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Symmetry of codes in graphs

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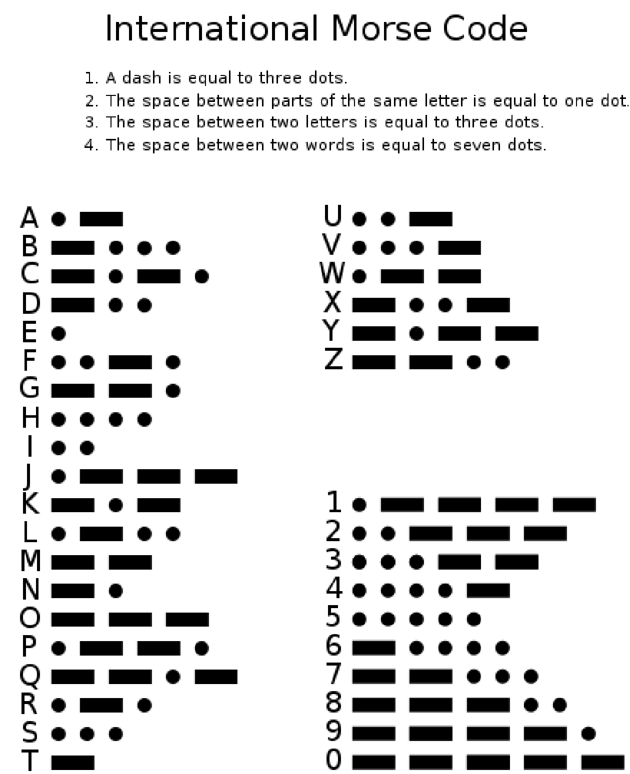
Communicating Information

Electronically brings   danger of introducing errors



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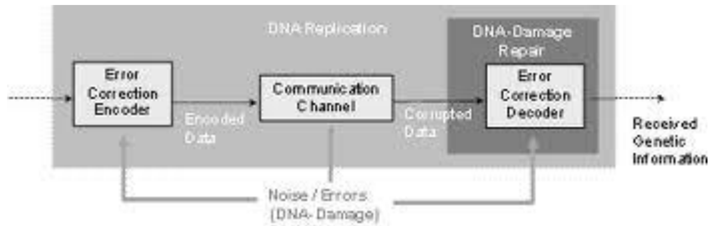
Standard representation

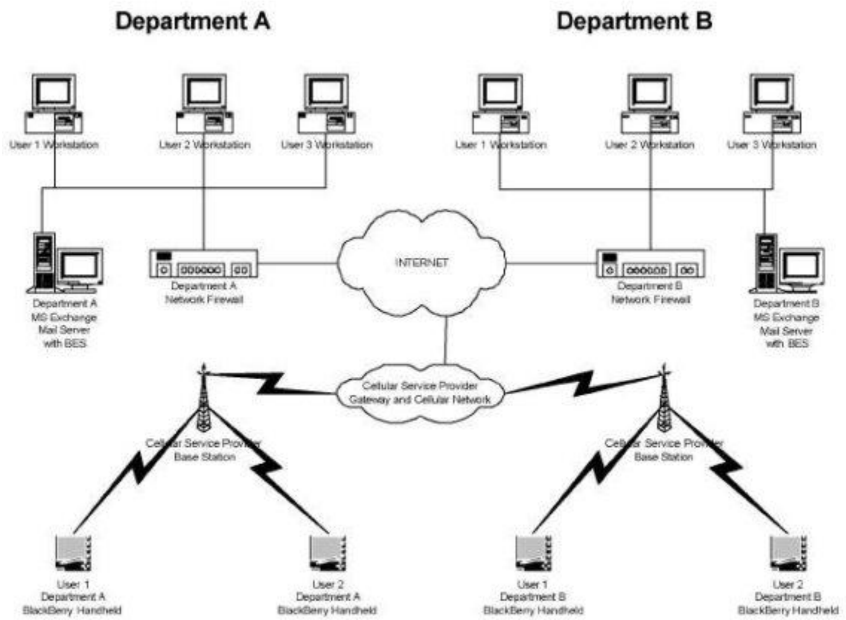


Block codes:

Codewords are strings

Errors are incorrect entries

Distance(sent, received)
= number of errors



Codes in Graphs

1973 Delsarte

•Interpret vertex subsets C of any graph X as codes

•vertices in C are codewords

•Introducing a “single error” into a codeword v gives vertex u at distance 1 from v in X

