Graph Detachments

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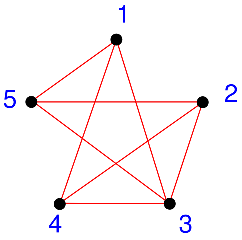
University of Dundee, U.K.

35ACCMCC, Monash University, 6 Dec 2011

Detachments

Definition

A **detachment** of a graph *G* is formed by splitting each vertex into one or more subvertices, and sharing the incident edges arbitrarily among the subvertices.



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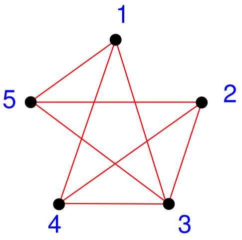

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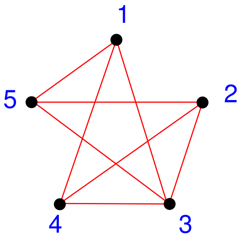

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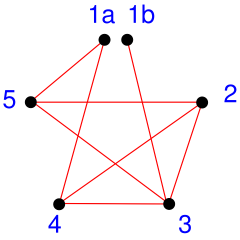

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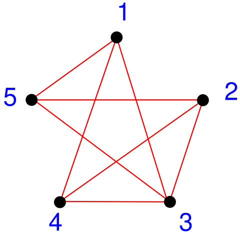

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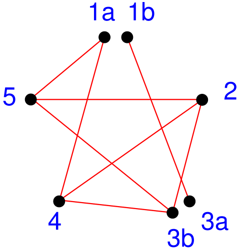

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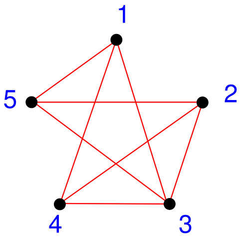

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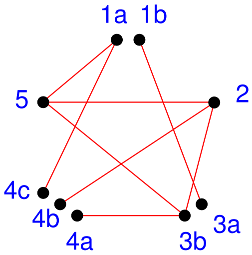

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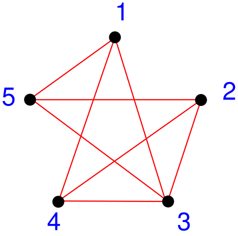

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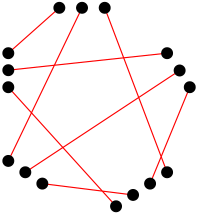

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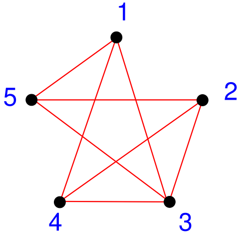

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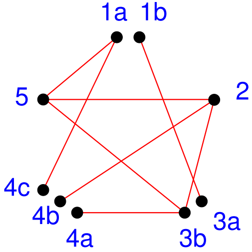

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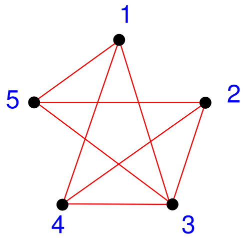

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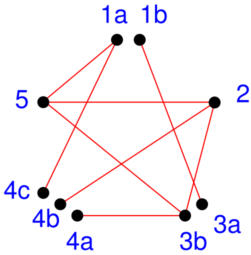

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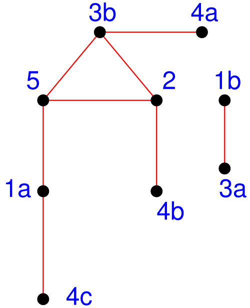

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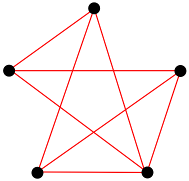

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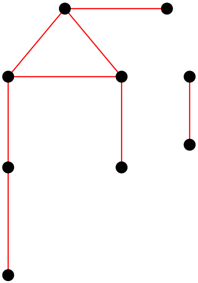

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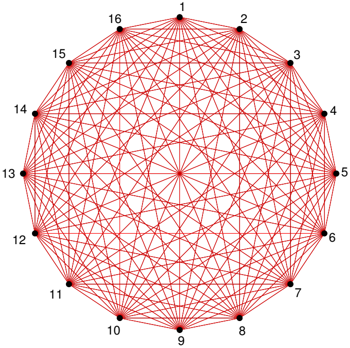

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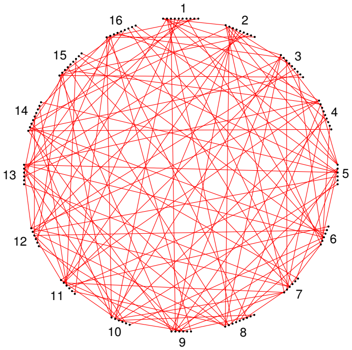

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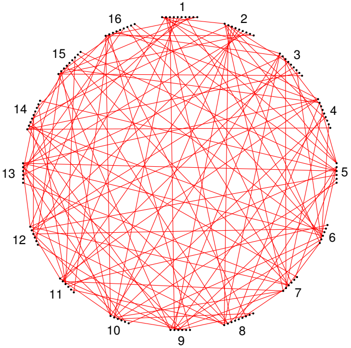
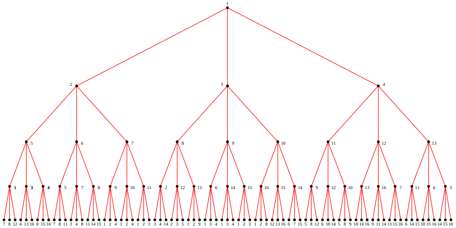
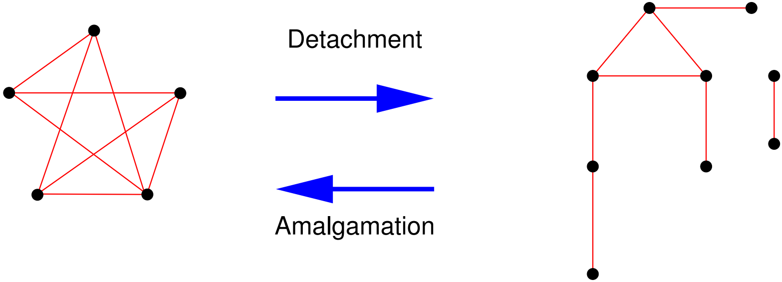
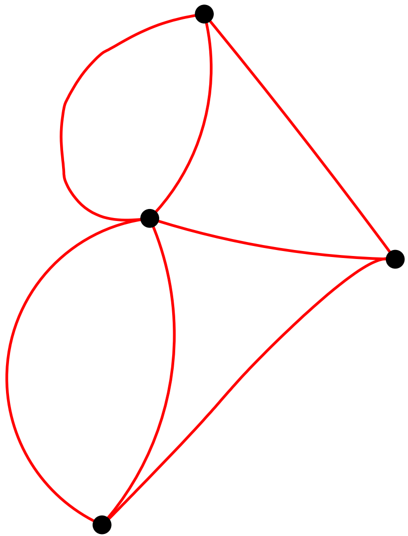




Detachment ↔ Amalgamation



Euler's Königsberg Bridges Problem (1736)



