Models of chaos in dimensions 2 and 3.

A. Hammerlindl

Joint w/ C. Bonatti, A. Gogolev, R. Potrie

Consider a flow $\{\varphi^t\}_{t\in\mathbb{R}}$.

What is the long term behaviour?

Define the ω-limit set

$$\omega(x) = \lim_{T\to\infty} \overline{\{\varphi^t(x) : t > T\}}.$$

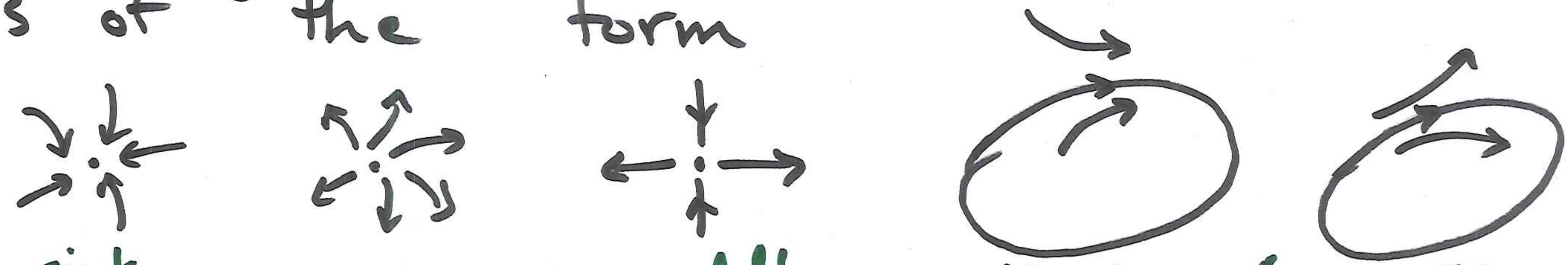
What are the possible ω-limit sets?

After perturbation, what are the possible ω-limit sets?

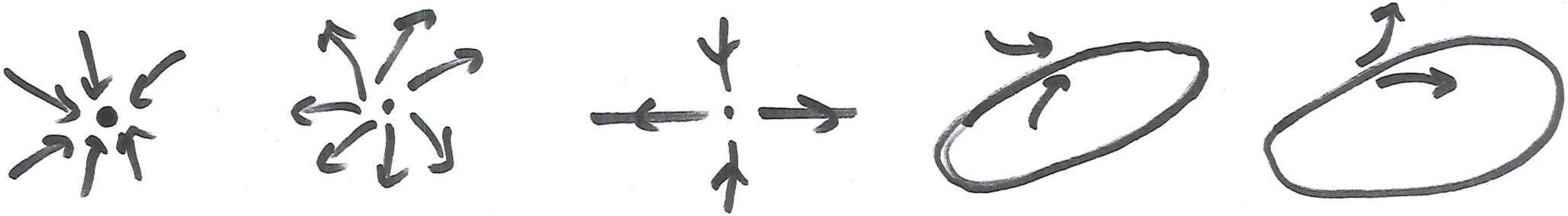
Thm [Peixoto 1962]

Let φ^t be a flow on a compact surface.