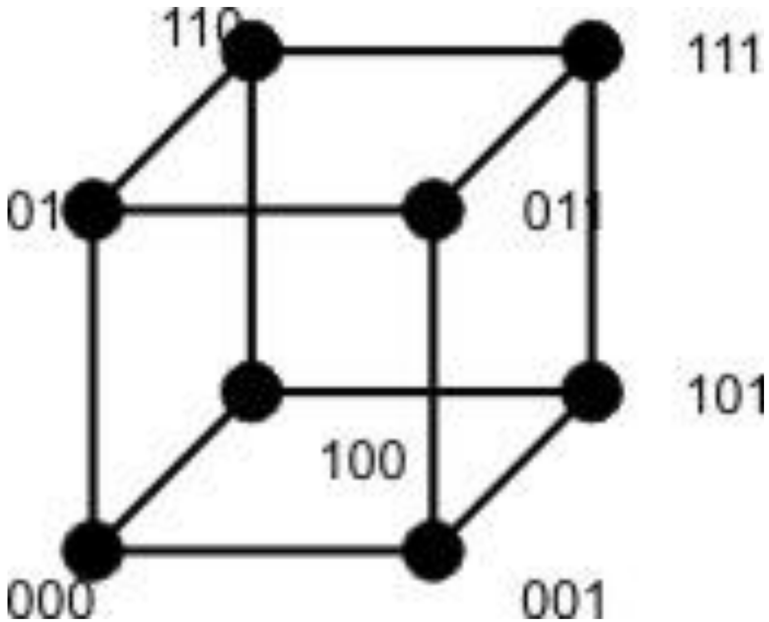
•if u not in C then call u a neighbour of code C

Classical setup:
X = H(m,q) a Hamming graph

In H(3,2) take

C={ 000, 111 } so δ = 3

- VX = m-tuples from alphabet of size q
- { x, y } edge in X if x, y differ in one entry
- Distance d(x,y) = number of different entries
- Minimum distance δ for C is least d(x,y) for x,y in C



Delsarte suggested: take X a completely regular graph

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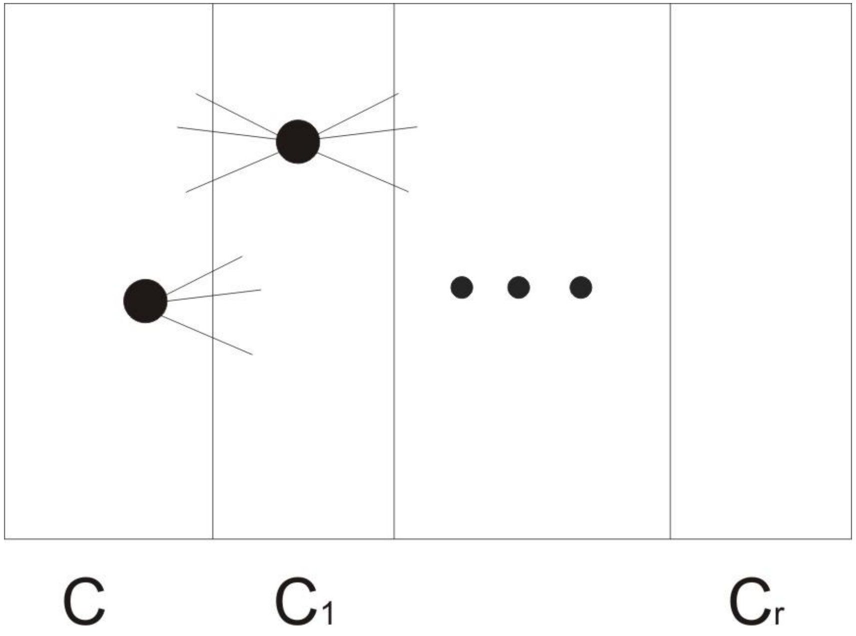
Distance partition of C



C_1 = neighbour set of C
r = covering radius

Introduced completely regular codes

- Distance partition is equitable
- For v in C_i numbers of edges from v to vertices in C_j depend only on i and j – independent of v

Example of completely regular code

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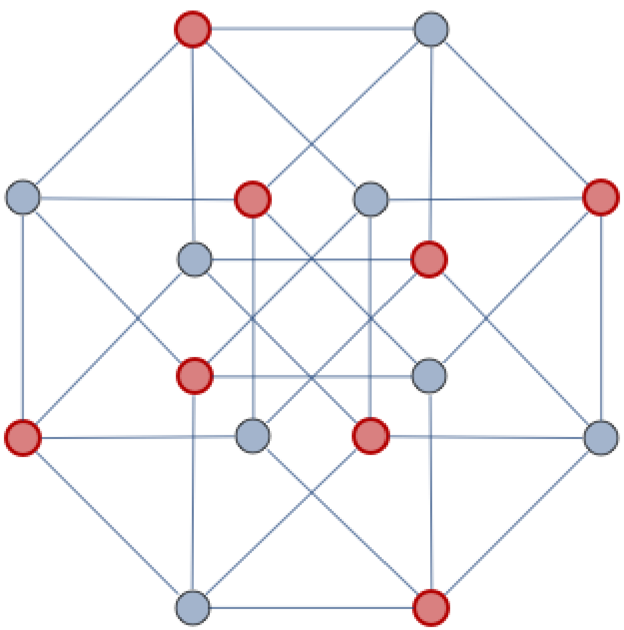
C in H (4 , 2)



