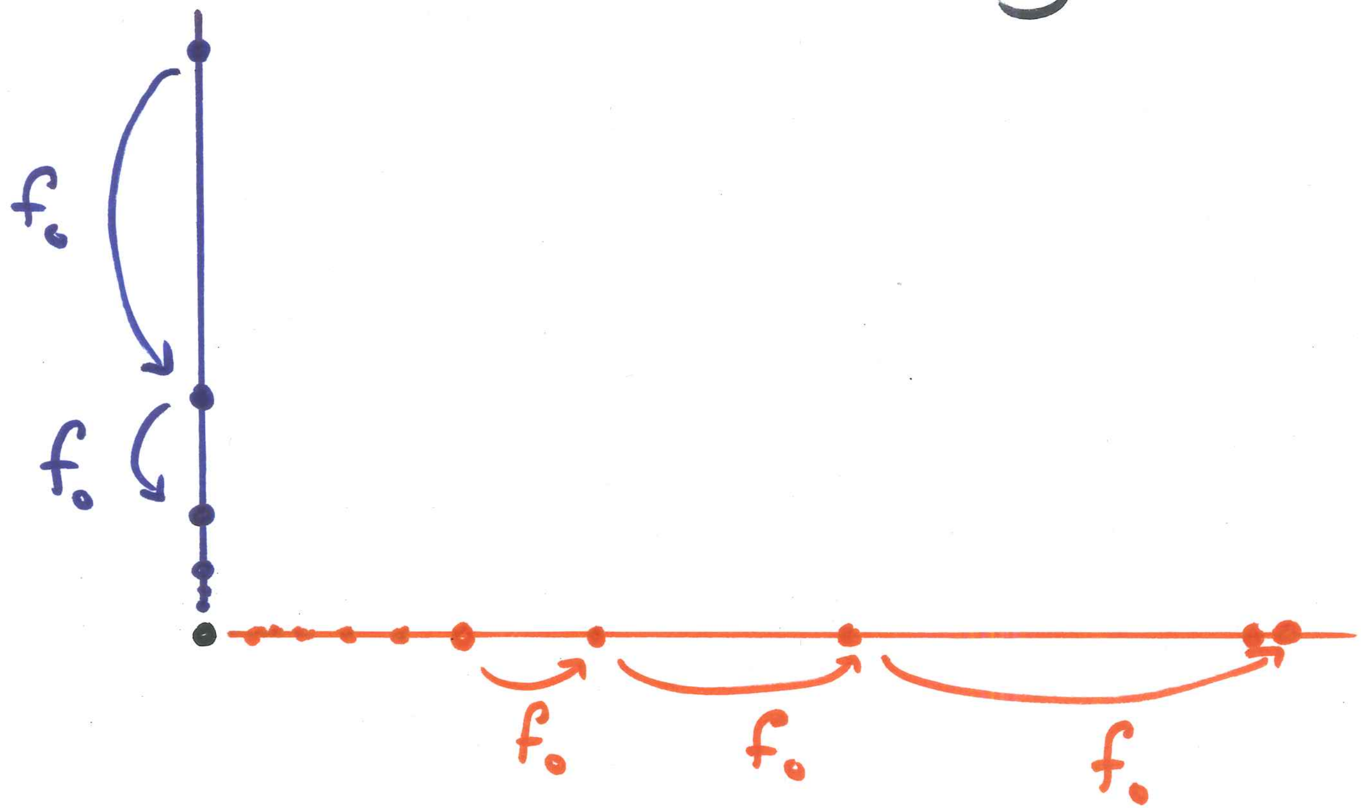
can be suspended to flow

φ^t in dim 3

For diffeos in dim 2, chaos is possible.