After a C^r-small perturbation, every omega-limit set $\omega(x)$ is of the form



sink source saddle attracting / repelling periodic orbit

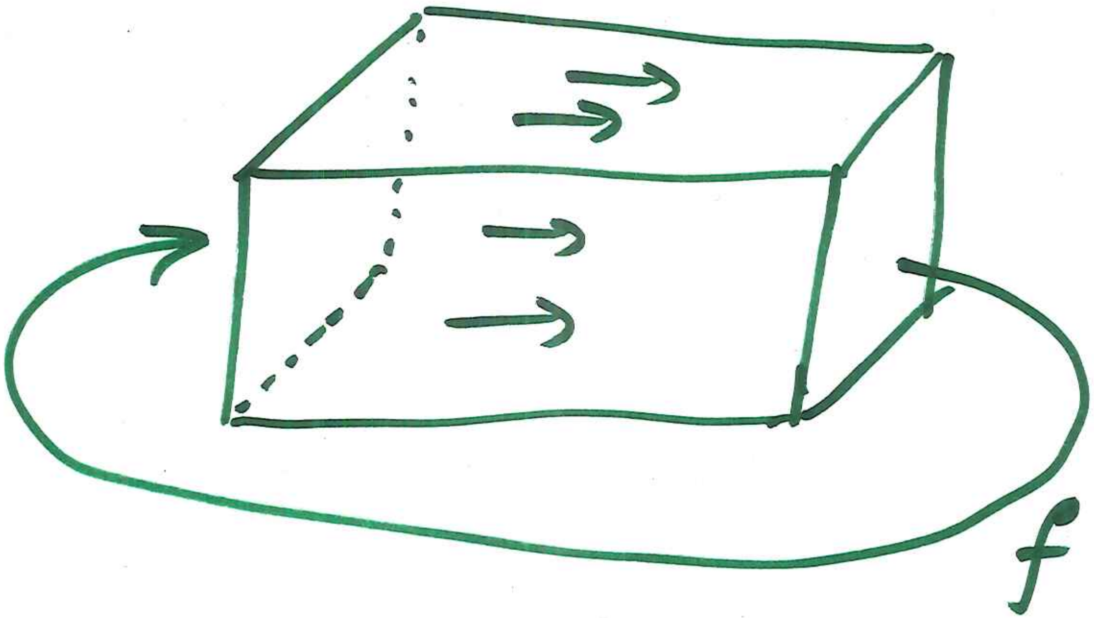


Note: all of these are uniformly hyperbolic.

What about flow in dimension 3?

First consider diffeomorphisms of surfaces.

