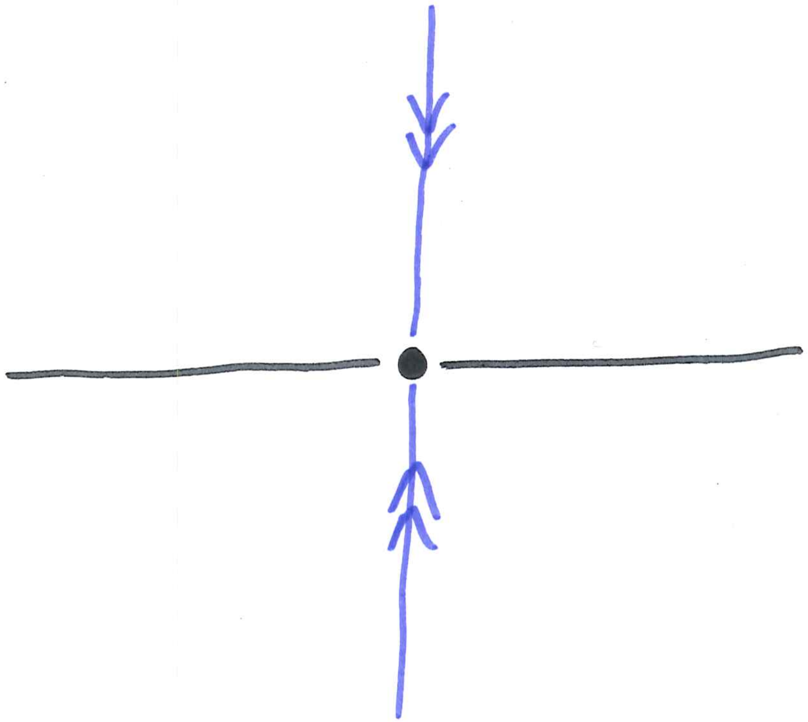
Compact center-stable submanifolds.

Andy Hammerlindl

Consider $f(x,y) = (\frac{1}{2}x, \frac{1}{4}y)$ on $\mathbb{R}^2$.

The origin has a well-defined strong stable manifold.

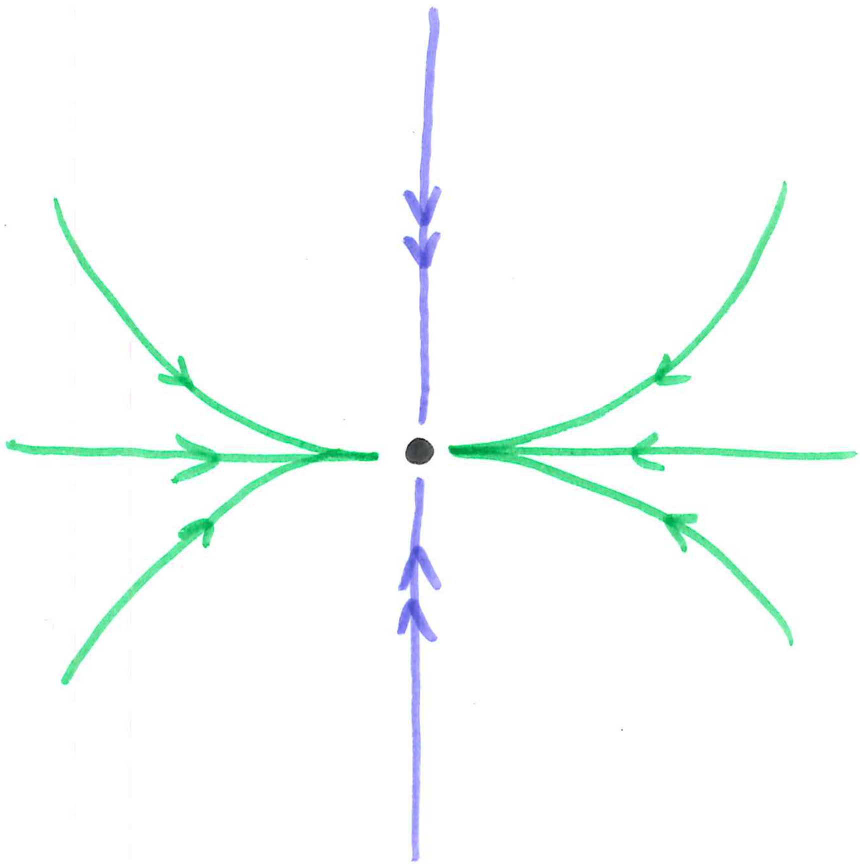
$$W^s(0) = \left\{ p \in \mathbb{R}^2 : \lim \frac{d(f^n(p), 0)}{\lambda^n} = 0 \right\}$$



where $\frac{1}{4} < \lambda < \frac{1}{2}$.