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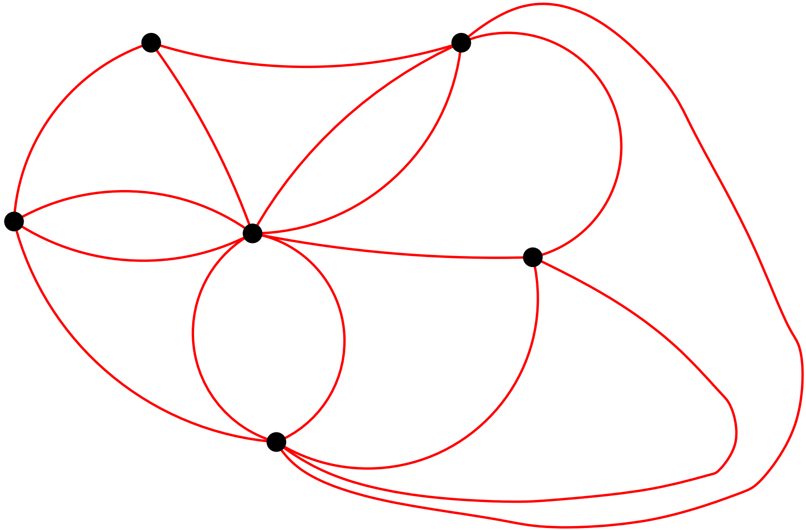

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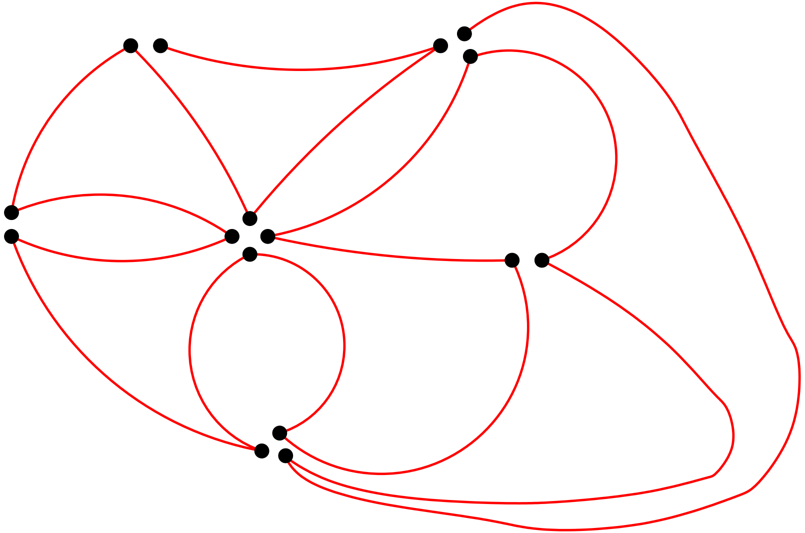


Euler's Königsberg Bridges Problem (1736)

Question

When does a graph G have an Eulerian path/trail, i.e. a path which uses each edge exactly once?

Equivalently (detachment formulation)

When is the path with $|E(G)|$ edges a detachment of G?

Answer:

Precisely if G is connected, and has at most 2 vertices of odd degree.

Similarly, the cycle $C_{|E(G)|}$ is a detachment of G if and only if G is connected and every vertex has even degree.

Detachments

General Question

Given two graphs G and H, when is H a detachment of G?

When H is a path or a cycle, there are simple necessary and sufficient conditions.

Problem

Can we get necessary and sufficient conditions when H is more complicated, e.g. maximum degree 3, 4, etc.?

Necessary condition 0

Given two graphs G and H, when is H a detachment of G?

Necessary condition 0

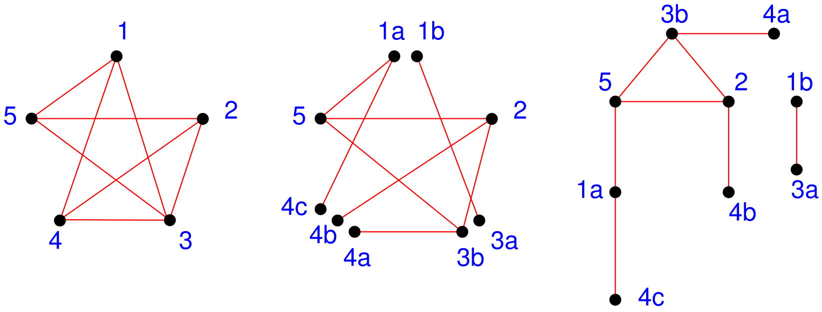
G and H must have the same number of edges.

Necessary condition 1

Given two graphs *G* and *H*, when is *H* a detachment of *G*?

Necessary condition 1

The vertices of *H* can be partitioned into sets so that the sums of the vertex degrees in each set give the degrees of the vertices of *G*.



G | Detachment of *G* | *H*

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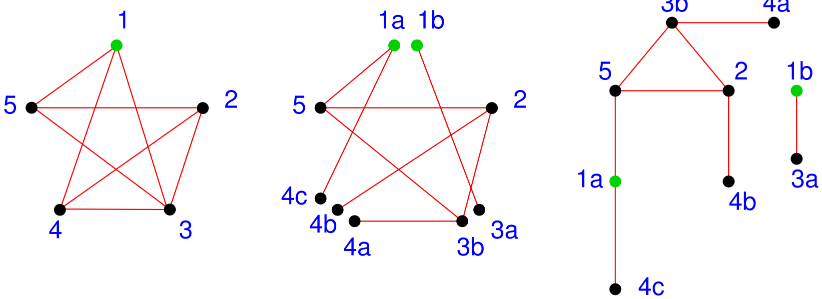

G — Detachment of *G* — *H*

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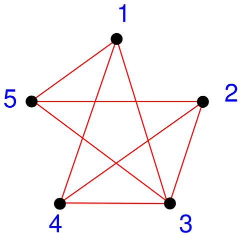

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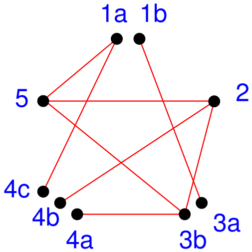
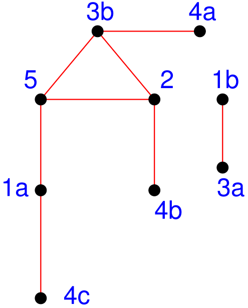


Detachment of G



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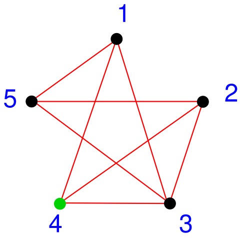

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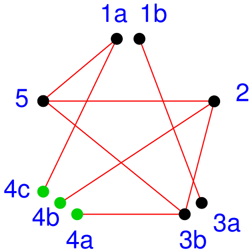

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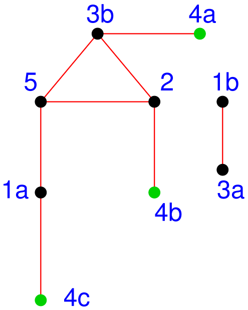

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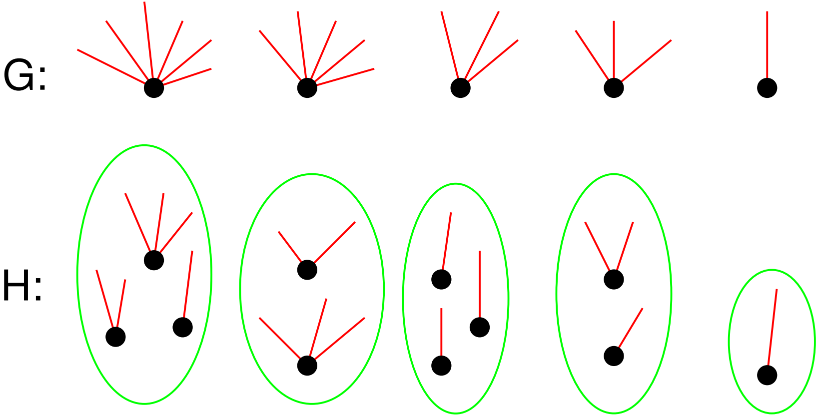


Computational problem

How difficult is it to determine if one graph is a detachment of another?