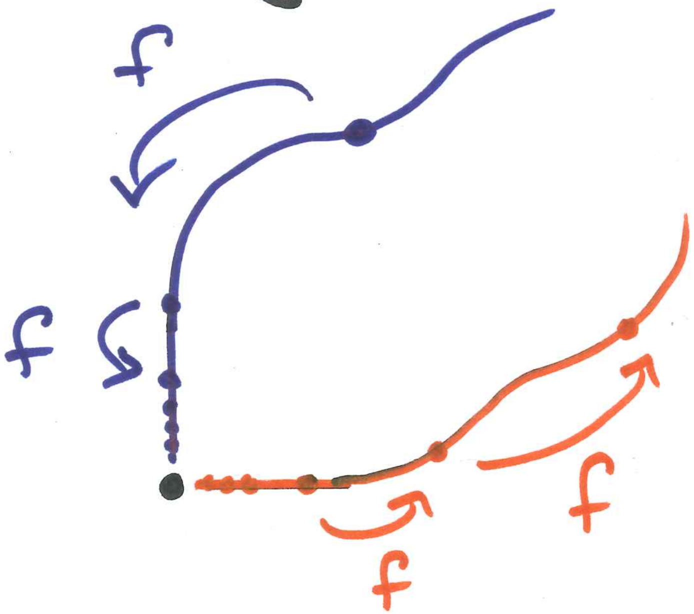
Ex

In $\mathbb{R}^2$, $f_0(x,y) = (2x, \frac{1}{3}y)$



Deform f_0 away from origin

to get f



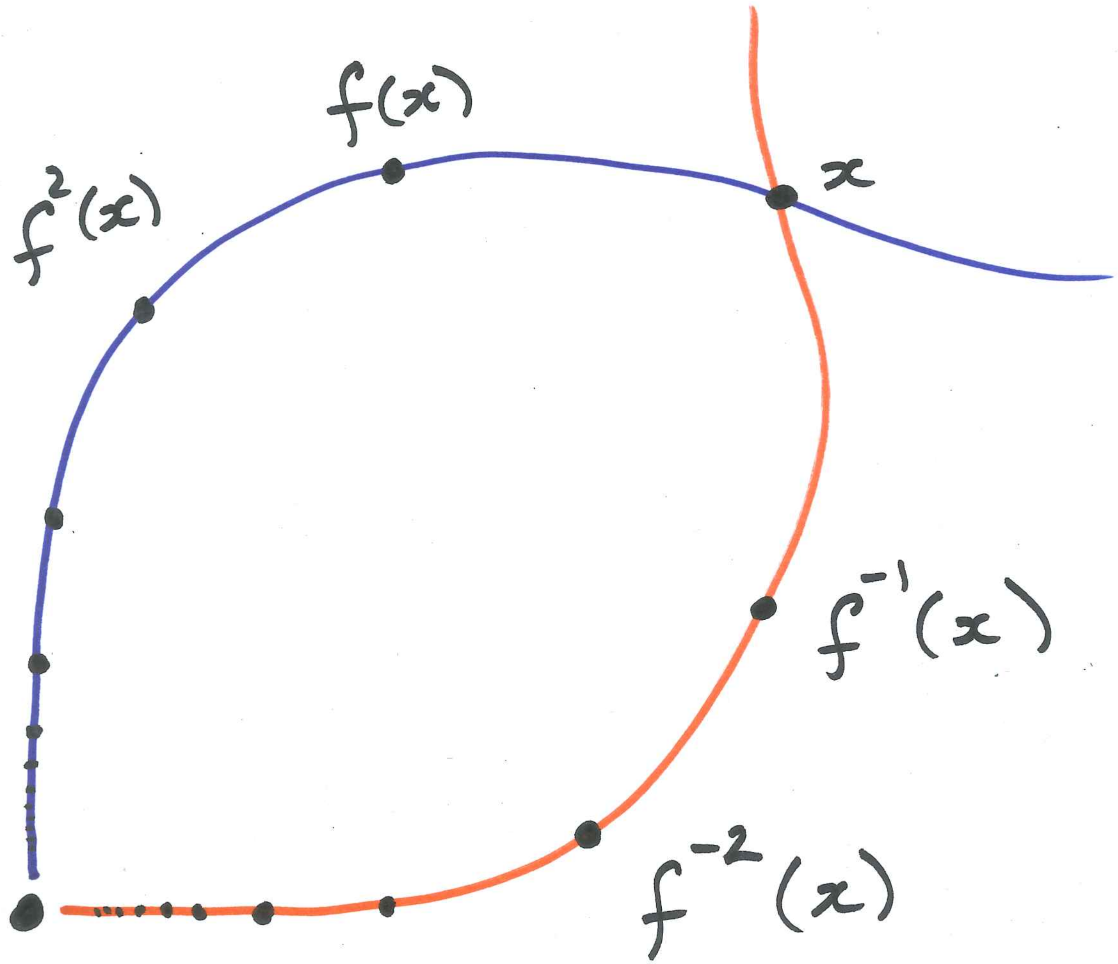
Still have invariant curves.

Don't cross the streams.

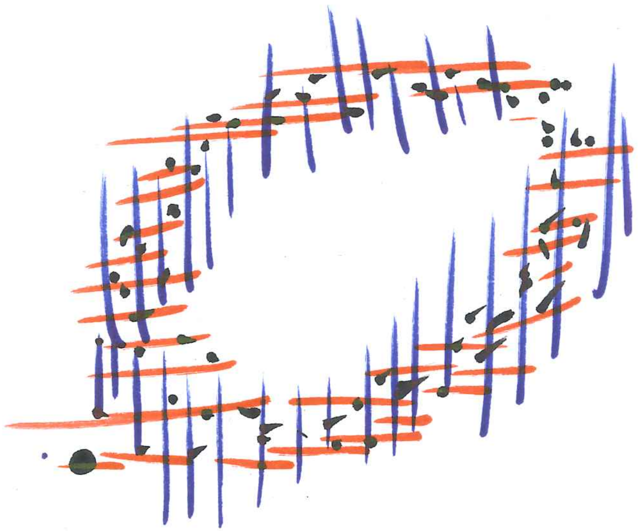
Why?

It would be bad.

- Ghostbusters (1984)



Take the closure Λ of all the intersections in the picture.



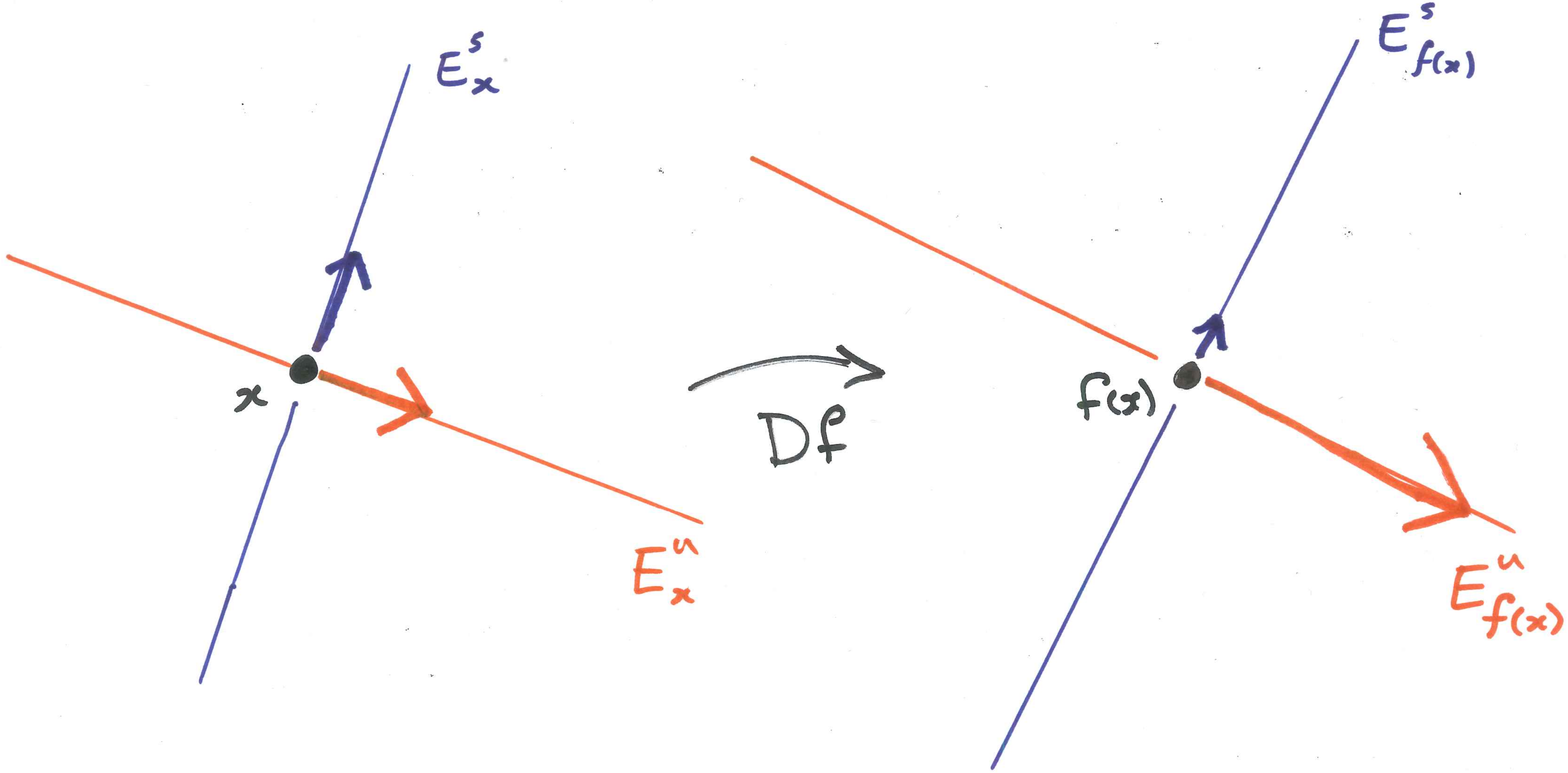
This is a <u>uniformly hyperbolic</u> set.