$$f(x,y) = \left(\frac{x}{2}, \frac{y}{4}\right)$$

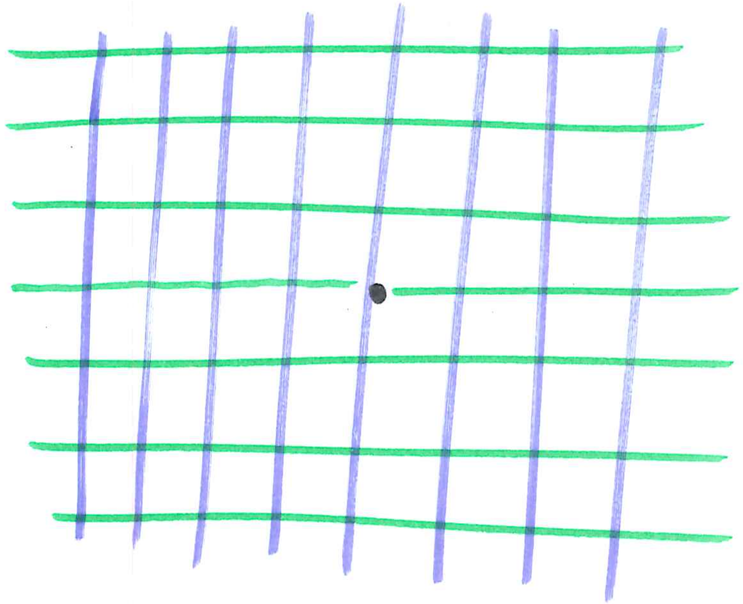
The <u>center</u> or <u>slow</u> <u>manifold</u> is not well-defined.



For any $c \in \mathbb{R}$, consider the curve

$$y = cx^2.$$

We want a well-defined center direction everywhere in the phase space.

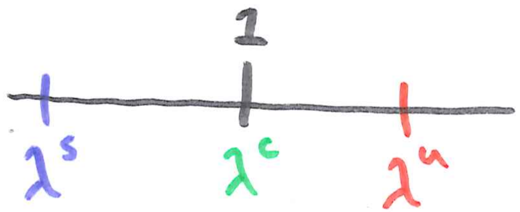


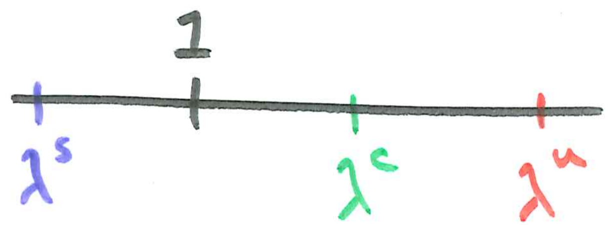
Look at systems $f: M \to M$ where M is compact.

Consider a 3×3 matrix A with integer entries, $\det(A) = 1$, and eigs $\lambda^s < \lambda^c < \lambda^u$

Ex

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{OR} \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$



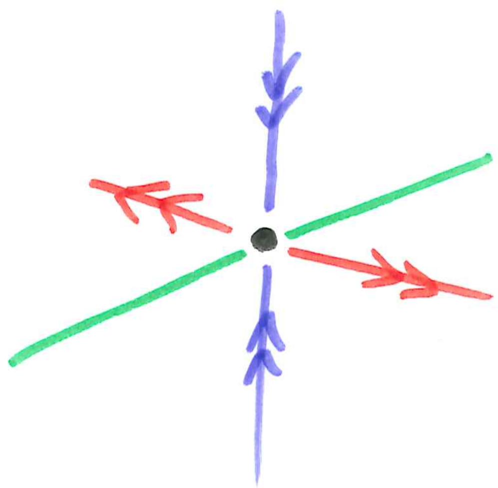


This defines a map $f : \mathbb{R}^3/\mathbb{Z}^3 \longrightarrow \mathbb{R}^3/\mathbb{Z}^3$.

3×3 matrix $\longrightarrow$ $f : \mathbb{R}^3/\mathbb{Z}^3 \circlearrowleft$

Eigenvalues $\lambda^s < \lambda^c < \lambda^u$

Then, the eigenspaces yield well-defined stable, center, and unstable directions at every point.



<u>Def</u> A diffeomorphism $f: M \to M$ is <u>partially hyperbolic</u> if there is a Df-invariant splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u$$