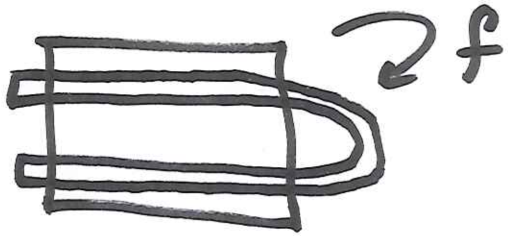
$f: M \to M$



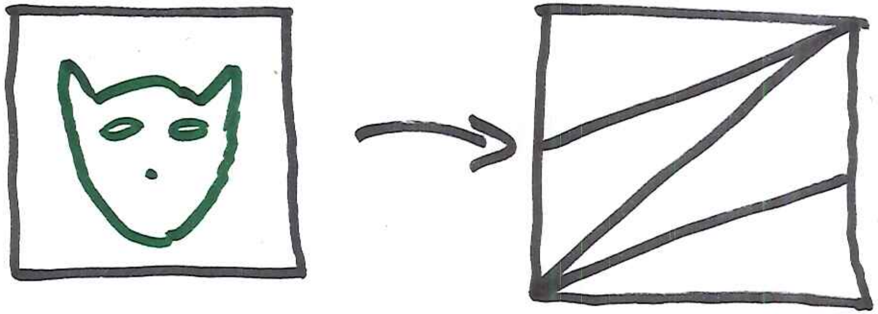
Hyperbolic sets in dim 2 for diffeos

periodic points

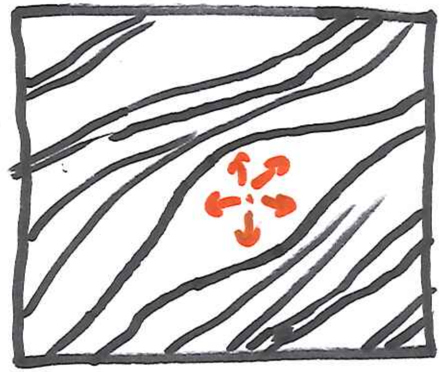
attracting/repelling periodic circles



horseshoes



Anosov maps

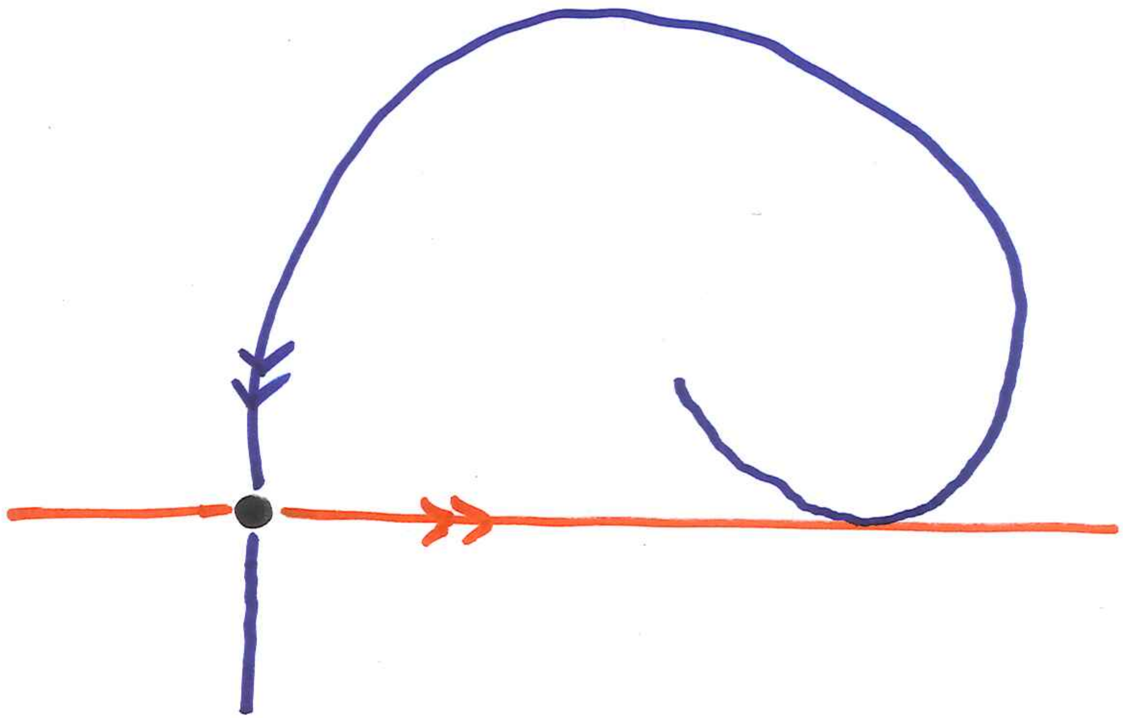


Derived-from-Anosov maps

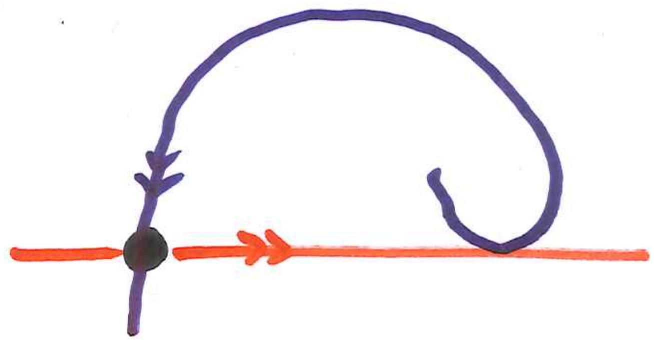
For diffeos in dim 2

After perturbation, is every $\omega(x)$ hyperbolic?

A possible obstruction: a homoclinic tangency



Newhouse 1970: $\exists$ a surface diffeo f s.t. every g C^2-close to f has a tangency.



Open question for surface diffeos in C^1.

Say a system is "far from tangencies" if $\exists$ a C^1-nbhd with no tangencies.

<u>Thm</u> [Pujals - Sambarino 2000]

"Far from tangencies" a surface diffeo may be C^1-perturbed so that for every x, $\omega(x)$ is hyperbolic.

Flows in dim 3.