For a diffeomorphism $f : M \to M$, a subset Λ is uniformly hyperbolic if $f(\Lambda) = \Lambda$ and at each $x \in \Lambda$, $T_x M$ splits as

$$T_x M = E^u_x \oplus E^s_x$$

unstable expanded by Df

stable contracted by Df



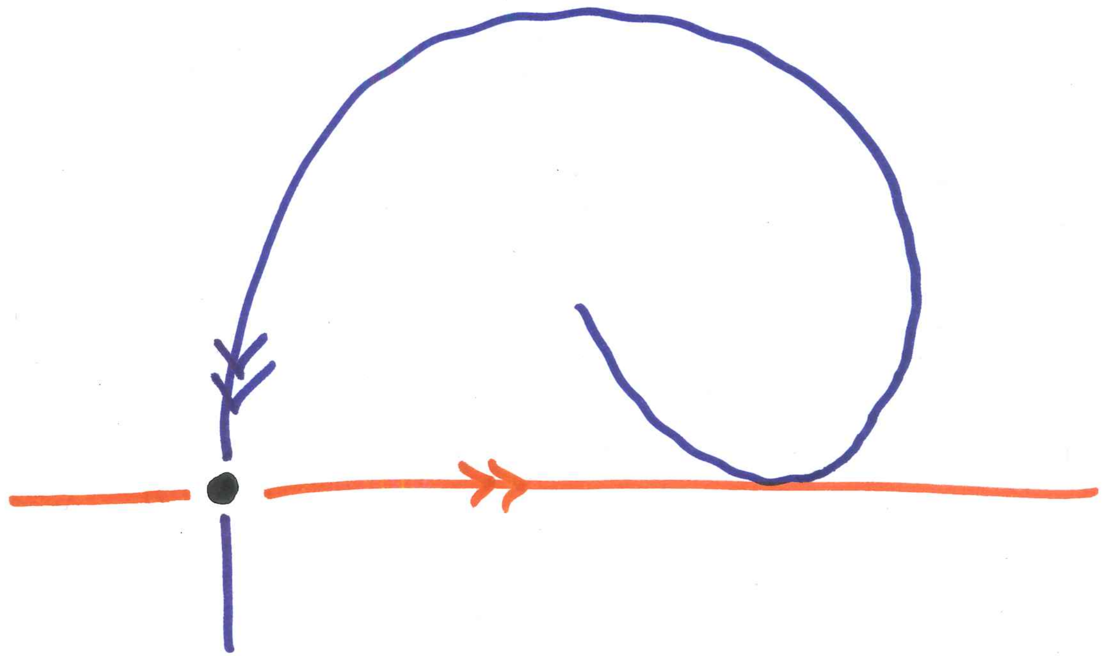
E^s_x
x
E^u_x
Df
$E^s_{f(x)}$
$f(x)$
$E^u_{f(x)}$

For diffeos in dim 2:

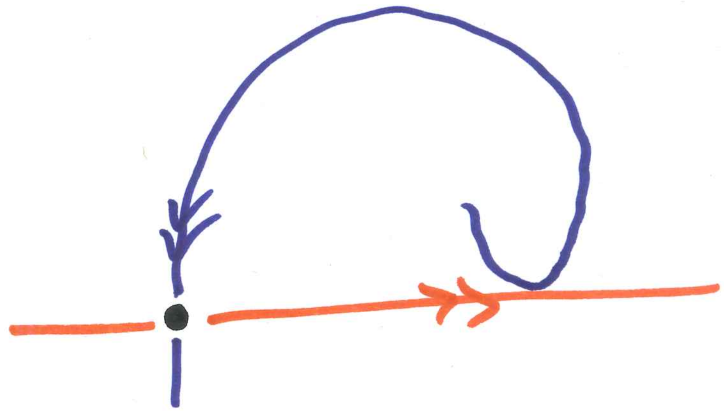
After perturbation, is every $\omega(x)$ uniformly hyperbolic?

A possible obstruction:

a homoclinic <u>tangency</u>



Newhouse 1970 : $\exists$ a surface diffeo f so that every g C^2-close to f has a tangency.



Open question for surface diffeos in the C^1-topology.

Say a system is "far from tangencies" if $\exists$ a C^1-nbhd with no tangencies.

<u>Thm</u> [Pujals - Sambarino 2000]

"Far from tangencies"

a surface diffeo may be C^1 - perturbed so that every $\omega(x)$ is uniformly hyperbolic.

For a diffeo $\omega(x) = \lim_{N\to\infty} \overline{\{f^n(x) : n > N\}}$

Recall: A system is transitive if there is a dense orbit

i.e. there is $x \in M$ such that

$$\omega(x) = M$$

Are there robustly transitive diffeos?

__Ex__ the "cat map"

$$A(x,y) = (2x+y,\ x+y)$$

on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Here, the entire mfld $\mathbb{T}^2$ is a uniformly hyperbolic set.