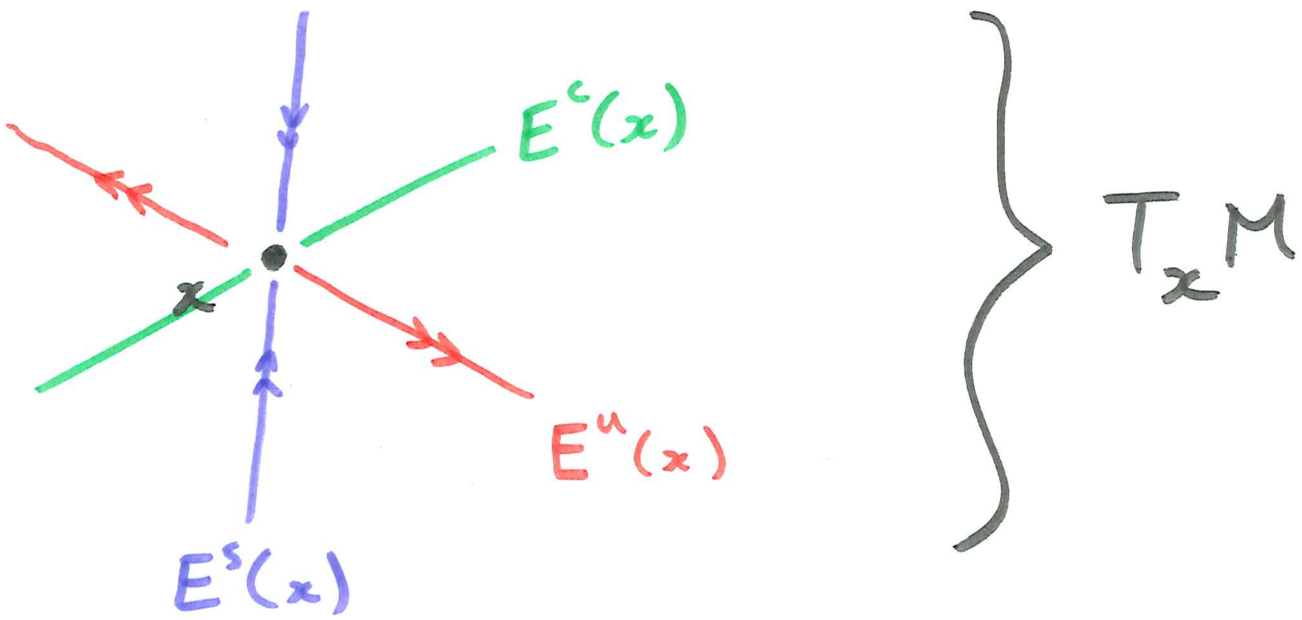
E^s: contracted by Df

E^c: at most mild contraction or expansion

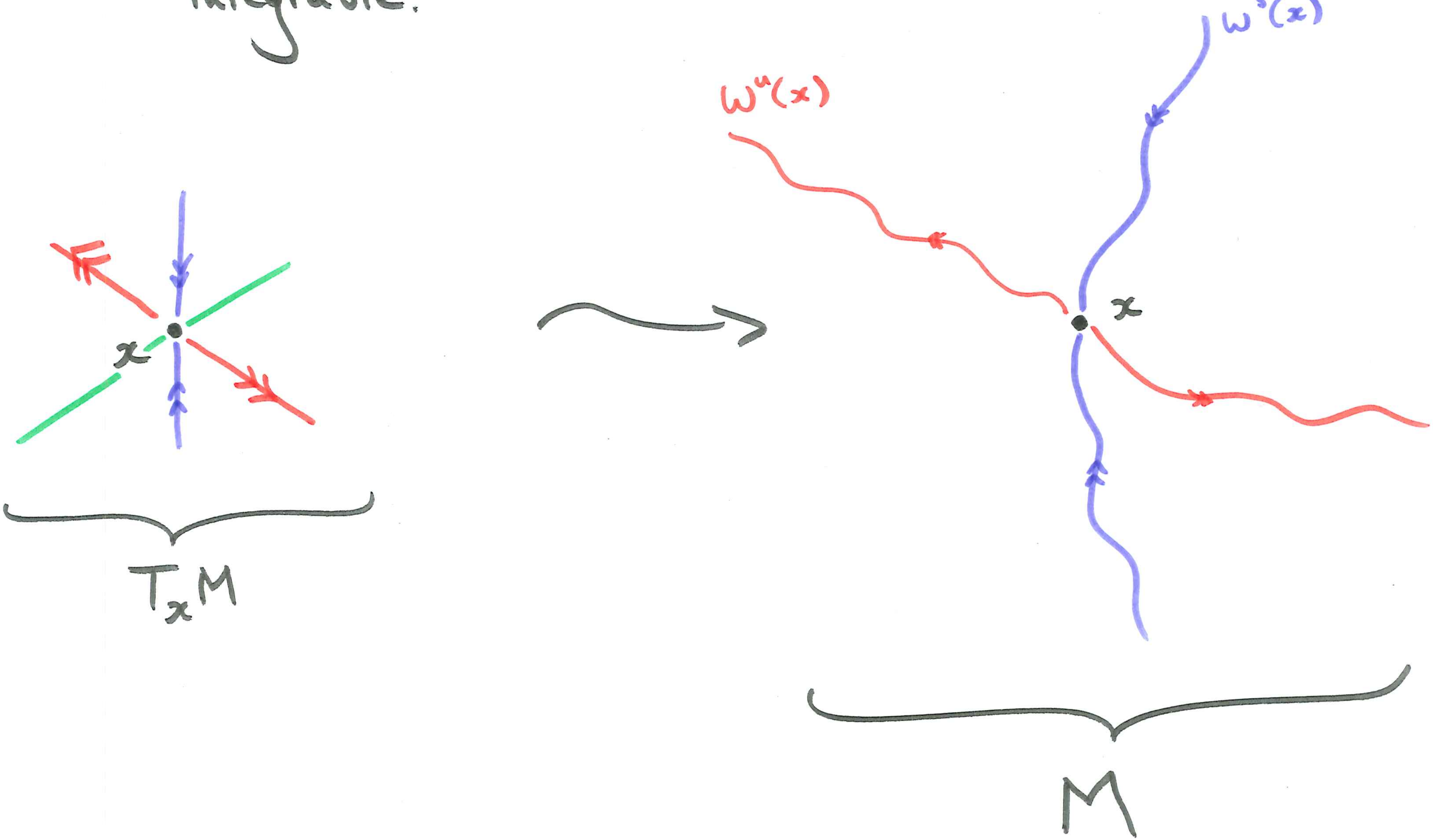
E^u: expanded by Df

$$TM = E^s \oplus E^c \oplus E^u$$

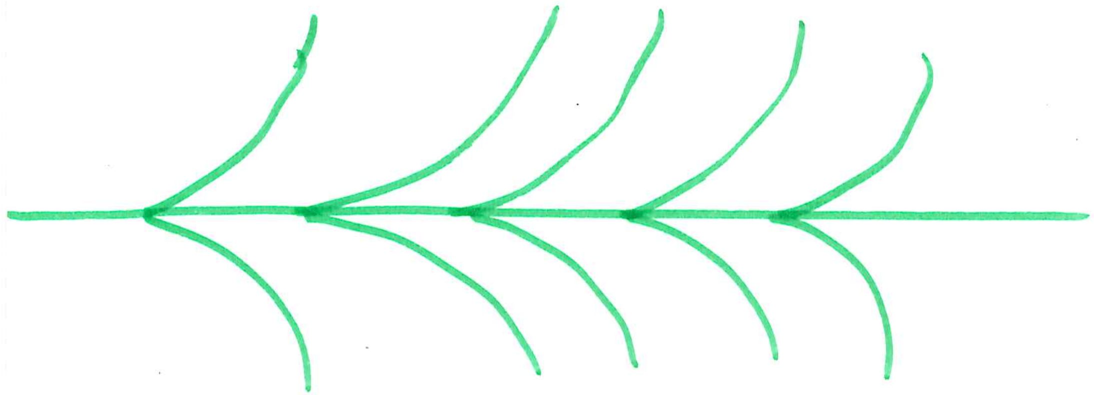
For now, assume $\dim M = 3$ and each of E^s, E^c, E^u is one-dim'l.



The bundles E^s and E^u are uniquely integrable.



E^c is only Hölder continuous, not C^1 in general.