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Minimum distance $\delta = 2$
covering radius = 1

Completely regular codes

- Delsarte: Generalising perfect codes
- Disappointingly not many CR codes known with large minimum distance δ
- Led to Conjectures for CR codes in H(m,q)

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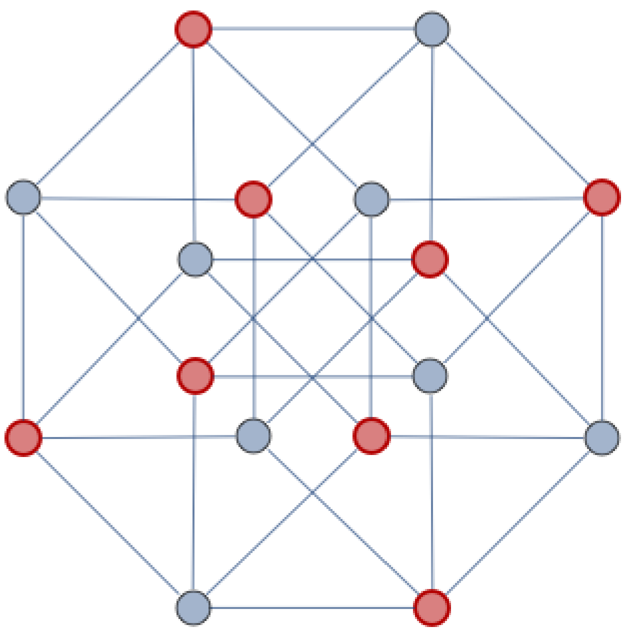
Conjectures for CR codes in H(m,q)

C in H (4 , 2)



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

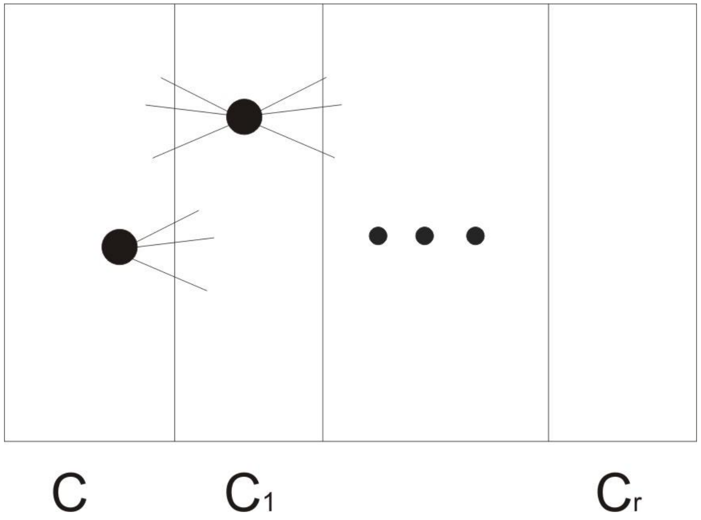
Minimal distance $\delta = 2$
covering radius = 2

Conjectures

- Neumaier 1992 only CR code in an H(m,q) with $\delta = 8$ is the binary Golay code
- Borges, Rifa, Zinoviev 2001 every CR code in an H(m,q) has δ at most 8

Two directions for further study using symmetry

Automorphism group Aut(C):
Setwise stabilser of C in Aut(X)



For all codes C:

- Aut(C) leaves each C_i invariant

C is completely transitive:

- Aut(C) is transitive on C_i for each i

Warning: Some use **more restrictive definition of Aut(C) !!**

Work on completely transitive codes in graphs

In H(m,q)

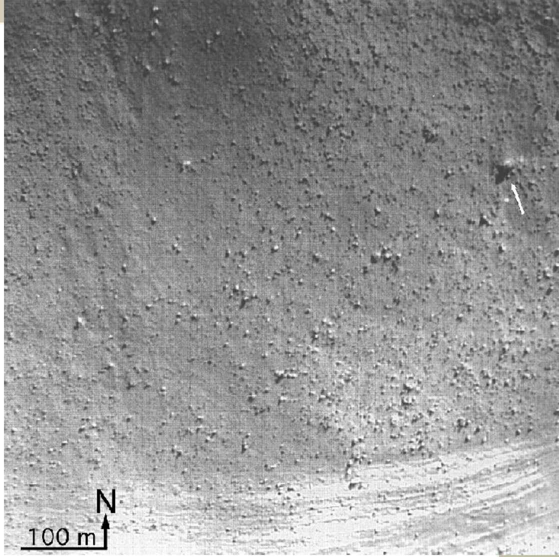
- Patrick Sole
- Michael Giudici and CEP
- Rifa and Zinoviev: with restrictive Aut(C) show δ at most 8
- Neil Gillespie PhD 2012