In such a case, the dynamical system is called __Anosov__

<u>Anosov</u>: $\exists$ global splitting $TM = E^u \oplus E^s$

<u>Thm</u> [Mañé] For a diffeo of a cmpt surface
1982

C^1-robustly transitive $\iff$ Anosov.

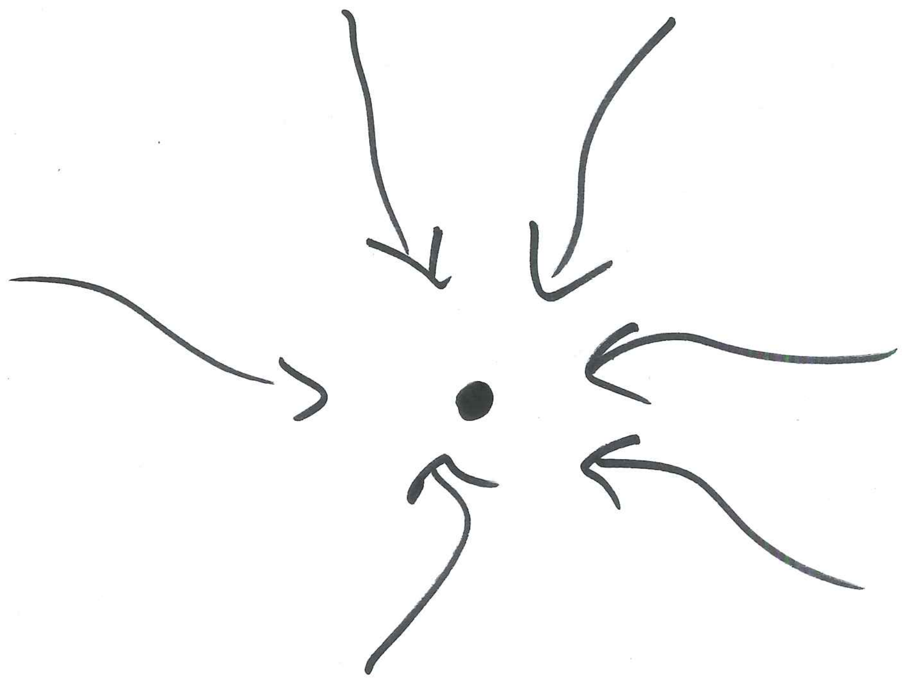
Note: Anosov $\Rightarrow$ line field on M