(Think of $\dot{y} = \sqrt[3]{y}$)

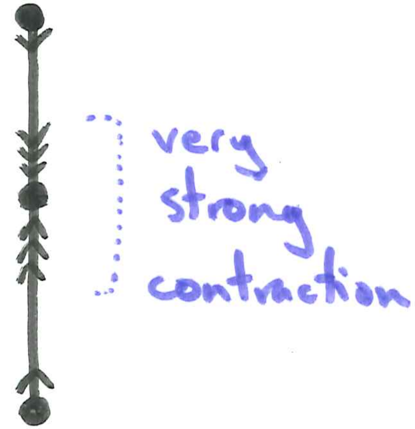
For a long time, it was an open question if $\dim E^c = 1$ implies that E^c is uniquely integrable.

Counterexample by F. Rodriguez-Hertz, J. Rodriguez-Hertz, and R. Ures (2016).

There is a partially hyperbolic diffeomorphism on $\mathbb{T}^3 = \mathbb{R}^3 / \mathbb{Z}^3$ such that E^c is not uniquely integrable.

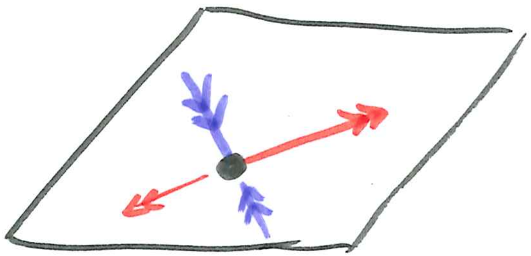
Idea of construction:

$g : S^1 \circlearrowleft$

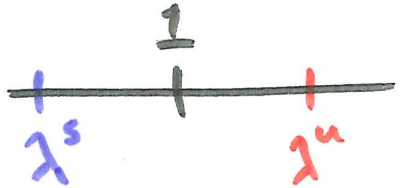


$A : \mathbb{R}^2/\mathbb{Z}^2 \circlearrowleft$

given by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

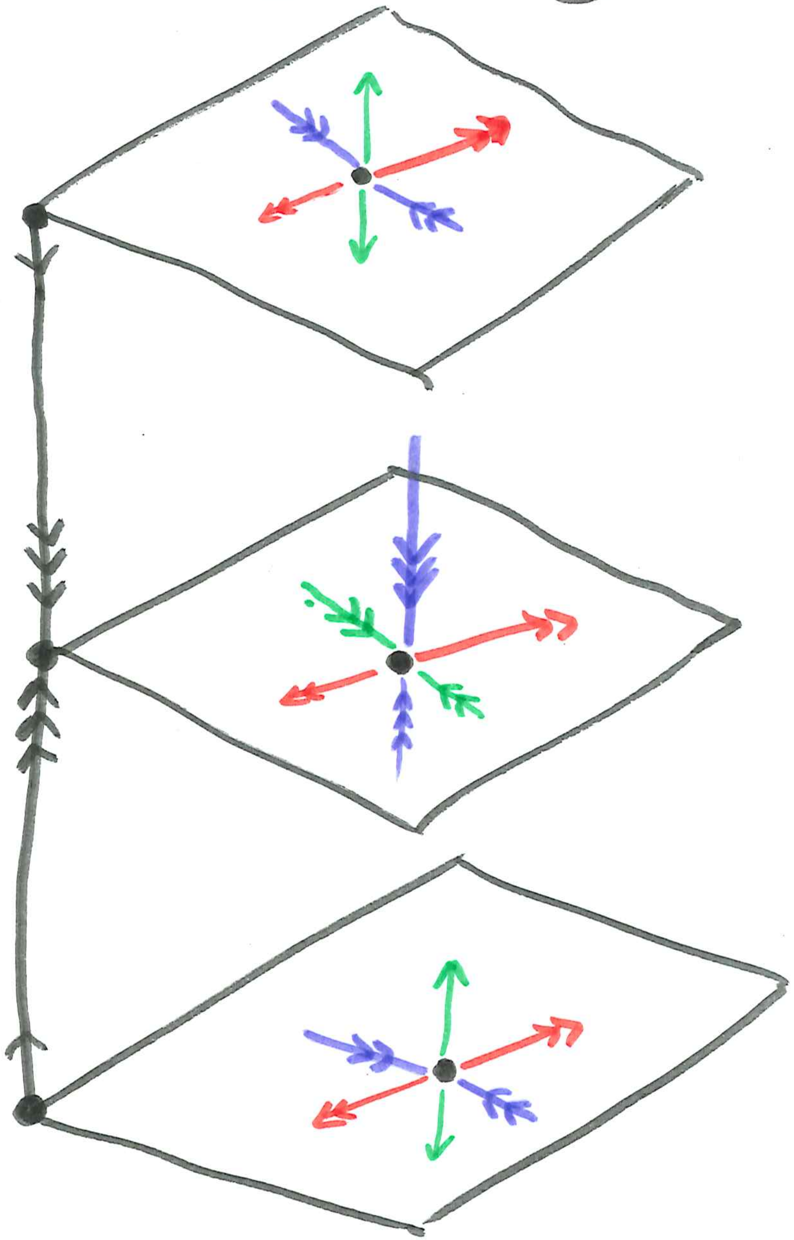


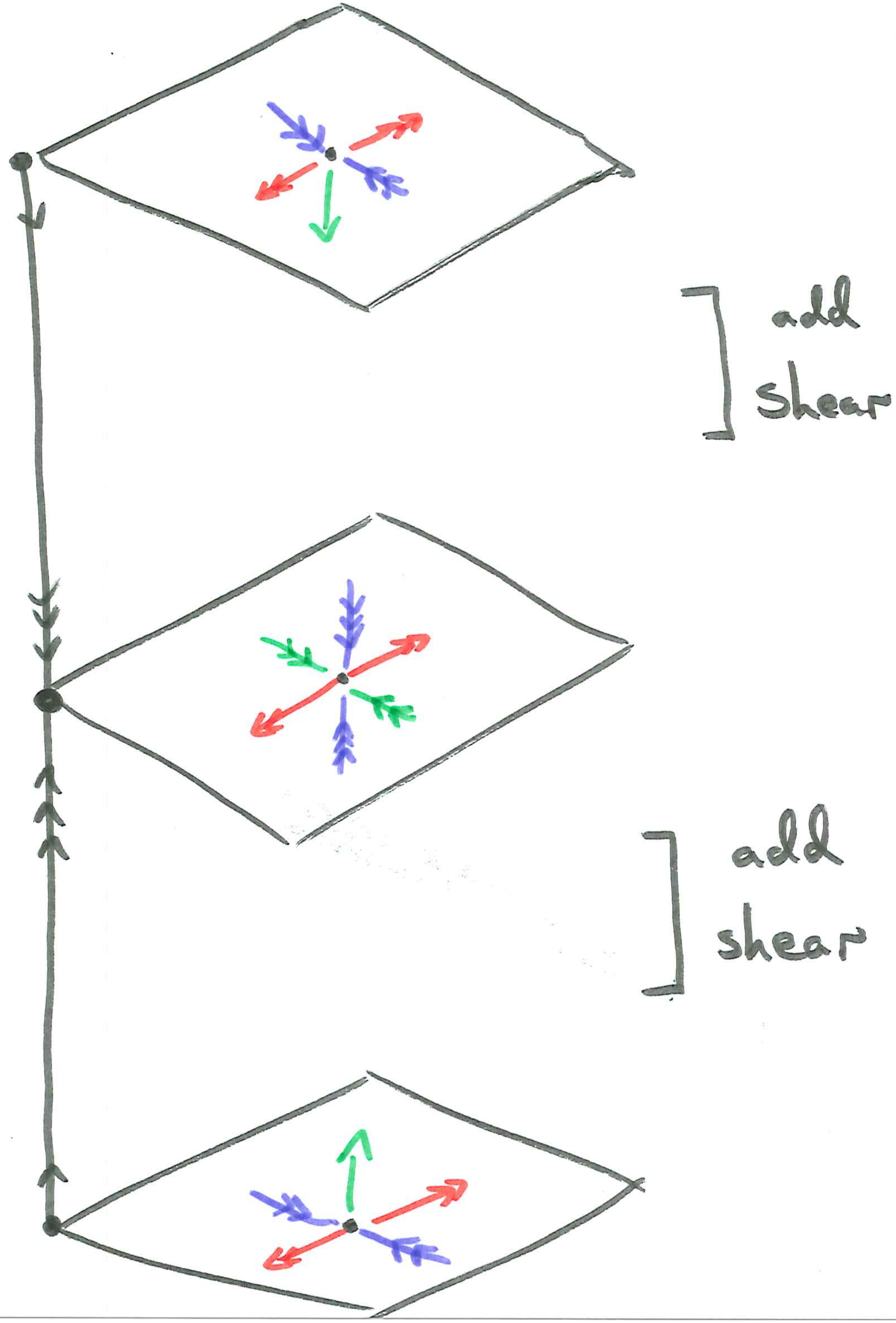
"cat map"