In Johnson graphs J(v,k)

- Bill Martin
- Chris Godsil and CEP

An example:

NASA *space* probe Mariner 9 in 1971 used the Hadamard code n=32 to transmit *photos* of Mars back to Earth



Hadamard codes

- Take Hadamard matrix H
- “Double and negate”
- Change -1 to 0
- Code(H) in H(n,2)
- Automorphism (P, Q) with H=PHQ with P, Q monomial
- Aut(H) = Aut(Code(H))
- size 2n , $\delta = n/2$

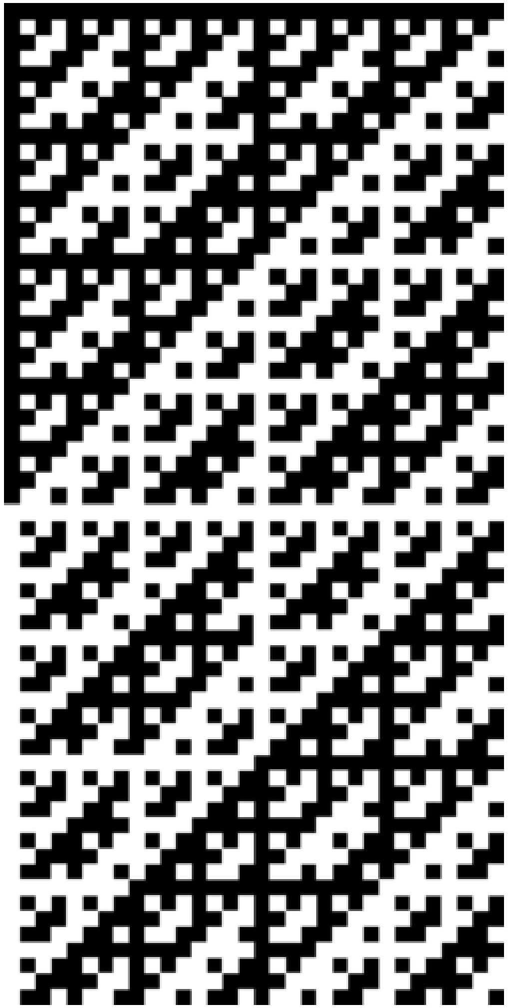
Tiny Example:

$$\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \Rightarrow \begin{matrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{matrix} \Rightarrow \begin{matrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix}$$

A completely transitive Hadamard code

Neil Gillespie and CEP

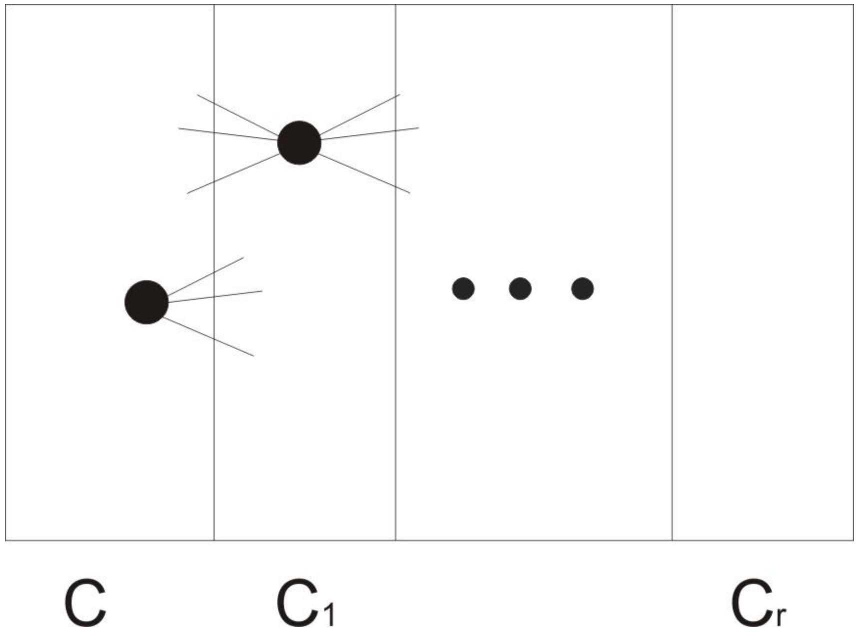
- Unique12 x 12 matrix H
- 1962 M Hall Aut(H)=2.M_{12}
- Code(H) is completely transitive!
- δ = 6
- Covering radius = 3