$\Rightarrow M = \mathbb{T}^2$

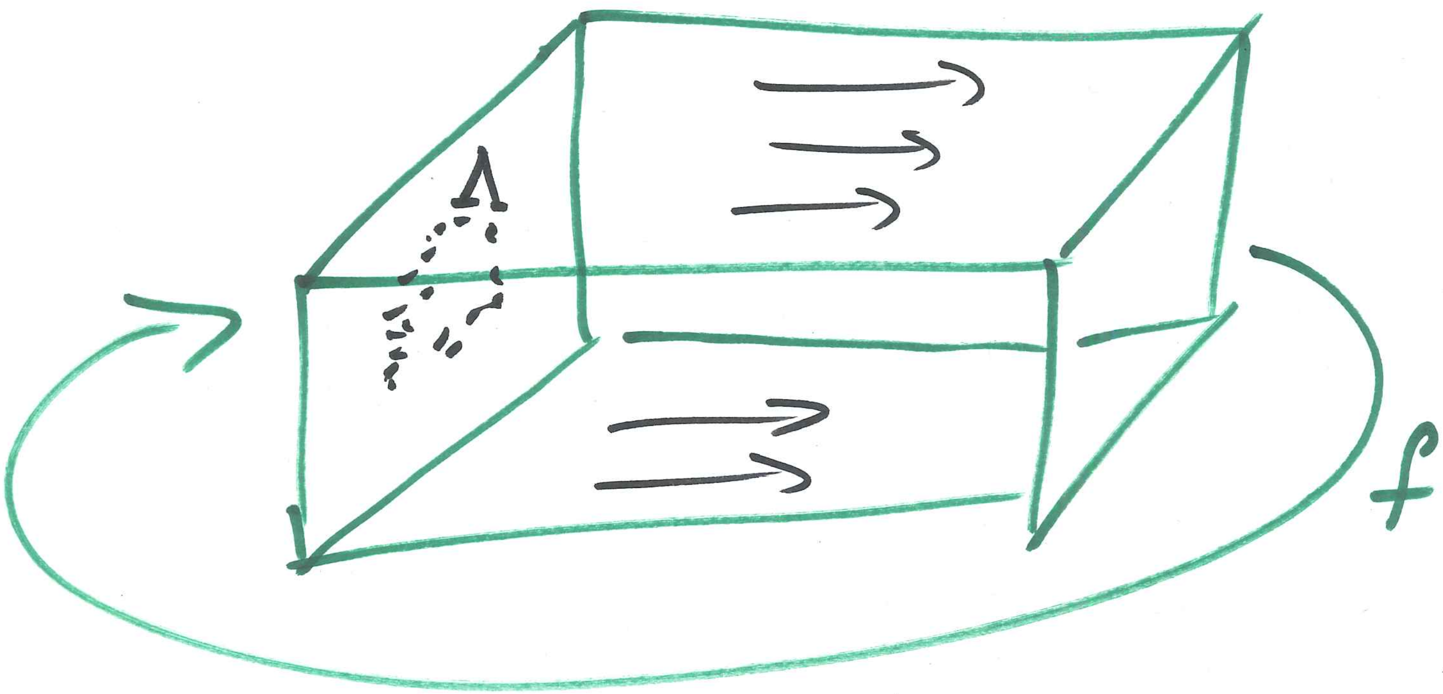
flows in dim 3

Examples

$$\omega(x) = \{\text{point}\}$$

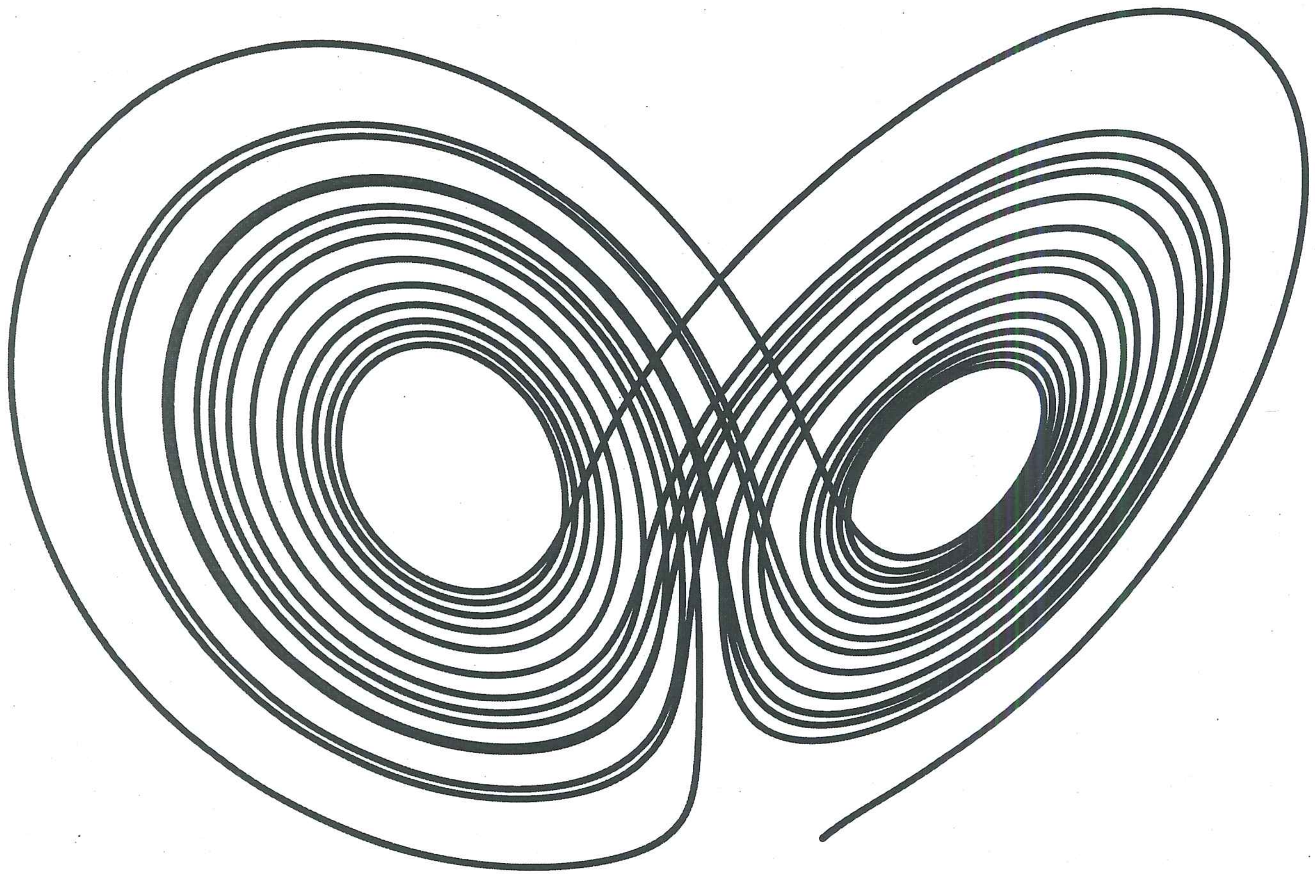


Example Take a uniformly hyperbolic set Λ for a surface diffeo f and suspend it.

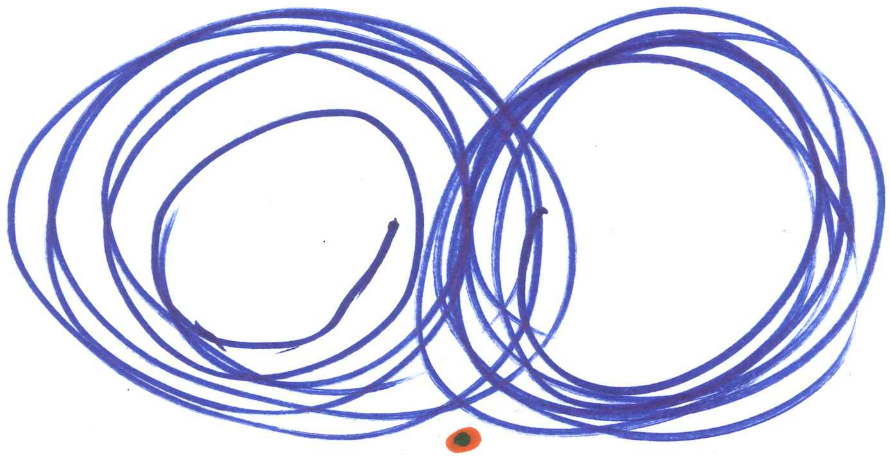


This is robust under perturbation.