Thm [S. Crovisier - D. Yang 2015]

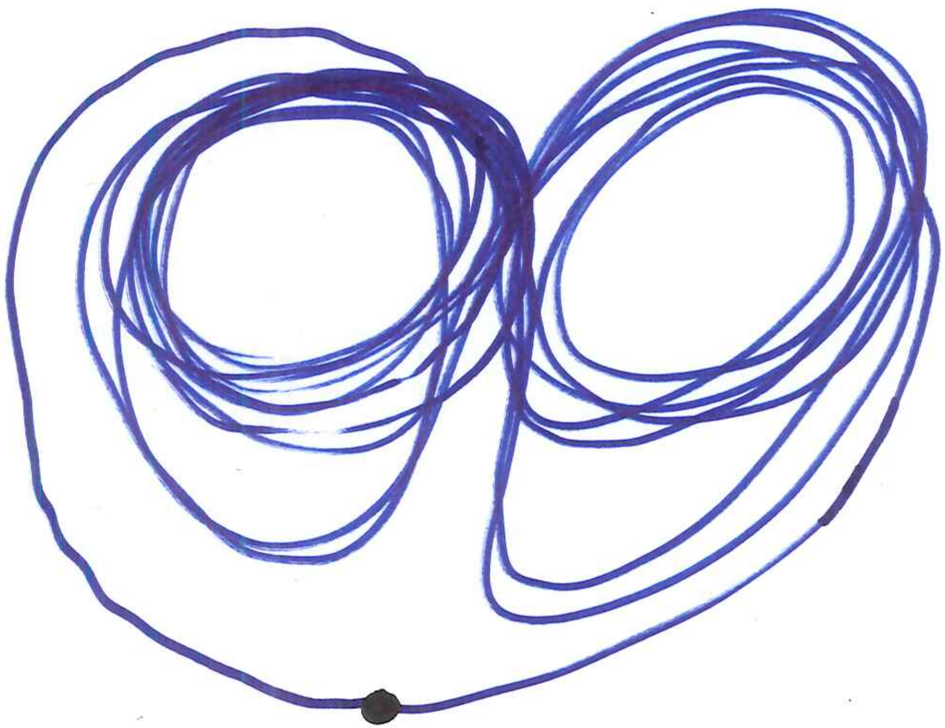
"Far from tangencies" a flow φ^t in dim 3 may be C^1-perturbed so that for every $x \in M$, $\omega(x)$ is either