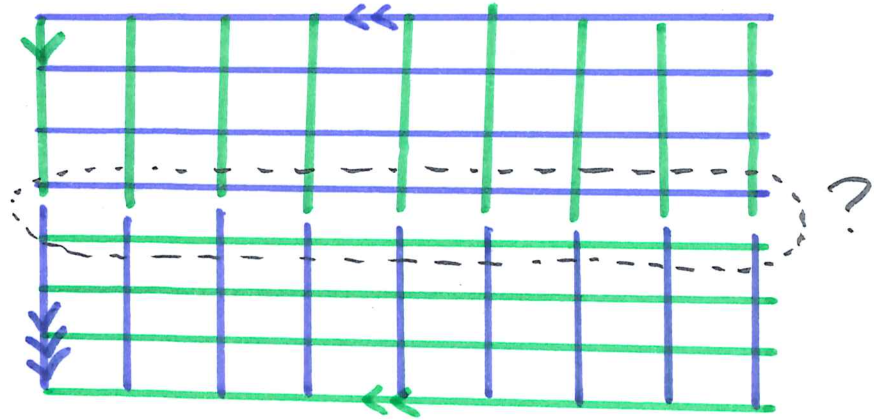
Consider the product

$g \times A : S^1 \times \mathbb{T}^2 \circlearrowleft$

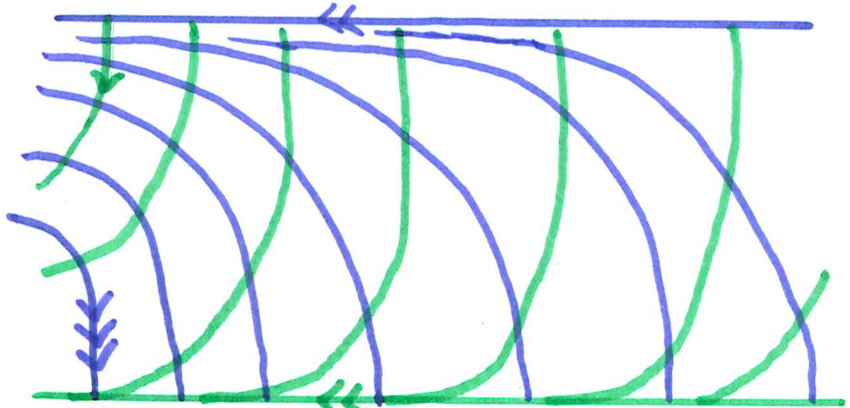




before shear

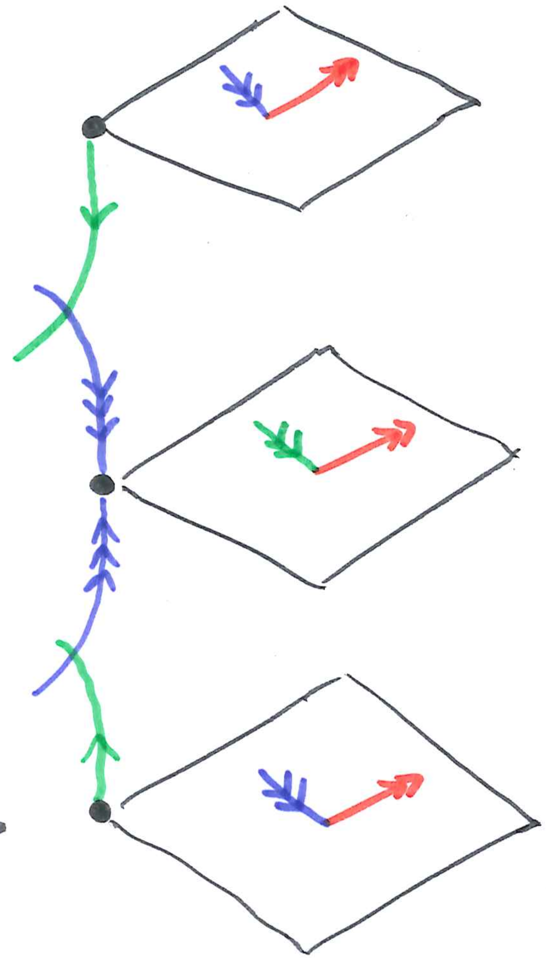


after shear



$f: M \to M$ with globally defined splitting $TM = E^s \oplus E^c \oplus E^u$