Second direction: neighbour-transitive codes

Aut(C) transitive on C & C_1

- Gillespie: C in H(m,q)
- Liebler & CEP: C in J(v,k)



We don't care about the "far-away" vertices

Neil Gillespie's work

Constructions & Classifications

- Remarkable new family of codes C(T)
- Building blocks for large class of neighbour-transitive codes

Neil's C(T) codes

Choose favourite permutation group T

- Each x in T becomes a codeword:
- E.g. If $T=S_3$ then (123) sometimes written as

  1 2 3
  2 3 1

- Take associated codeword as (2 3 1) = $(1^x\ 2^x\ 3^x)$

Gillespie & CEP

For $T = S_3$ on { 1, 2, 3 }

- C(T) in H(3,3)
- |T| = 6 codewords
- Length 3, Alphabet {1,2,3}
- Distance between

$(1^x\ 2^x\ 3^x)$ and $(1^y\ 2^y\ 3^y)$

Is number of points moved by xy^{-1} so $\delta = 2$ for $C(S_3)$

In Neil's classification T is simple "socle" of 2-transitive group

- δ = minimal degree(T)
- Aut (C(T)) contains T x T and is neighbour-transitive
- Proof uses 2-transitivity

Gillespie & CEP

T=PSL(2,29) on PG(1,29)

- C(T) in H(30,30)
- Size |T| ≈ 13K
- Length 30 = |PG(1,29)|
- Alphabet PG(1,29)
- δ = 28 = minimal degree(T)
- So corrects 13 errors!

N. Tr. codes in Johnson graphs

Johnson graph J(v,k) **Based on a v-set *V***

Johnson graph $J(v, k)$ based on v-set $\mathcal{V}$

- vertex set $\binom{\mathcal{V}}{k}$
- arc set J: all vertex pairs (α, α_1) with $|\alpha \cap \alpha_1| = k - 1$
- distance in $J(v, k)$: $d_J(\alpha, \beta) = i \Leftrightarrow |\alpha \cap \beta| = k - i$

View code C as subset of vertices of $J(v, k)$

Neighbour set $C_1 = \{\gamma_1 \notin C \mid d_J(\gamma, \gamma_1) = 1 \ \exists\, \gamma \in C\}$
Minimum distance $\delta(C) = $ minimum of $d_J(\gamma, \gamma') = k - |\gamma \cap \gamma'|$
where $\gamma, \gamma' \in C, \gamma \neq \gamma'$

$\mathrm{Aut}(C) < \mathrm{Aut}(J(v,k)) = \mathrm{Sym}(V)=S_v$

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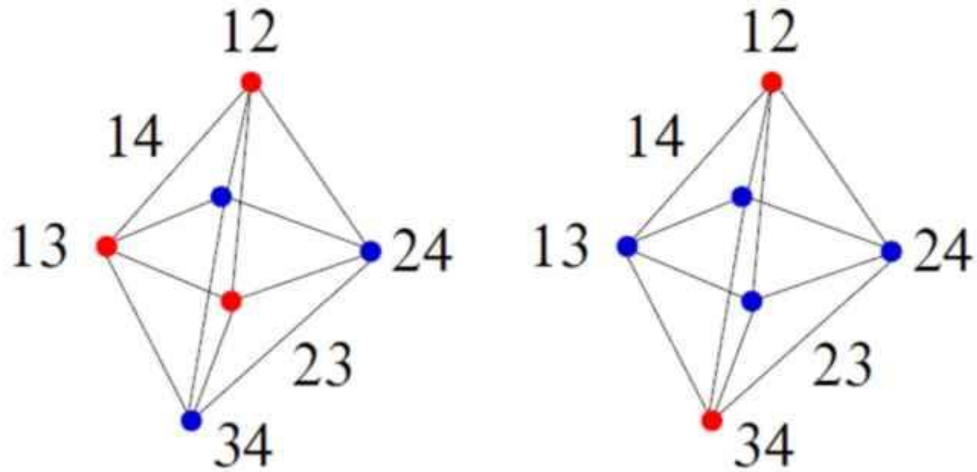


C in J(4,2)

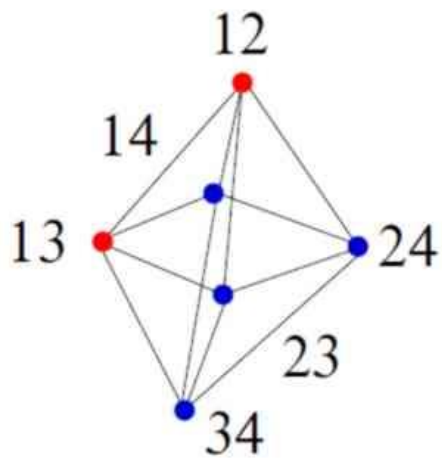
Example 1. $C = \{12, 13, 23\}$, $C_1 = \{14, 24, 34\}$,