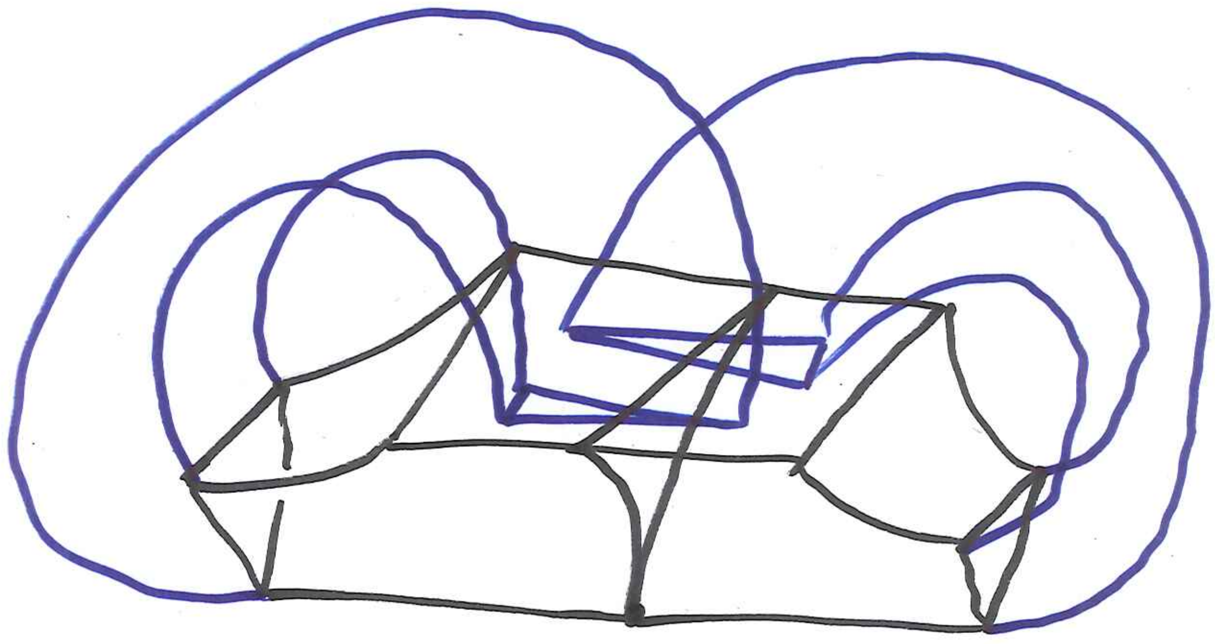
hyperbolic or

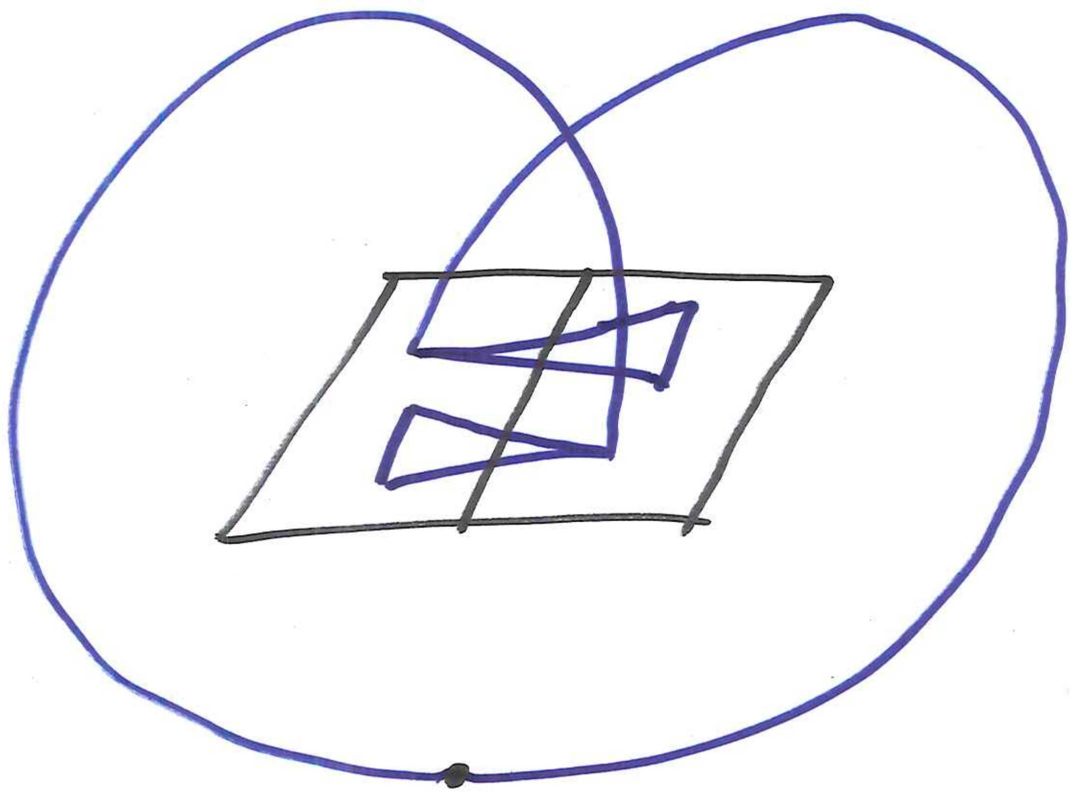
singular hyperbolic.