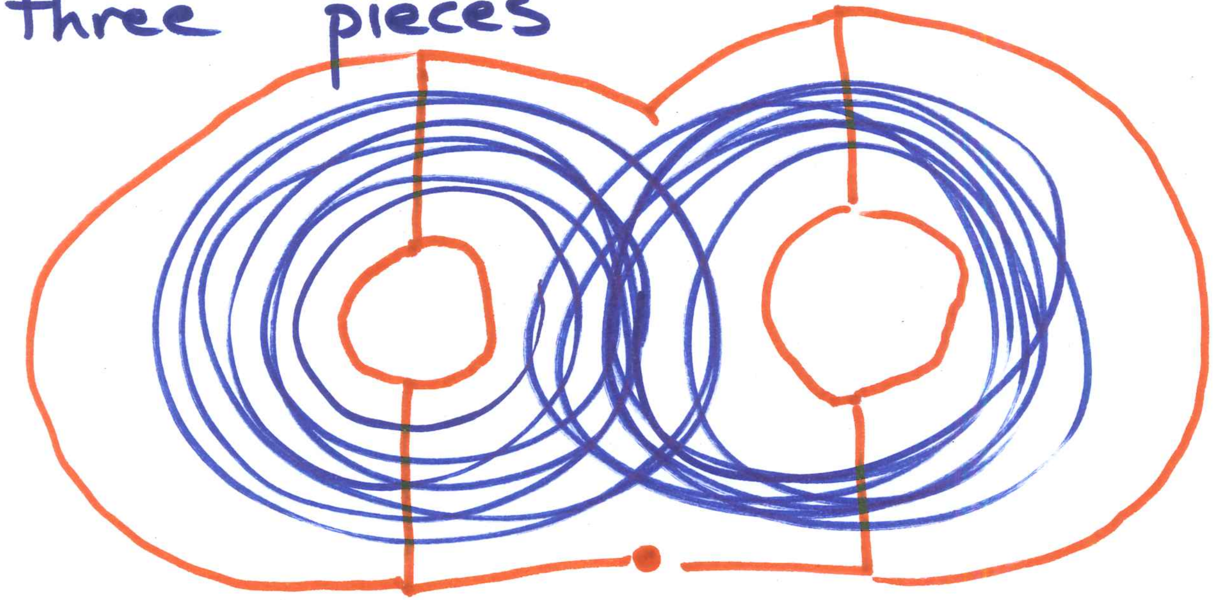
Ex The Lorenz attractor



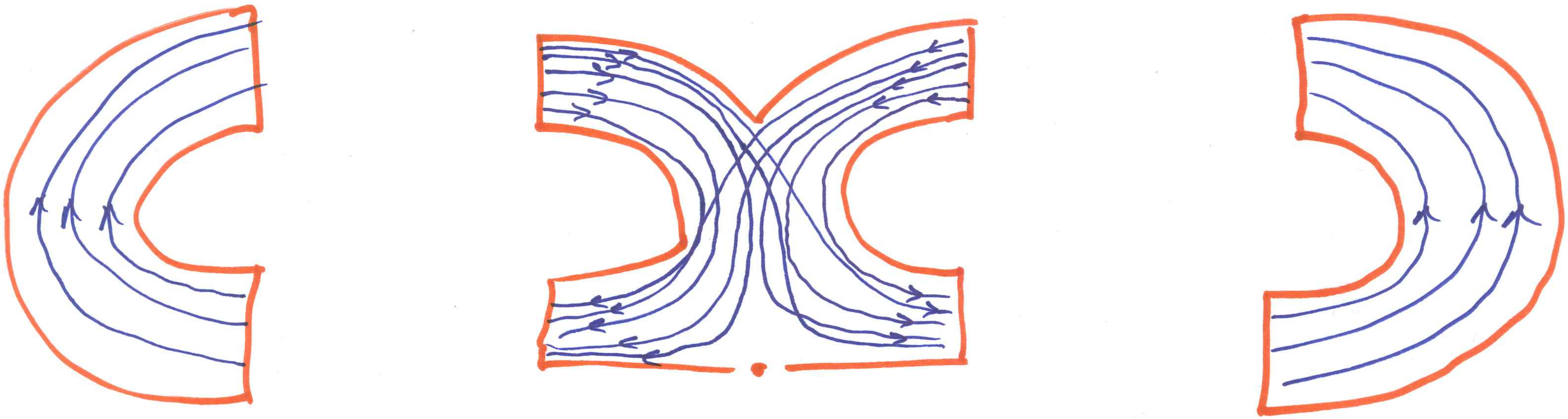
Split the Lorenz attractor



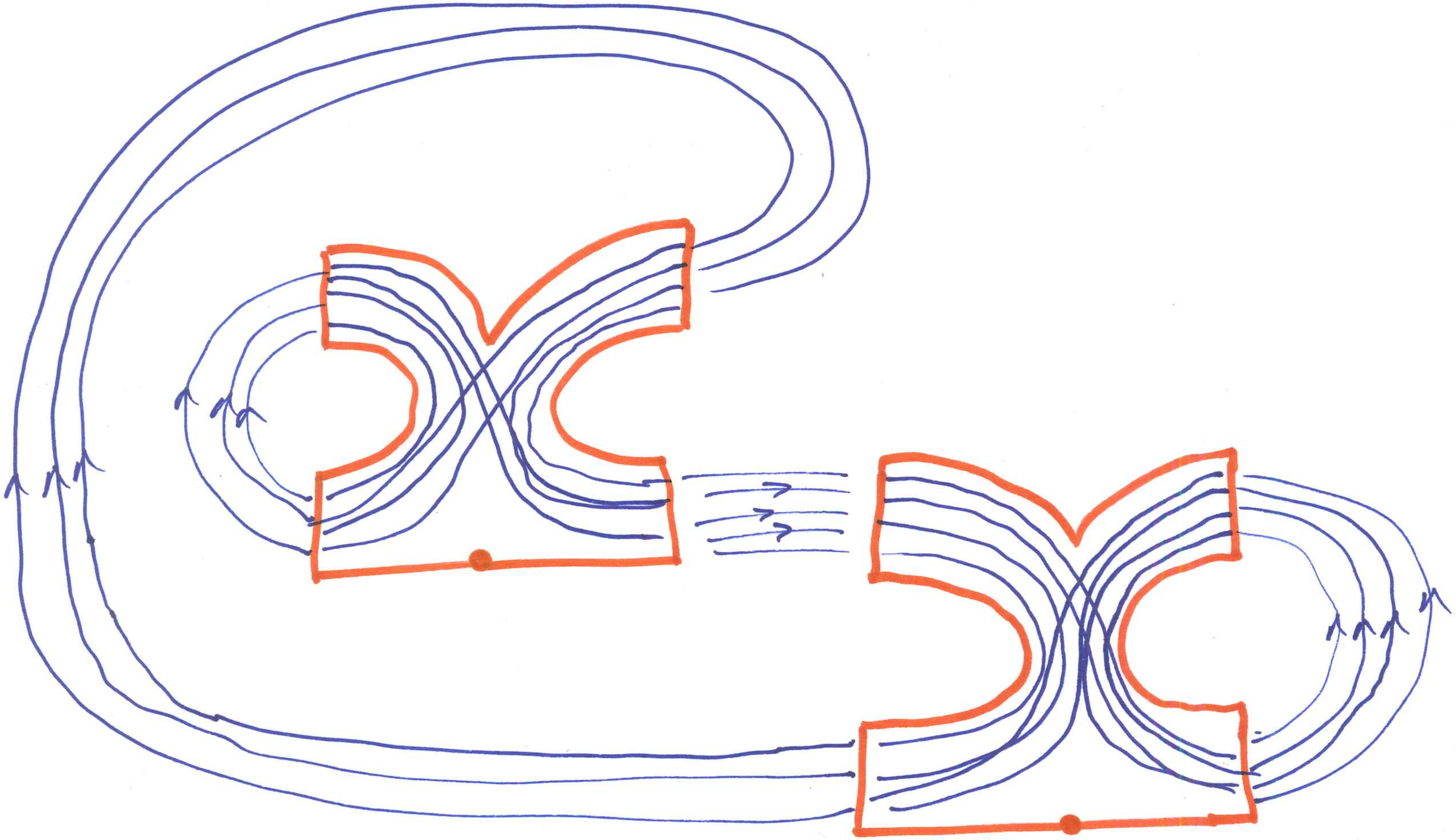
into three pieces