DETACHMENT

Instance	Two graphs G, H.
Question	Is H a detachment of G?

Since it is essential for G and H to have the same number of edges, assume this from now on.

Detachment into *two* cycles

The case when the graph H is a cycle is easy. However, when G is 4-regular, and H consists of two cycles, the problem is NP-complete.

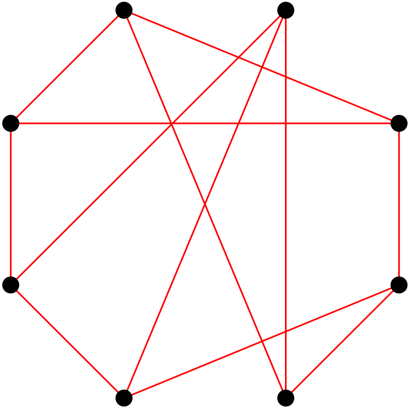
Proof

By reduction from HAMILTONIAN CIRCUIT for cubic graphs.

Detachment into *two* cycles

HAMILTONIAN CIRCUIT for cubic graphs

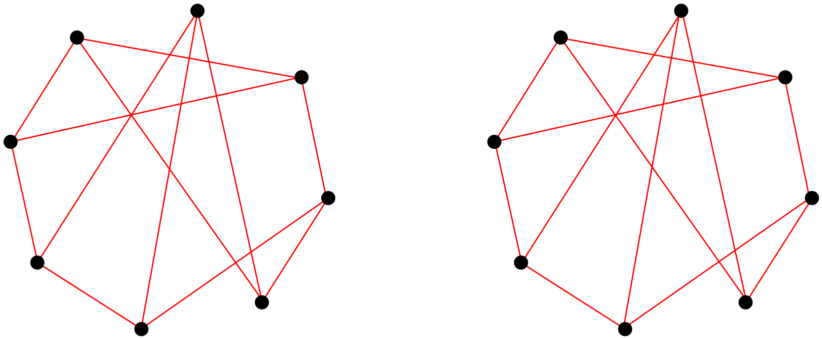
n vertices



Does G have a Hamiltonian Circuit?

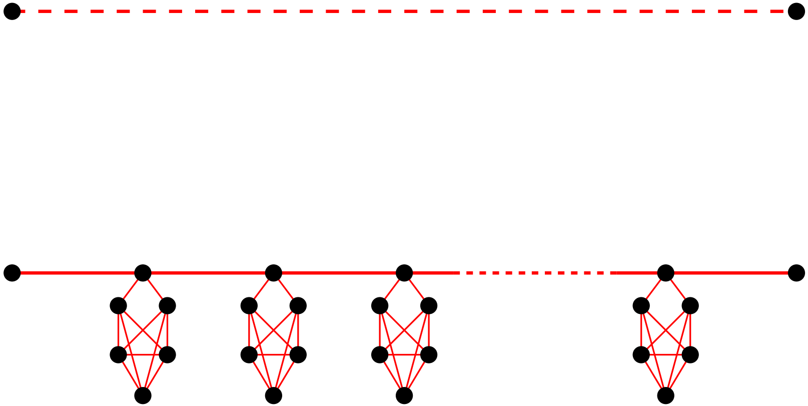
Detachment into *two* cycles

G



H : two cycles, of lengths n and $m - n$

Detachment into *two* cycles



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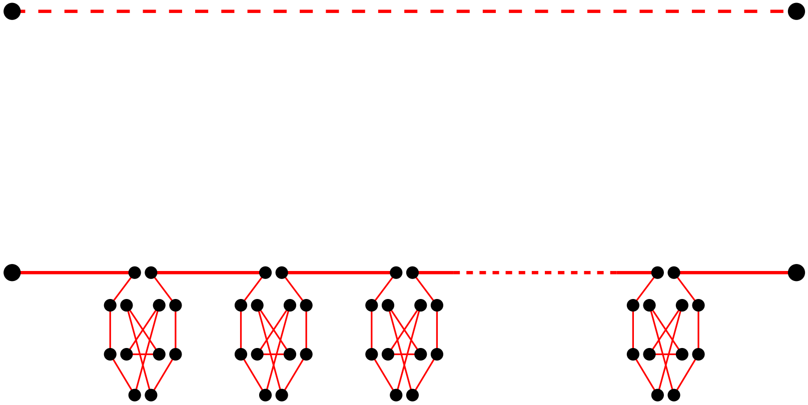

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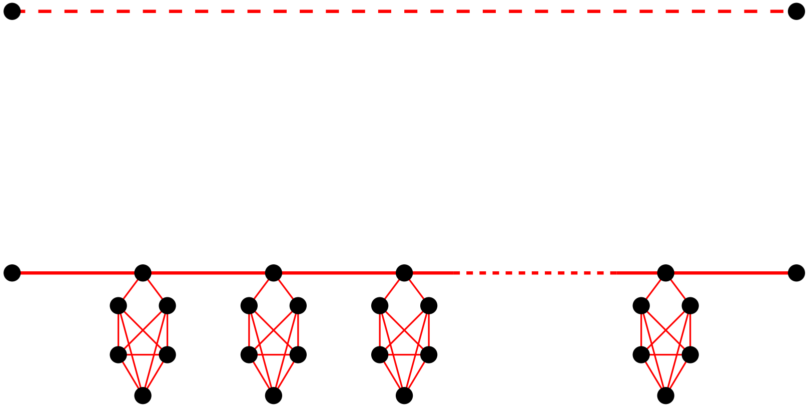

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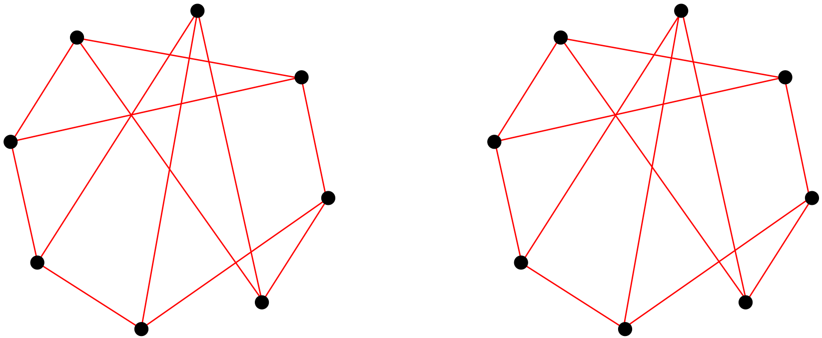