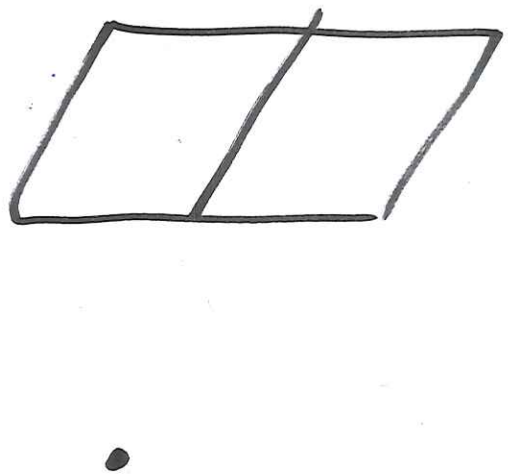
For flows in dim 3, hyperbolic sets are well-understood.

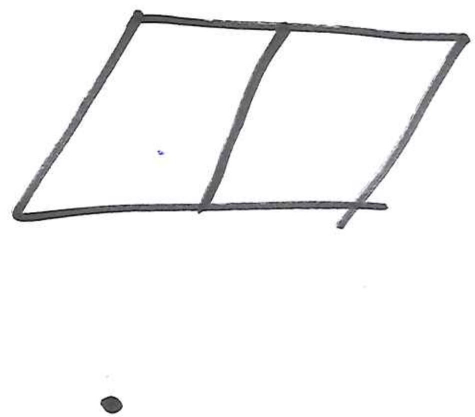
Still some questions about transitive Anosov flows...