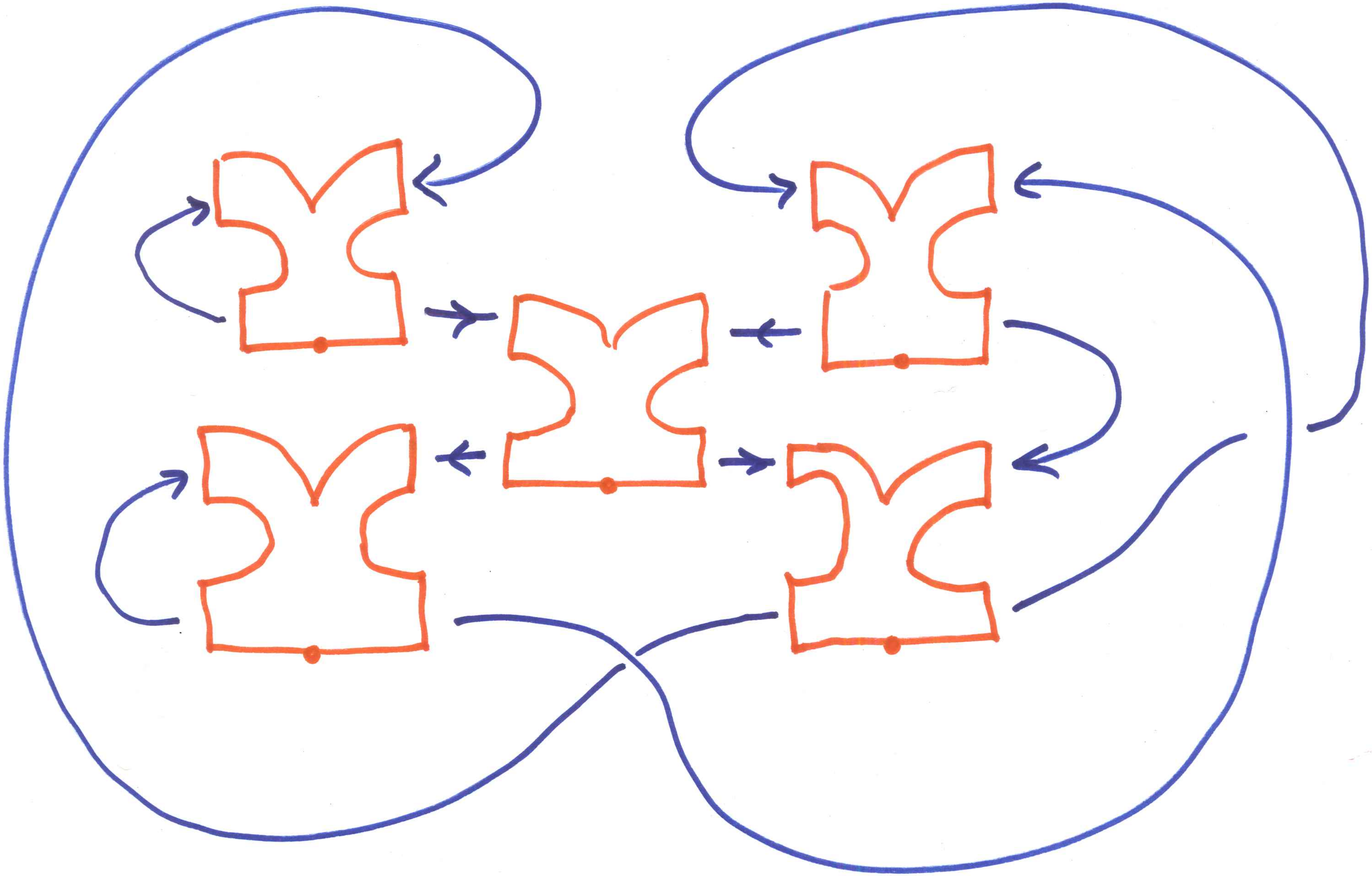


Only the middle piece is chaotic.



Multiple copies of this middle piece may be connected together.



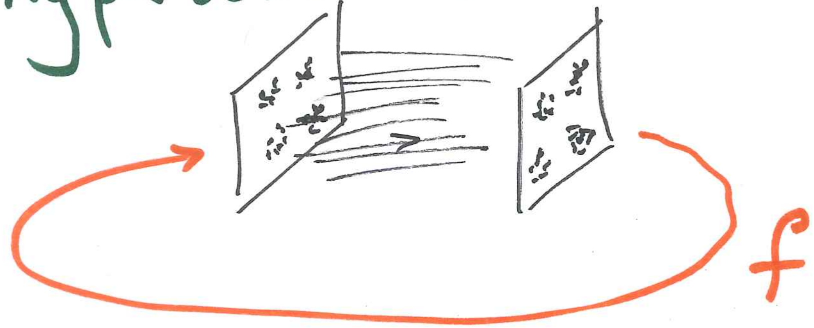


These are examples of singular hyperbolic sets.