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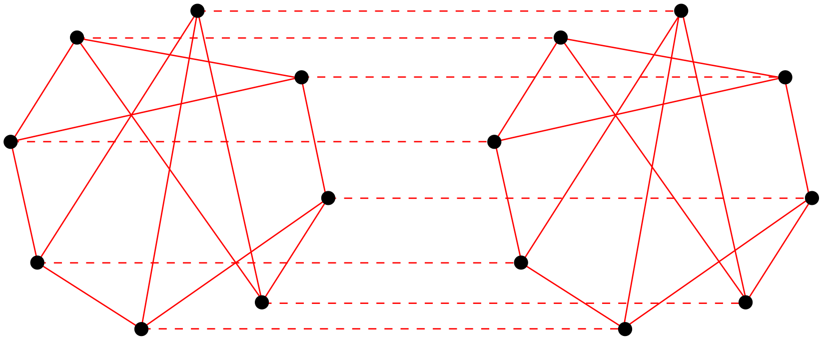

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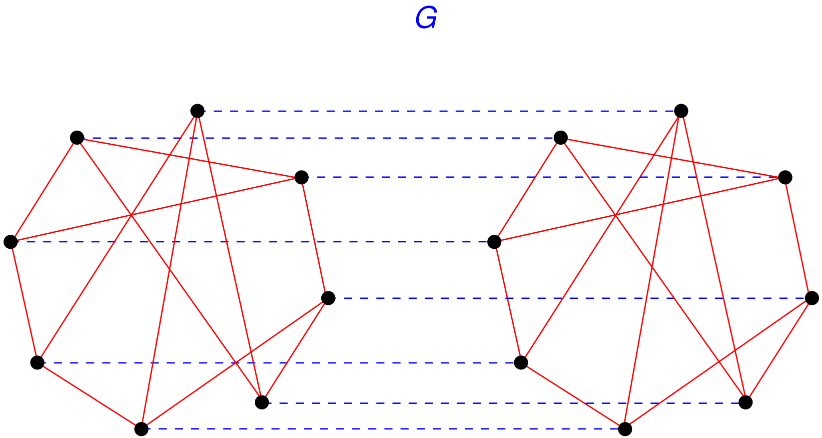

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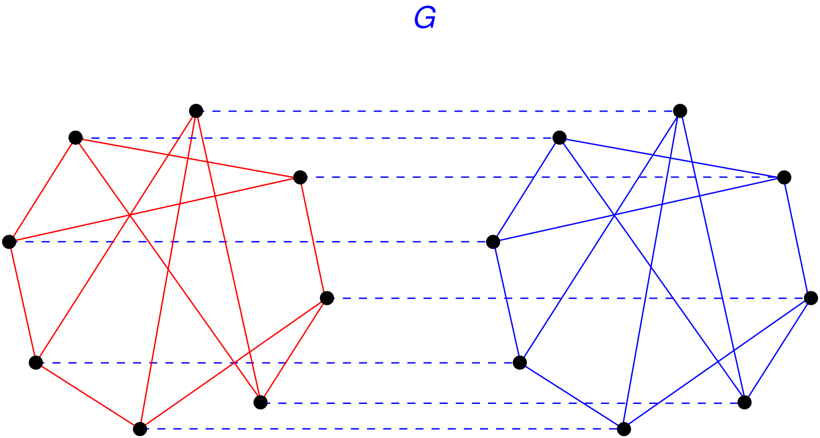

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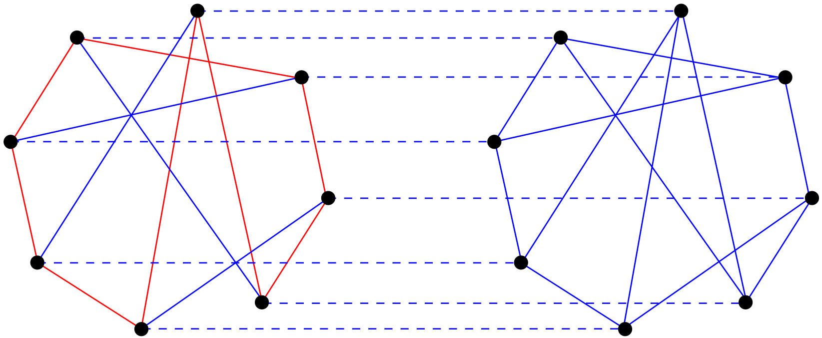

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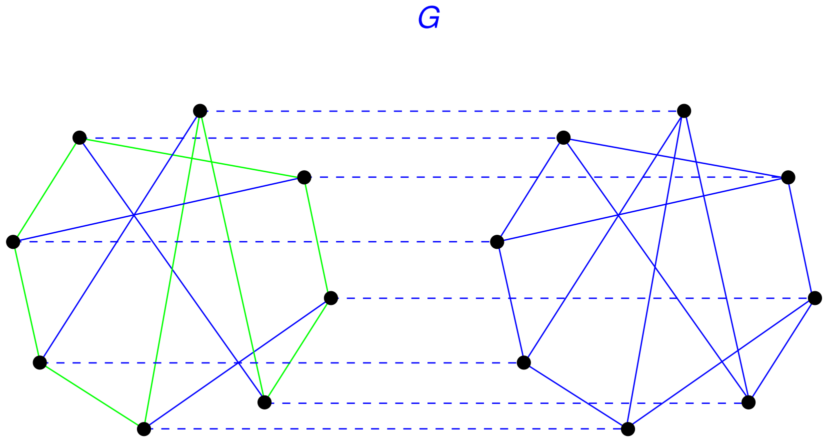
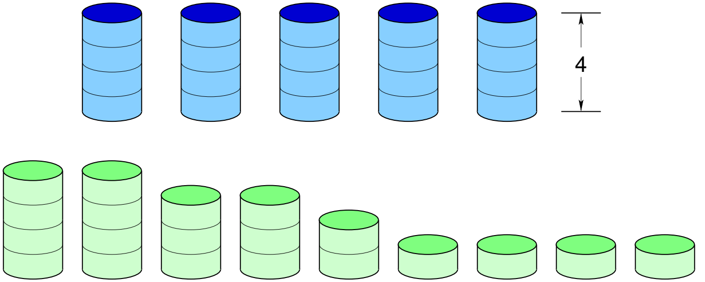


H : two cycles, of lengths n and $m - n$

Detachment of Linear Forest is NP-complete

Bin Packing: 5 BINS, each of capacity 4



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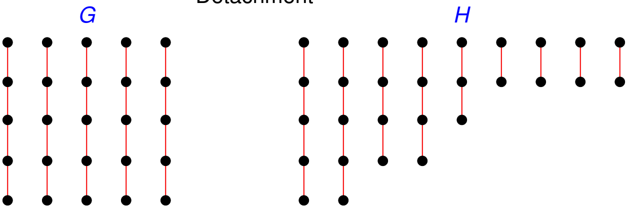

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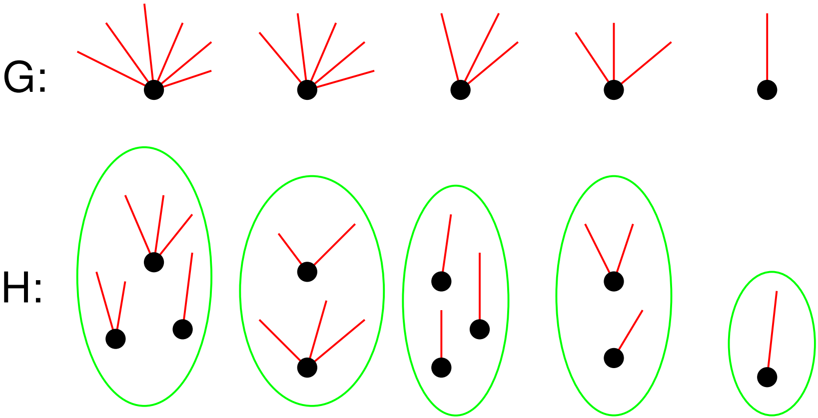


Detachments of complete graphs

Now consider the special case when the graph G is a complete graph K_n. This turns out to be a bit more tractable.