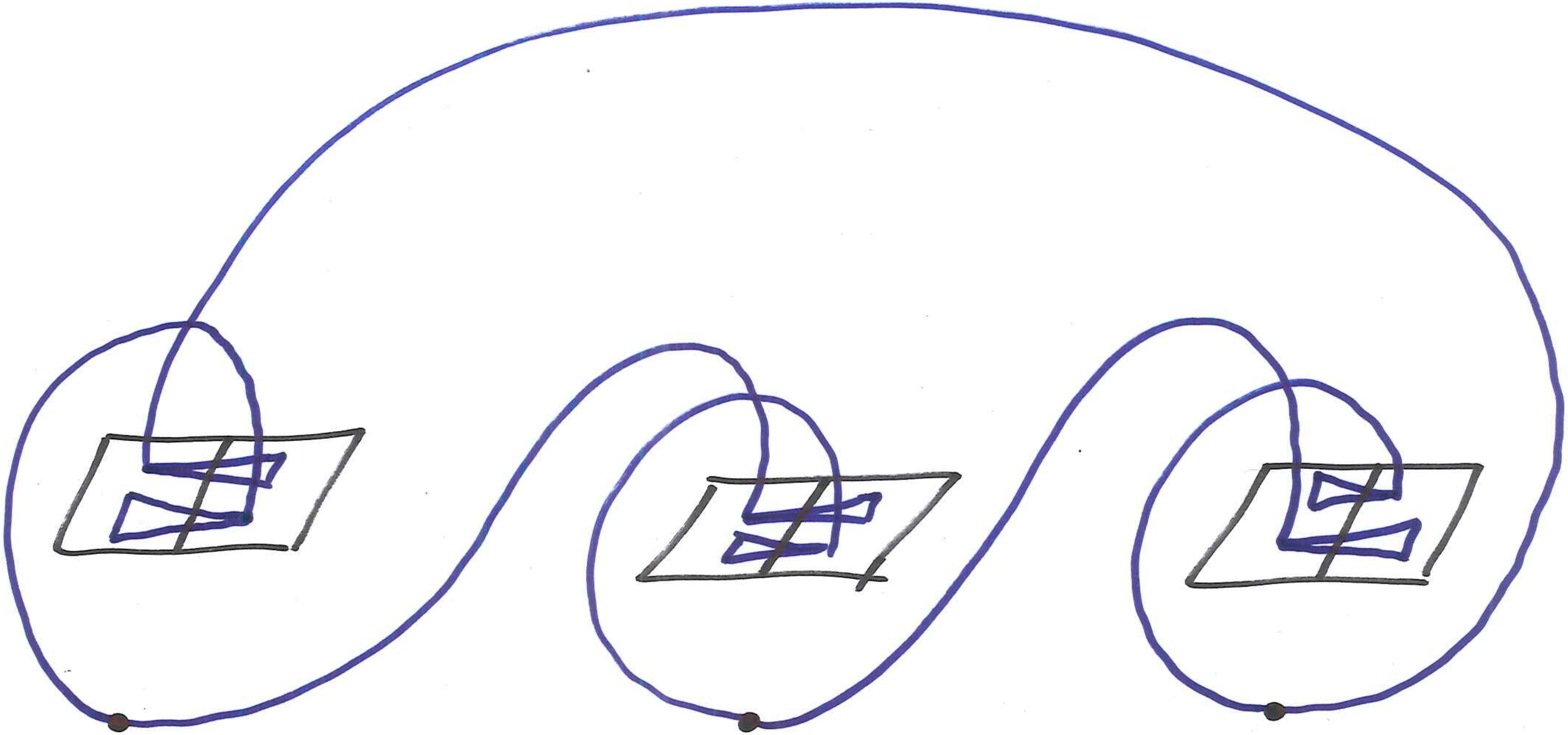








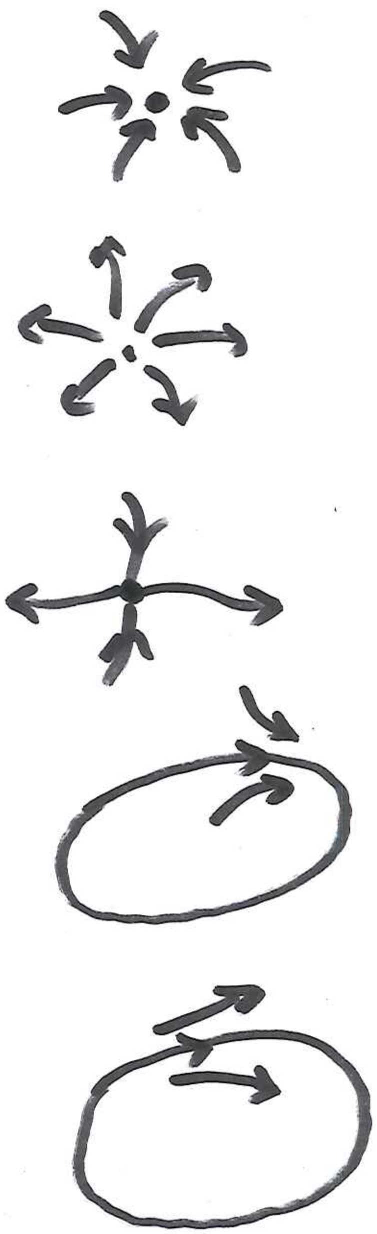




Thm [Araújo - Galatolo - Pacifico 2012]

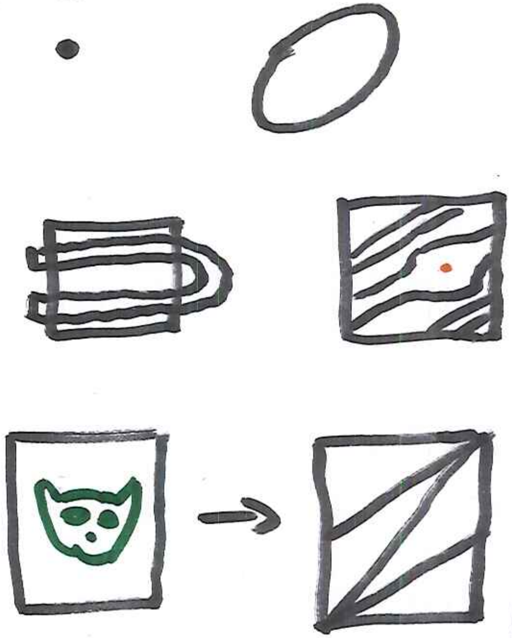
A singular hyperbolic set is a finite number of Lorenz attractors glued together.