Thm (2015) "Far from tangencies" a flow φ^t on a 3-mfld M may be C^1-perturbed so that every ω-limit set $\omega(x)$ is one of:

1) $\omega(x)$ is a single point •

2) $\omega(x)$ is the suspension of a 2-dim'l uniformly hyperbolic set



3) $\omega(x)$ is singular hyperbolic

"Lorenz-like"

4) $\omega(x) = M$ and φ_t is a robustly transitive Anosov flow

Thm Credits

Doering (1985)

Morales, Pacifico, Pujals (1999, 2004)

Arroyo, F. Rodriguez Hertz (2003)

Araújo, Pacifico, Pujals, Viana (2009)

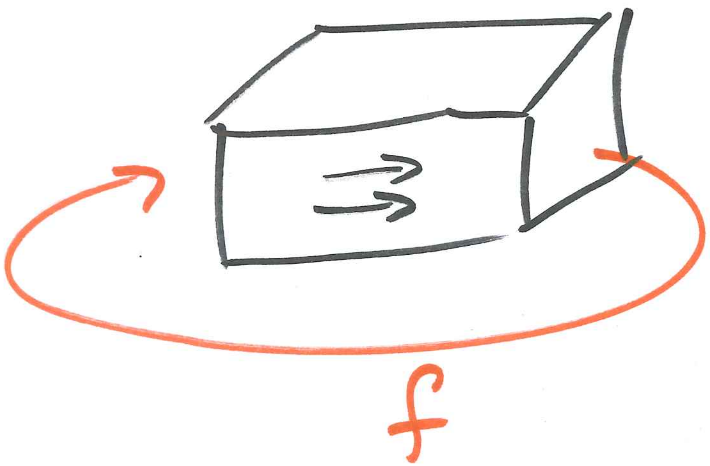
Crovisier, D. Yang (2015)

Also Tucker (2002) and others...

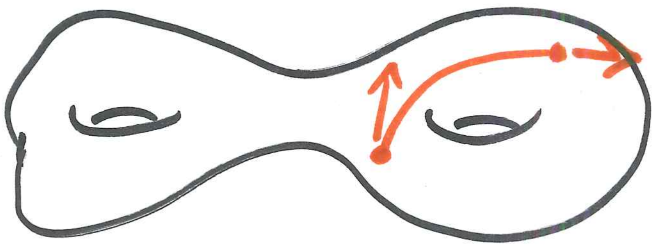
Thm (Doering 1985) Any robustly transitive flow on a 3-mfld is Anosov

Main Examples of Anosov flows:

1) Suspension of Anosov diffeo



2) geodesic flows (negative curvature)



3) surgery on these examples

There are no Anosov flows on

S^3, $\mathbb{T}^3$, most Seifert fiber spaces,

the Weeks mfld, reducible manifolds, ...

For flows in dim 3

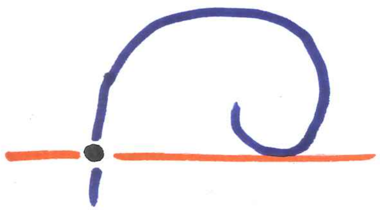
Robust Transitivity $\Longrightarrow$ Anosov $\Longrightarrow$ Restrictions on the geometry.

The Great Unknown

Diffeos in dim 3

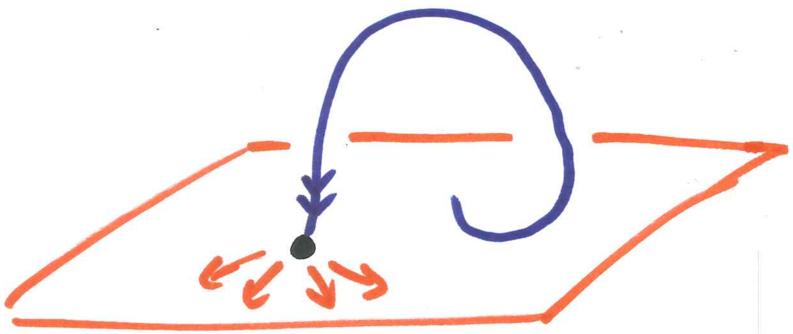
Are tangencies robust in the C^1 topology?

Open question for



diffeos in dim 2
and
flows in dim 3.

There are known examples of



robust tangencies
for diffeos
in dim 3.

Recall: A system is transitive if there is a dense orbit ($\omega(x) = M$ for some x)

For diffeos in dim 2

robustly transitive $\Rightarrow$ Anosov $\Rightarrow$ $M = \mathbb{T}^2$

For flows in dim 3

robustly transitive $\Rightarrow$ Anosov $\Rightarrow$ restrictions on M

For diffeos in dim 3

robustly transitive $\overset{?}{\Rightarrow}$ any restrictions on M?

For diffeos in dim 3

$$\text{robustly transitive} \overset{\text{Mañé 1978}}{\not\Longrightarrow} \text{Anosov}$$

Díaz-Pujals-Ures 1999

$\Downarrow$

f is either weakly partially hyperbolic

or

strongly partially hyperbolic

Def A diffeo $f: M^3 \to M^3$ is (strongly) partially hyperbolic if there is a Df-invariant splitting

$$TM = E^u \oplus E^c \oplus E^s$$

E^u: expanded by Df

E^c: no strong contraction or expansion

E^s: contracted by Df

(Exact inequalities available on request.)

Examples of partial hyperbolicity

1) linear maps A on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$

with eigenvalues $\lambda^s < \lambda^c < \lambda^u$.

Then eigenspaces give splitting $E^s \oplus E^c \oplus E^u$

2) algebraic maps on nilmanifolds

3) the time-one map φ_1 of an Anosov flow φ_t

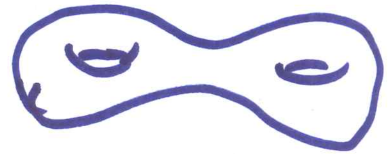
Thm (H - Potrie 2013)

If M is a 3-mfld with solvable fundamental group, and $f : M \to M$ is partially hyperbolic, then

f is "leaf conjugate" to

1) a linear map on $\mathbb{T}^3$,
2) an algebraic map on a nilmanifold, or
3) the time-one map of an Anosov flow.

Surface diffeo $\sigma : S \to S$ (genus ≥ 2)



Derivative $D\sigma : TS \to TS$

Normalize $D\sigma : T^1S \to T^1S$

unit tangent bundle

Thm (Bonatti - Gogolev - H - Potrie 2015)

For any surface diffeo $\sigma : S \to S$ there is a robustly transitive, partially hyperbolic diffeo $g : T^1S \to T^1S$ isotopic to $D\sigma$.