$\mathrm{Aut}(C) = S_3 = \langle (12), (123) \rangle$

Example 2. $C = \{12, 34\}$, $C_1 = \{13, 14, 23, 24\}$,
$\mathrm{Aut}(C) = D_8 = \langle (12), (1324) \rangle$

Aut(C) < Aut(J(v,k)) = Sym(V)=S_v

C in J(4,2)



Non-Example 3. $C = \{12, 13\}$, $C_1 = \{14, 22, 24, 34\}$,

$\mathrm{Aut}(C) = \langle (23) \rangle$ transitive on C (code-transitive) but not on C_1

Non-Example 4. $C = \{14, 22, 24, 34\}$, $C_1 = \{12, 13\}$,
$\mathrm{Aut}(C)$ transitive on C_1 but not on C.

Some Questions:

- Are there many examples?
- Are most completely transitive?
- How large can $\delta(C)$ be?
- $\exists$ examples involving interesting ‘geometry’?
- Is any kind of classification feasible?

Comment on “design” interpretation

- Since vertices in J(v,k) are k-sets
- Natural to interpret codes C in J(v,k) as designs
- Nice examples arise from nice designs!

A few nice neighbour-transitive examples

- Blocks of 2-(11, 5, 2) biplane in J(11,5) with group PSL(2,11)
- Blocks of the Witt designs for Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24} and other goodies!
- Quadrics in the Higman-Sims graph with group HS i.e. a 2-(176,50,14) design
- Exactly four examples with group Co_3 and v=276, k = 6, 36, 100, 126

This classification problem comes from a reduction to problem about 2-transitive permutation groups

- Case of sporadic 2-transitive permutation groups (such as Mathieu groups, HS, Co_3)
- Finite problem solved using theory and GAP
- Collaboration with Max Neunhoeffer
- Complete list of sporadic examples [21 of them] along with their minimum distances δ

Comment on "code" interpretation arising from discussions with Max

- Since vertices in J(v,k) are k-sets
- Another natural interpretation: codeword = binary v-tuple [characteristic function of the k-set]
- Then C becomes a constant weight binary code in Hamming graph H(v,2)
- Distance between code words in H(v,2)
  = 2 x distance in J(v,k)
- Group of C in Aut(J(v,k)) contained in group of C in Aut(H(v,2)) – neighbour transitivity does not go through

Comment on “complements”

- Given code C in $J(v, k)$

  for $\gamma \subset \mathcal{V}$ write $\overline{\gamma} = \mathcal{V} \setminus \gamma$
- Define $\overline{C} = \{\overline{\gamma} \mid \gamma \in C\}$ $\quad \overline{C}_1 = \{\overline{\gamma}_1 \mid \gamma_1 \in C_1\}$
- Then $\overline{C}$ is a code in $J(v, v-k) \cong J(v, k)$ with ‘block size’ $v - k$

  $\overline{C}$ has neighbour set $\overline{C}_1$ and $\mathrm{Aut}(C) = \mathrm{Aut}(\overline{C})$

  $\overline{C}$ is neighbour-transitive $\Leftrightarrow$ C is neighbour-transitive

  so assuming $k \leq \frac{v}{2}$ really no restriction

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Comment on “geometrical” interpretation

- Regard (C, C_1) as an incidence structure. For $\gamma \in C, \gamma_1 \in C_1$, (γ, γ_1) incident $\Leftrightarrow d_J(\gamma, \gamma_1) = 1$
  - Let $\textsc{Inc}(C, C_1)$ be set of incident (γ, γ_1) with $\gamma \in C, \gamma_1 \in C_1$
  - C is called incidence-transitive if $\mathrm{Aut}(C)$ is transitive on $\textsc{Inc}(C, C_1)$
- if $\delta(C) \geq 2$, then C incidence-trans $\Rightarrow$ C neighbour-trans
- if $\delta(C) \geq 3$, then C incidence-trans $\Leftrightarrow$ C neighbour-trans

More neighbour-transitive examples

- Fix $\mathcal{U} \subset \mathcal{V}$, $k \leq |\mathcal{U}|$, and write $\overline{\mathcal{U}} = \mathcal{V} \setminus \mathcal{U}$.
- Let $C = C(\mathcal{U}, k) := \binom{\mathcal{U}}{k}$, so $\mathrm{Aut}(C) = \mathrm{Sym}(\mathcal{U}) \times \mathrm{Sym}(\overline{\mathcal{U}})$
  - $C_1 = \{\gamma_1 \in \binom{\mathcal{V}}{k} \mid |\gamma_1 \cap \mathcal{U}| = k - 1\}$
  - C is neighbour-transitive
- Also if $|\mathcal{U}| < k$ then $C(\mathcal{U}, k) := \{$ all k-subsets containing $\mathcal{U}\}$ also neighbour-transitive

Comments on these [work with Bob Liebler]

Theorem These are all the neighbour-transitive examples with Aut(C) intransitive on V

1993, 2003 Meyerowitz classified all completely regular codes in J(v,k) of “strength zero” – they are precisely the intransitive neighbour-transitive examples!