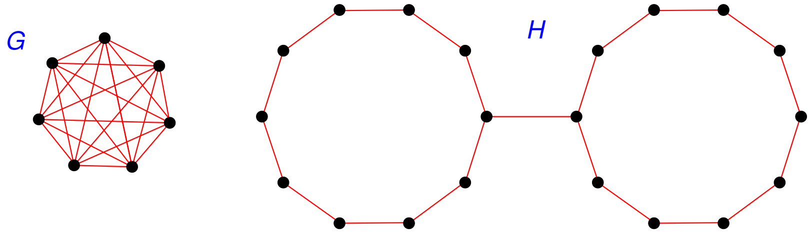
Necessary condition 1

The vertices of H can be partitioned into sets so that the sums of the degrees in each set is $n - 1$.

Sufficient when G is complete and

1. H is a collection of paths (Georges, 1995).
2. H is a collection of cycles (Balister, 1995).
3. H is a (large) bounded degree tree (E, 1996).
4. H consists of copies of some fixed graph (Wilson, 1975).

Necessary condition 1 is insufficent in general



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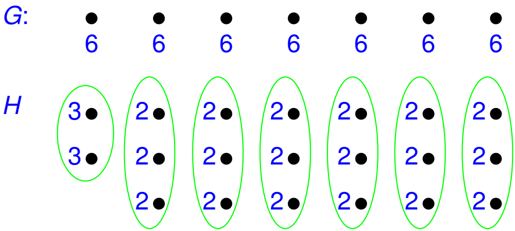


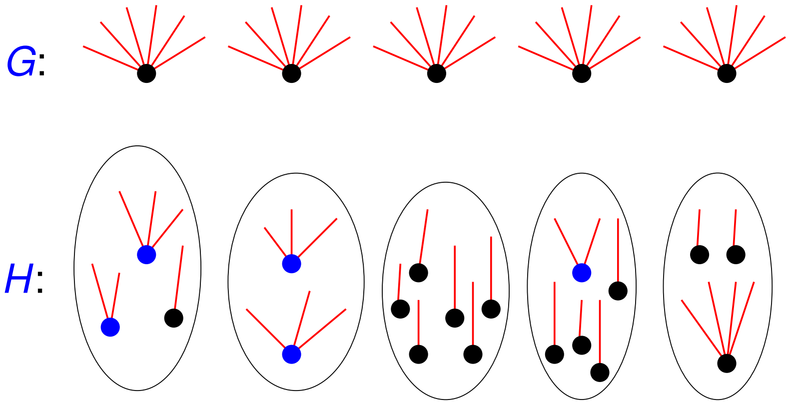
Necessary condition 2

Consider graphs of maximum degree d.
A vertex degree i is rare if the number of vertices of H of degree i is non-zero but less than a constant R_d.

Necessary condition 2

The vertices of H can be partitioned into sets with degree sum $n - 1$,
and
the edges incident with rare degree vertices do not form any loops or multiple edges between the sets.

Necessary condition 2



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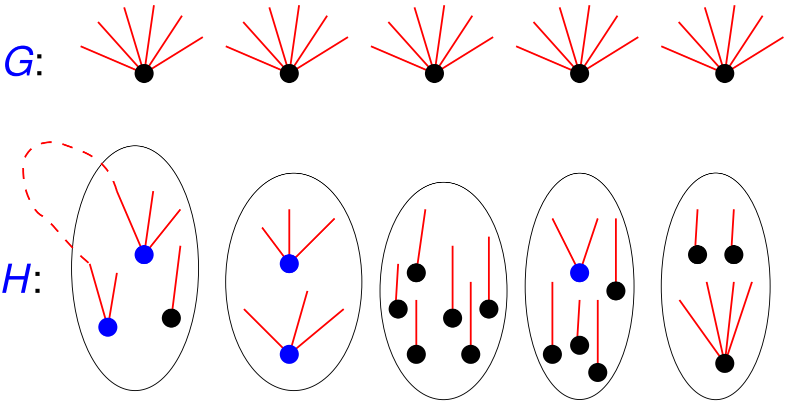

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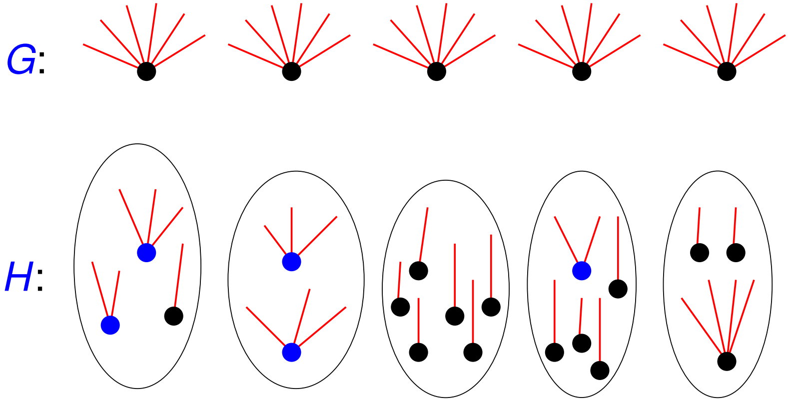

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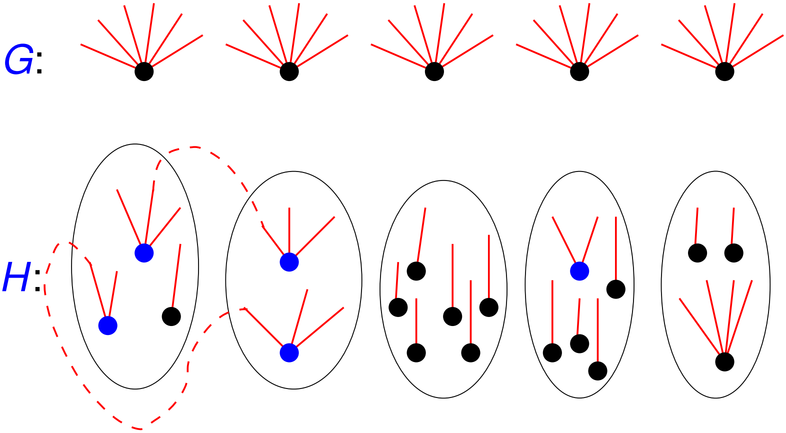