invariant torus tangent to $E^s \oplus E^u$



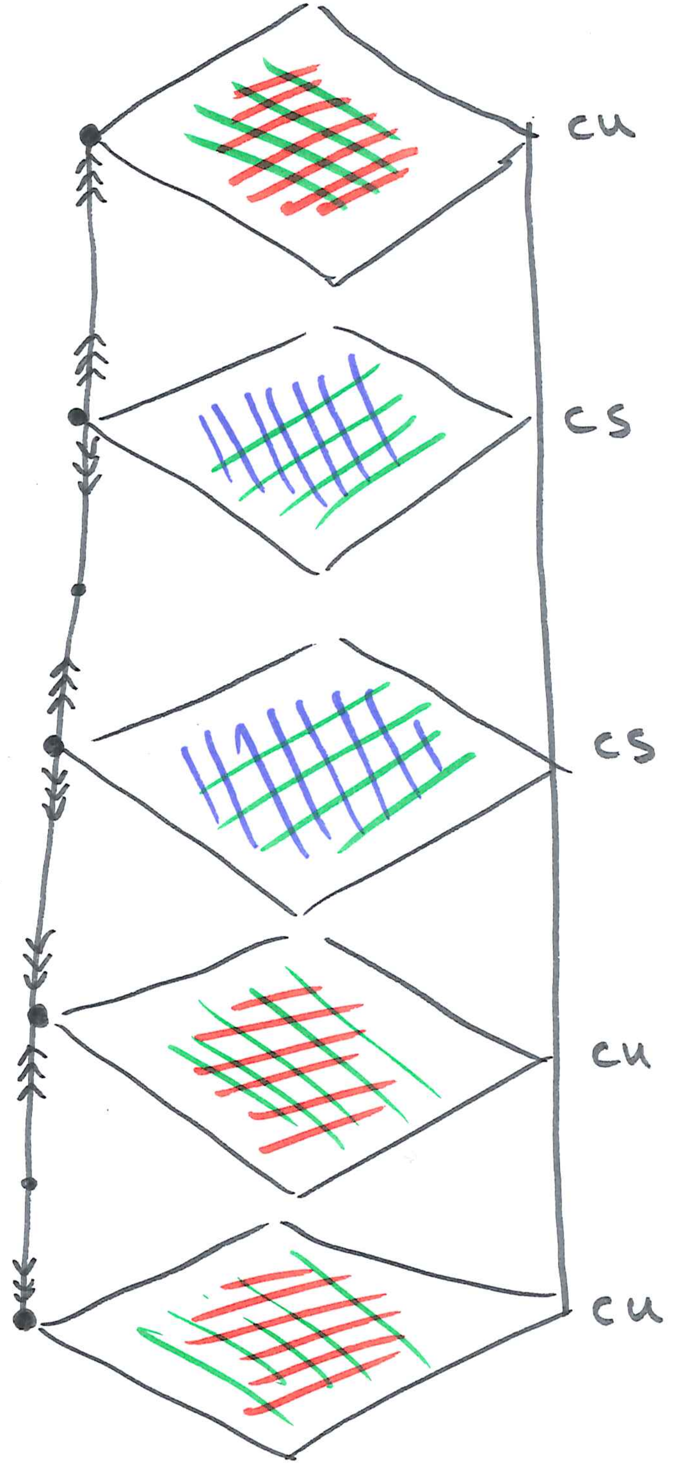
invariant torus tangent to $E^c \oplus E^u$

Question 1 In a 3 dim'l partially hyperbolic system, if E^c is not uniquely integrable, must there be an invariant torus tangent to $E^c \oplus E^u$ or $E^c \oplus E^s$?