Flows in dim 2

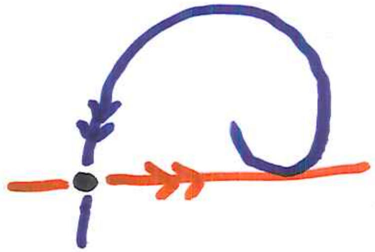


Diffeos in dim 2

hyperbolic sets:



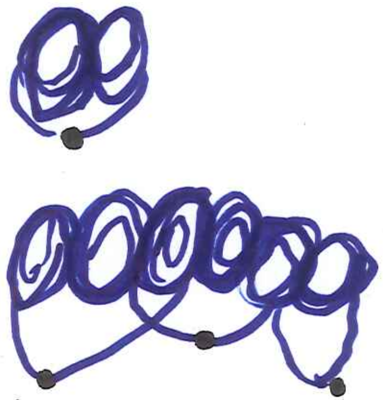
tangencies?



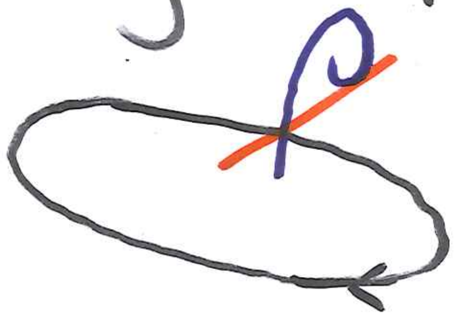
Flows in dim 3

hyperbolic sets

singular hyperbolic sets:



tangencies?



Diffeos in dim 3

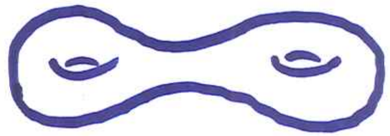
TANGENCIES!

OLD EXAMPLES

NEW EXAMPLES

WHAT ELSE?

Surface diffeo $\sigma : S \to S$ (genus ≥ 2)



Derivative $D\sigma : TS \to TS$

Normalize $D\sigma : T^1S \to T^1S$ $\leftarrow$ unit tangent bundle

<u>Thm</u> [Bonatti - Gogolev - H - Potrie]

For any surface diffeo $\sigma : S \to S$ there is a C^1-robustly transitive ($\exists x$ s.t. $\omega(x) = T^1S$) diffeo $g : T^1S \to T^1S$ isotopic to $D\sigma$.

Thank

You

In dim 2

$$C^1\text{-RT} \overset{\text{Mañé}}{\Longleftrightarrow} \text{Hyperbolic}$$

In dim 3

$$C^1\text{-RT} \Longrightarrow \text{weak Part. Hyp.}$$

$$\Uparrow$$

strong Part. Hyp. + Blenders

hyperbolic: $\Lambda \circlearrowleft f$ $\quad T_\Lambda M = \underset{\circlearrowleft Df}{E^u} \oplus \underset{\circlearrowleft Df}{E^s}$

$$\|Df\, v^u\| < 1 < \|Df\, v^s\|$$

for unit vectors $v^* \in E^*$

singular hyperbolic: $\Lambda \circlearrowleft \varphi_t$ $\quad T_\Lambda M = E^{ss} \oplus E^{cu}$

$$\|D\varphi_t|_{E^{ss}}\| < e^{-\lambda t}, \quad \|\mathrm{Det}\, \varphi_{-t}|_{E^{cu}}\| < e^{-\lambda t}$$

partially hyperbolic: $\Lambda \circlearrowleft f$ $\quad T_\Lambda M = E^u \oplus E^c \oplus E^s$

$$1 > \|Df\, v^s\| < \|Df\, v^c\| < \|Df\, v^u\| > 1$$

(weak p.h. if one of $E^u = 0$ or $E^s = 0$)