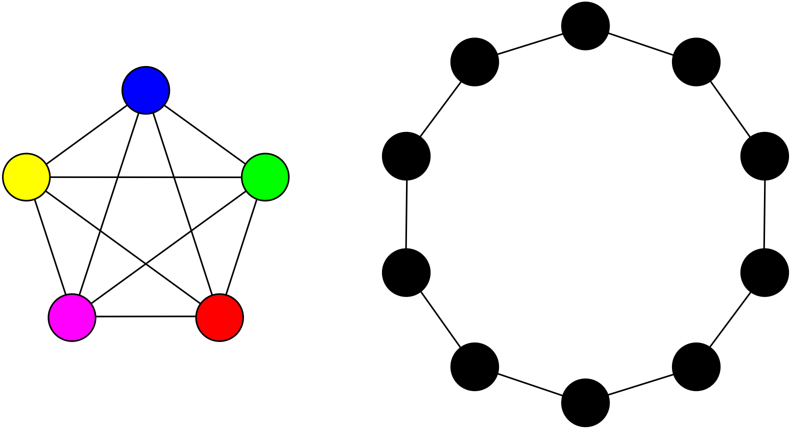
Detachments of complete graphs

Theorem

Let d be a integer. Let H be a triangle-free graph of maximum degree d, with $\binom{n}{2}$ edges. Then (for large enough n) H is a detachment of K_n (i.e. K_n is an amalgamation of H) if and only if:

1. The vertices of H can be partitioned into sets $V_1, \ldots, V_n$ with degree sum $n-1$;
2. the edges incident with rare degree vertices do not form any loops or multiple edges between the sets.

Detachments and Exact Colourings



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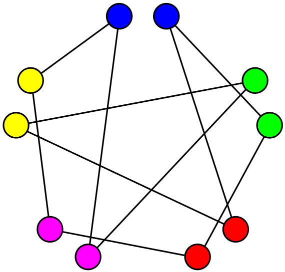
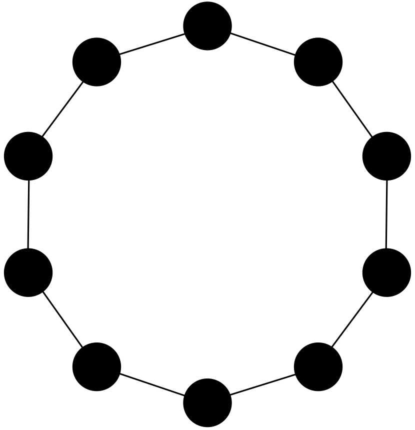
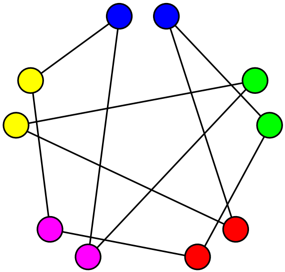
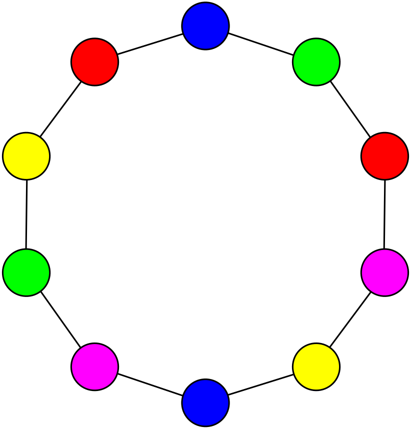




Detachments and Exact Colourings





Exact colouring with 5 colours

Detachments and Exact Colourings

Exact colouring

An exact colouring of a graph is a proper vertex colouring such that every pair of colours appears on exactly one edge.

A graph H has an exact colouring with n colours if and only if H is a detachment of K_n.

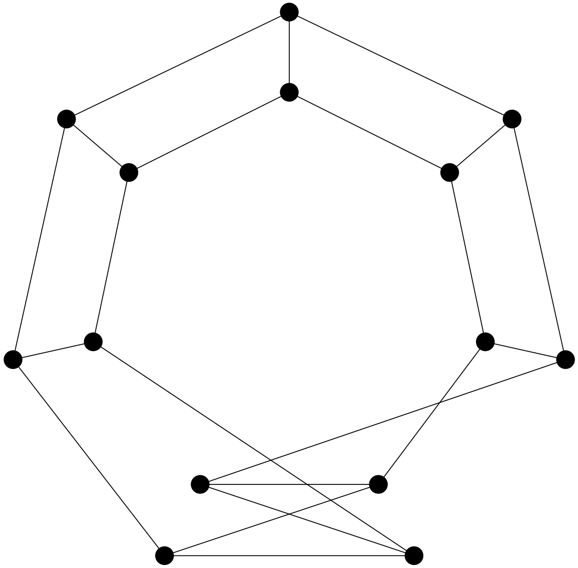
Detachments of complete graphs

Theorem

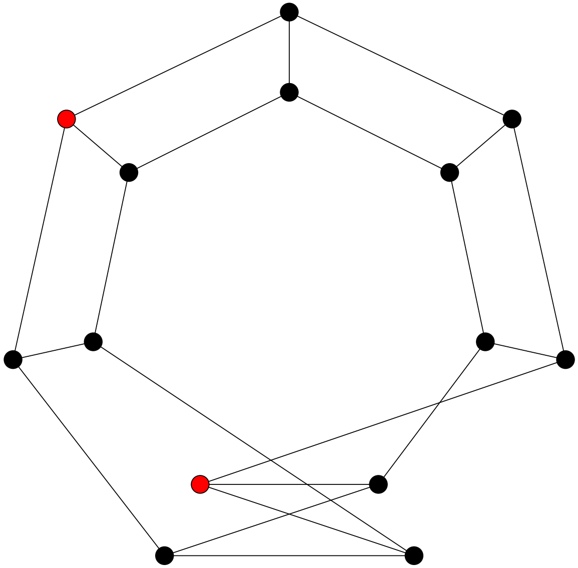
Let d be a integer. Let H be a triangle-free graph of maximum degree d, with $\binom{n}{2}$ edges. Then (for large enough n) H is a detachment of K_n (i.e. K_n is an amalgamation of H) if and only if:

1. The vertices of H can be partitioned into sets $V_1, \ldots, V_n$ with degree sum $n - 1$;
2. the edges incident with rare degree vertices do not form any loops or multiple edges between the sets.