Question 2 What kinds of systems have compact submanifolds tangent to $E^c \oplus E^u$ or $E^c \oplus E^s$?

R. Potrie and I showed that if a 3-dim'l partially hyperbolic system has at least one torus tangent to $E^c \oplus E^u$ or $E^c \oplus E^s$,

then it is "basically" a finite number of copies of this example glued together.



In these 3 dim'l systems, all cs and cu-tori are disjoint.

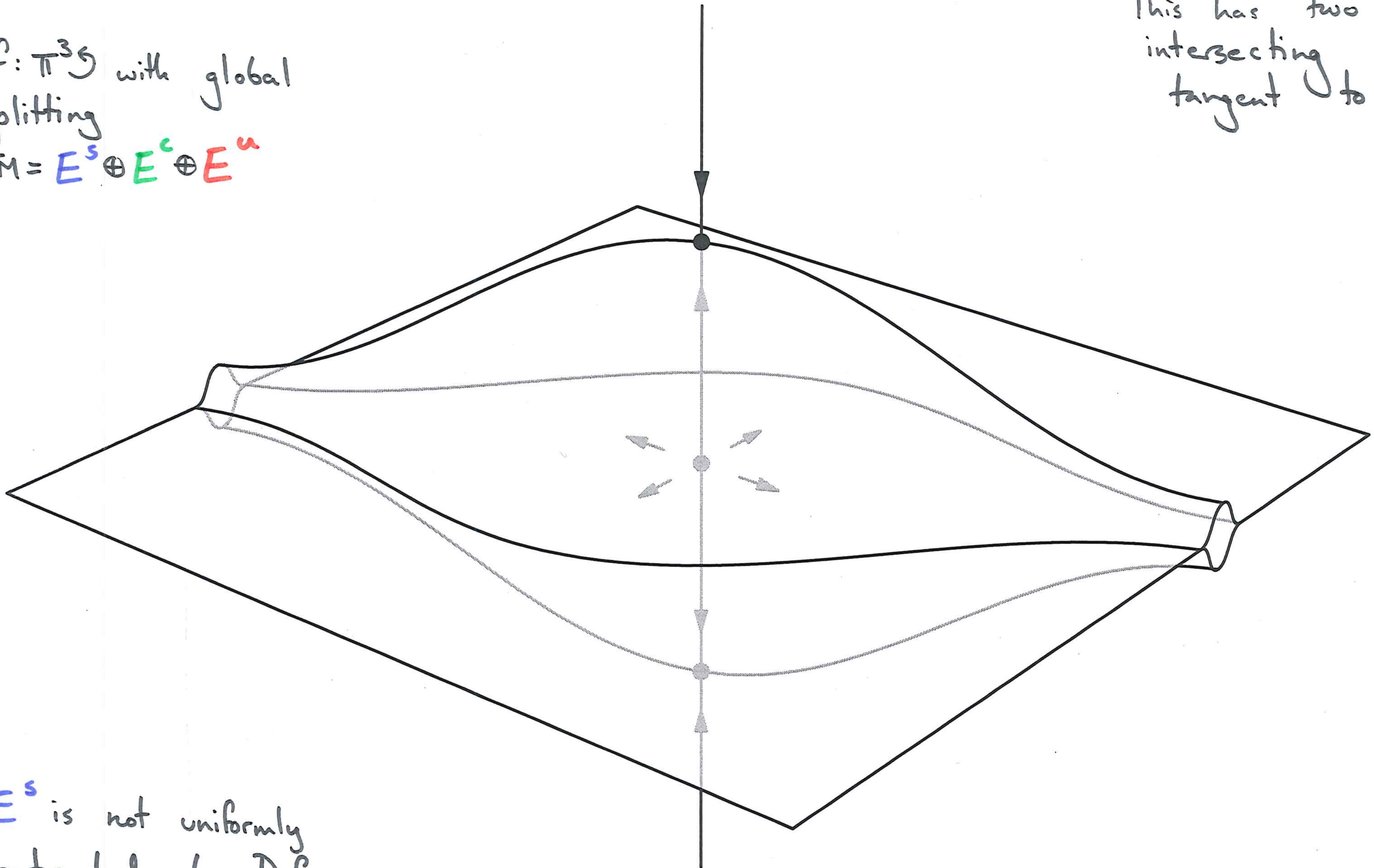
In higher dimensions, there is more flexibility in creating examples.

<u>Question 3</u> In higher dimensions, do compact submanifolds tangent to $E^c \oplus E^u$ have to be disjoint?

Almost countexample in dimension 3.

$f: \mathbb{T}^3 \circlearrowleft$ with global splitting
$TM = E^s \oplus E^c \oplus E^u$

This has two intersecting tori tangent to $E^c \oplus E^u$



E^s is not uniformly contracted by Df