Another set of known examples:

1994 Bill Martin "groupwise complete designs"

Partition $U = \{ U_1, U_2, \dots, U_b\}$ of V with $|U_i| = a$, and $b > 3$

Choose c with $1 < c$ at most $b/2$ and $k = bc$

Define C = all unions of c parts of U code in $J(v,k)$

1994 Bill determined which groupwise complete designs are completely regular codes in $J(v,k)$

Showed: if C in $J(v,k)$ completely regular and C is a 1-design but not a 2-design then C is a groupwise complete design

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From now on this is work with Bob Liebler

Group of groupwise complete design C: Stab(U) = S_a wr S_b

Stab(U): always neighbour transitive on C

Bob and I: generalised g.c.d. construction – take any code C_0 in J(b,c) based on the b-set U and define

$$C = \{ \text{ union of all parts in } x \mid x \text{ in } C_0 \}$$

Theorem C is neighbour-transitive if C_0 is "strongly incidence transitive"

From now on this is work with Bob Liebler

Bob and I: take any code C_0 in J(b,c) based on the b-set U and define

$$C = \{ \text{ union of all parts in } x \mid x \text{ in } C_0 \}$$

5 more explicit constructions based on partition U of V
[a couple are completely transitive – discovered with Chris Godsil]

Theorem If C is neighbour-transitive in J(v,k) and Aut(C) is imprimitive on V (preserves some partition U) then C is one of these examples

From now on assume Aut(C) primitive on V

Theorem

Given $\mathrm{Aut}(C)$ is primitive on $\mathcal{V}$ and neighbour trans on C
(a) $\delta(C) \geq 3$ implies $\mathrm{Aut}(C)$ is 2-transitive on $\mathcal{V}$
(b) $\delta(C) \geq 2$ and C inc-trans implies $\mathrm{Aut}(C)$ is 2-trans on $\mathcal{V}$

Idea of Proof Strategy

- Let $\mathrm{u} \in \mathcal{V}$ and $\Delta(\mathrm{u}) = \cap\{\gamma \in C | \mathrm{u} \in \gamma\}$.
- Prove $\Delta(\mathrm{u})$ block of imprimitivity for $\mathrm{Aut}(C)$. Conclude $\Delta(\mathrm{u}) = \{\mathrm{u}\}$
- 'Use incidence-transitivity' to prove $\mathrm{Aut}(C)$ 2-trans.

Opens possibility to classify incidence-transitive codes with $\delta(C) \geq 2$, using classification of 2-transitive groups.

This is the 2-transitive reduction for neighbour transitive codes in J(v, k)

Still not complete – significant questions remain!

Already showed you sporadic case – complete classification

This leaves essentially four cases

- Projective
- Affine
- Rank 1 groups (Sz, Ree, unitary)
- Symplectic

Projective groups: G = Aut(C)

$$\mathrm{PSL}(n,q) \le G \le \mathrm{P\Gamma L}(n,q) \text{ with } \mathcal{V} = \mathrm{PG}(n-1,q)$$

Example $n = 2$ $\quad q = q_0^2$, $\gamma =$ Baer sub-line $\mathrm{PG}(1, q_0)$, $C = \gamma^G$

Theorem for $n = 2$

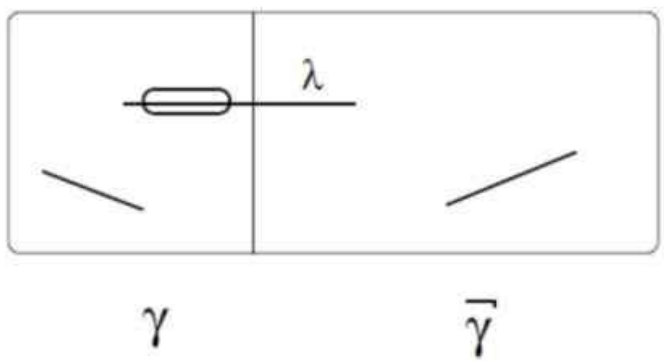
These are the only examples with $n = 2$, $k \ge 3$