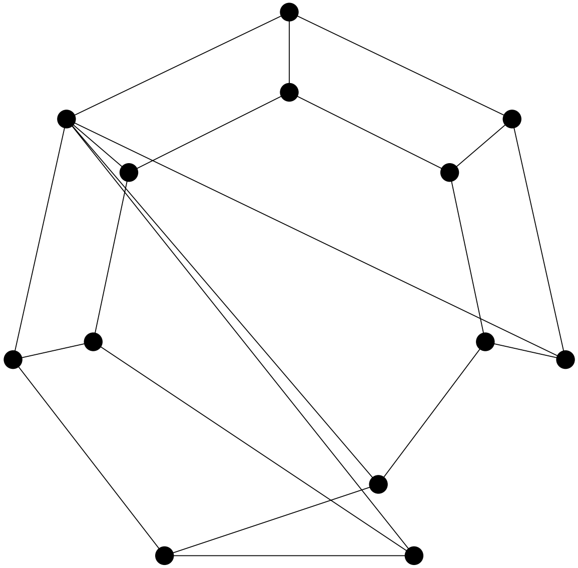
Example



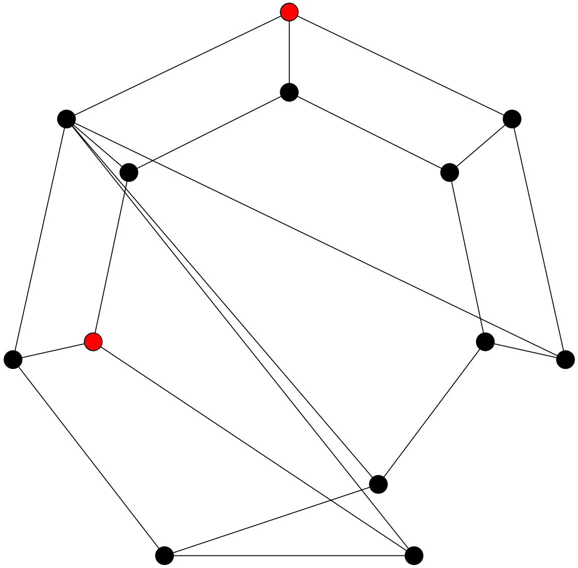
Example



Example



Example



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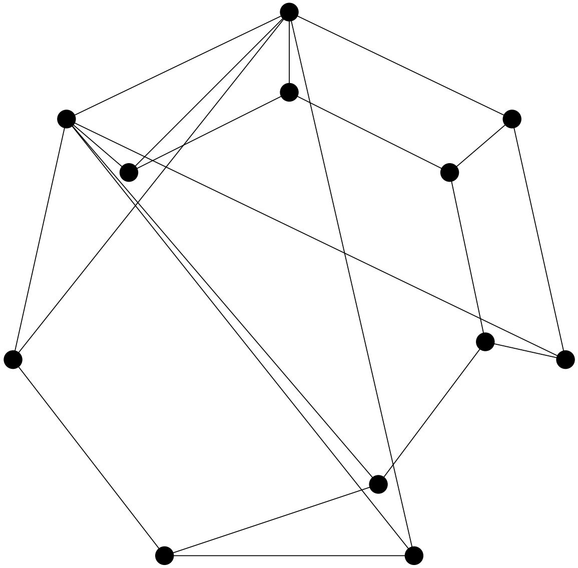
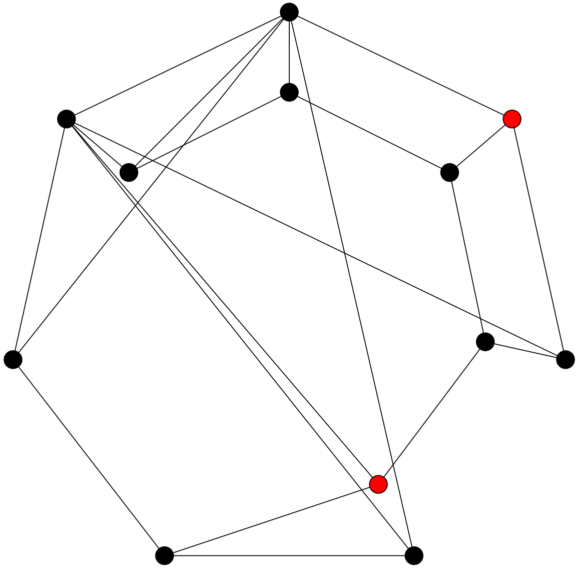


Example



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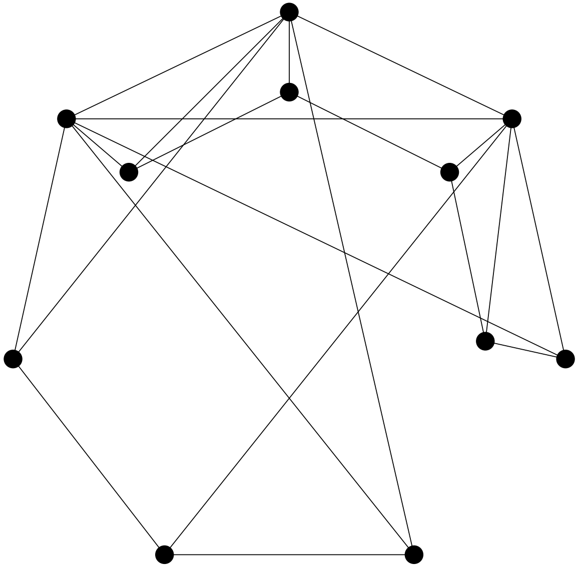

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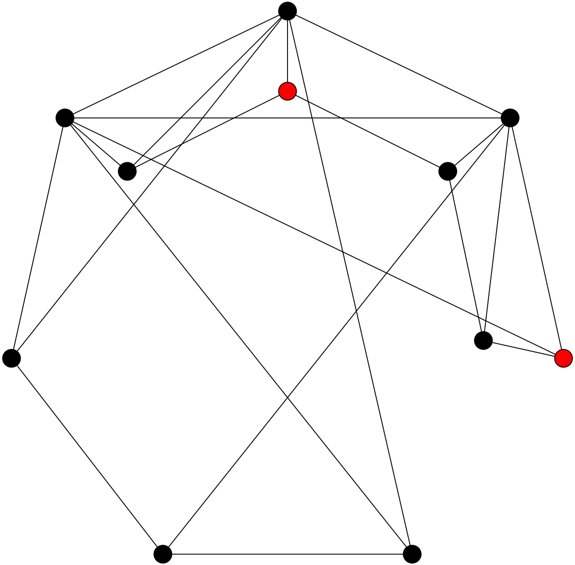
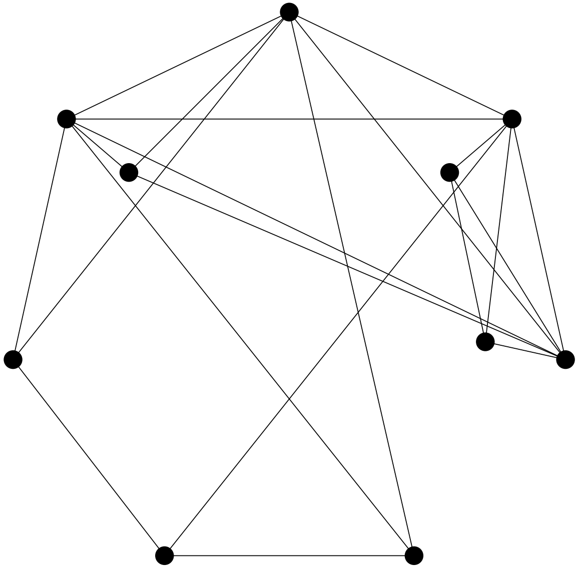
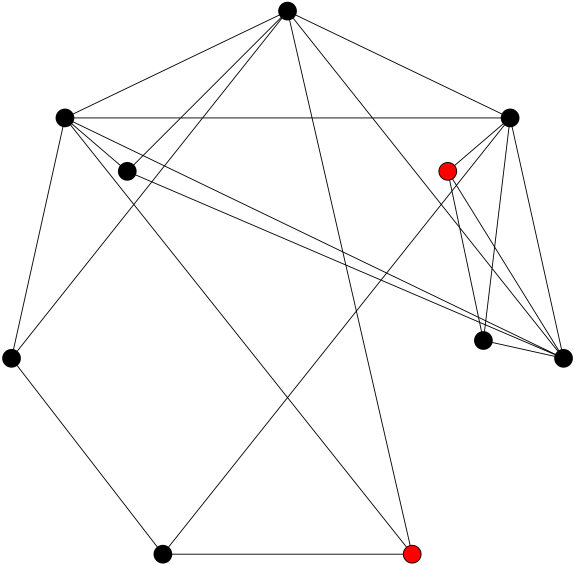