Examples $n \ge 3$ $\quad \gamma =$ a subspace of $\mathrm{PG}(n-1,q)$, $C = \gamma^G$

Projective groups: G = Aut(C)

Theorem for $n \geq 3$: Either C as in the examples or

for each line λ of $\mathrm{PG}(n-1, q)$, $|\lambda \cap \gamma| \in \{0, x, q+1\}$ for x fixed, and moreover, either $x = 2$, or $q = q_0^2$ and $x = q_0 + 1$



Examples with $x = 2$

$C =$ the complements of r-subspaces in $\mathrm{PG}(n, 2)$ - any others?
Any with $q = q_0^2, x = q_0 + 1$? – Baer sub-geoms are not

Affine actions: G = Aut(C) in AΓL(n,q)

$$G \leq \mathrm{A\Gamma L}(n,q) \text{ with } \mathcal{V} = \mathrm{AG}(n,q)$$

Example $n = 1$ $\quad q = 4,\ \mathcal{V} = \mathbb{F}_4,\ C = \{\gamma = \{0,1\} = \mathbb{F}_2\},$

$C_1 = \{\{0,\xi\},\{0,\xi+1\},\{1,\xi\},\{1,\xi+1\}\}$

G generated by $t_1 : x \mapsto x+1$ and $\sigma : x \mapsto x^2$

Theorem for $n = 1$

This is the only example with $n = 1$ with G incidence-transitive

Proof is surprisingly difficult

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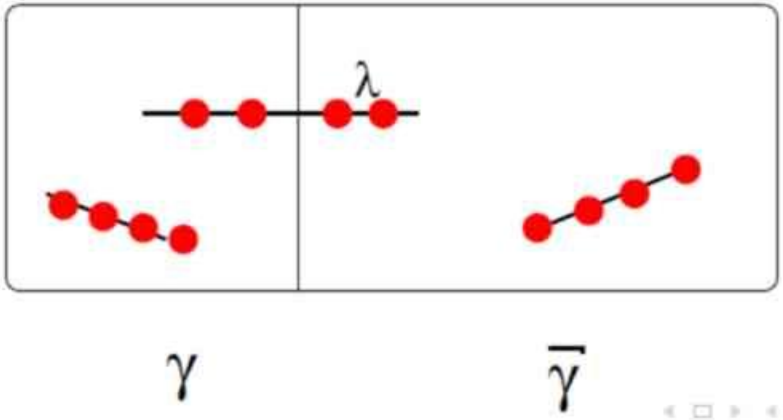
Affine actions: G = Aut(C) in AΓL(n,q)

Examples $n \geq 2$ $\quad \gamma =$ a subspace of $\mathrm{AG}(n, q)$, $C = \gamma^G$

Examples $n \geq 2$ $\quad q = 4$, $\gamma = \mathrm{AG}(n, 2)$ a Baer sub-geometry, $C = \gamma^G$

Theorem for $n \geq 2$: Either C as in the examples or

$q = 4$ and, for each line λ of $\mathrm{AG}(n, q)$, $|\lambda \cap \gamma| \in \{0, 2, 4\}$



Rank 1 groups: Ree, Sz, Unitary

- Sz(q) on V, $|V|=q^2+1$, $q=2^{2a+1}$ no examples

- Ree(q) on V, $|V|=q^3+1$, $q=3^{2a+1}$ no examples

- PSU(3, q) on V, $|V|=q^3+1$, examples from unital $k=q+1$

Symplectic groups G=Sp(2n,2) with |V|= $2^{n-1}(2^n+1)$ or $2^{n-1}(2^n-1)$

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$G = \mathrm{Sp}(n, 2)$ with $\mathcal{V} = \mathcal{Q}^\varepsilon$

set of non-degenerate quadratic forms of type $\varepsilon = \pm$ that polarise to the symplectic form preserved by G

Let $V(2n, 2)$ be the underlying space for natural G-action

Example $\mathcal{U}$ nonsingular so $V(2n, 2) = \mathcal{U} \perp \mathcal{U}^\perp$.

$\gamma = \{$ all forms $\varphi \in \mathcal{Q}^\varepsilon$ such that $\varphi|_\mathcal{U}, \varphi|_{\mathcal{U}^\perp}$ have types $\varepsilon_\mathcal{U}, \varepsilon_{\mathcal{U}^\perp}$ where $\varepsilon_\mathcal{U}\varepsilon_{\mathcal{U}^\perp} = \varepsilon \}$

So that's it:

- Several open problems
- Affine and projective cases: Very symmetrical geometrical configurations - do they exist?
- Symplectic groups: huge analytical issues – orthogonal model
- Ways forward?
  - Computation for small n
  - Use geometry and algebra for better understanding
  - Use knowledge of maximal subgroups to restrict possibilities