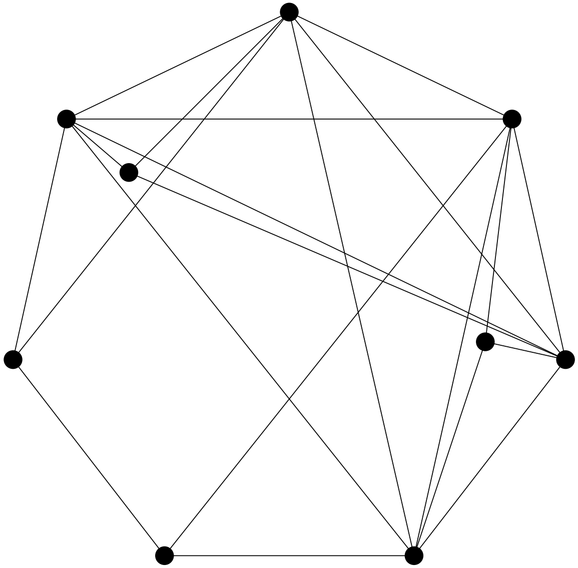
Example



Example



Example



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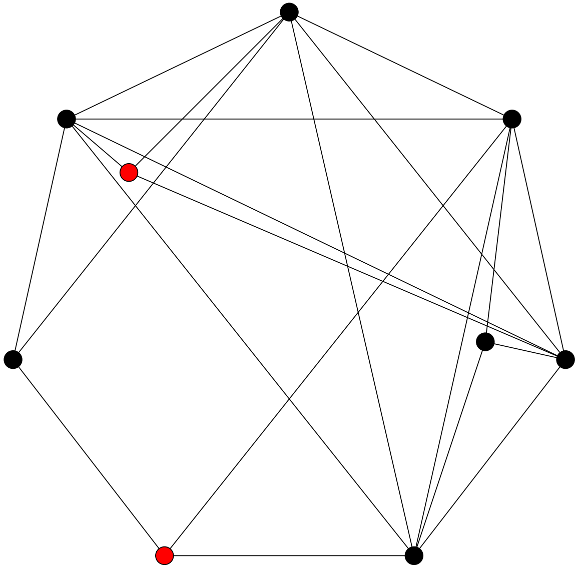
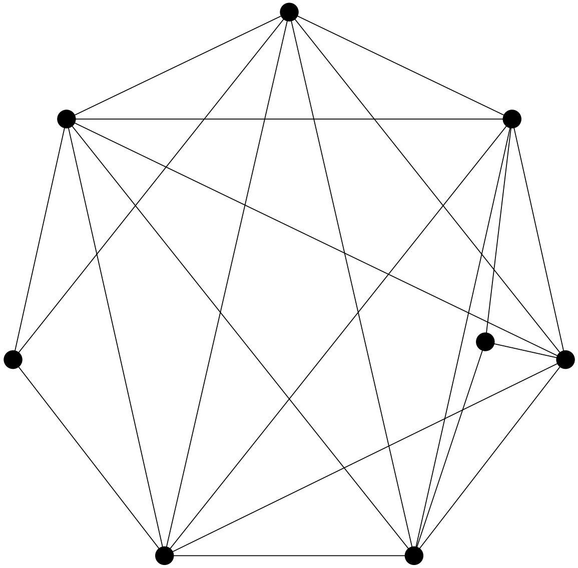
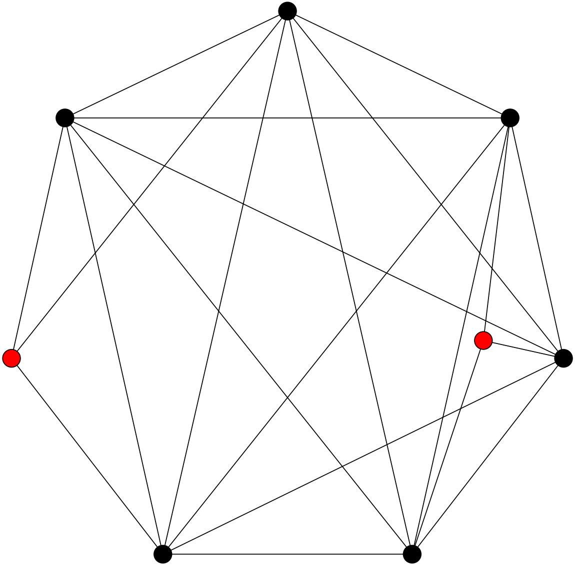


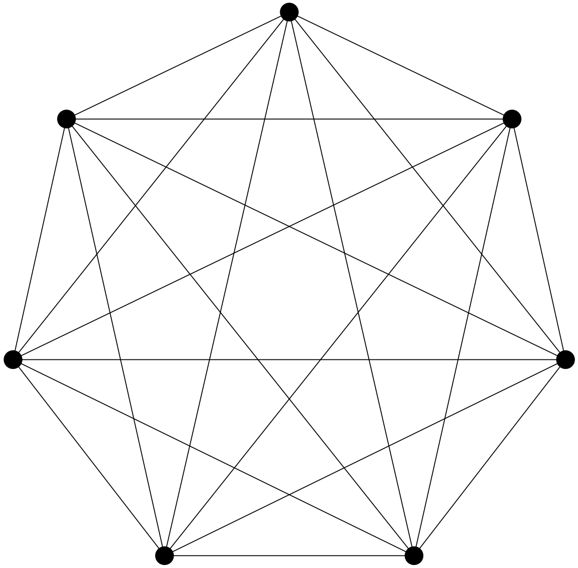
Example



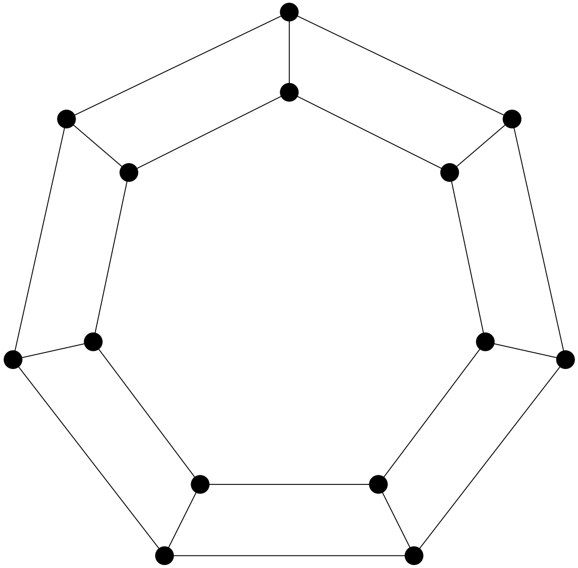
Example



Example



Non-example



Sketch of proof

Need to show that H can be amalgamated to form a complete graph, or, equivalently, that H has an exact colouring.

The amalgamation and exact colouring viewpoints are both useful because

- a partial amalgamation allows us to construct useful highly structured subgraphs;
- a partial colouring allows us to partially specify the required amalgamation/colouring without changing the graph.

In fact we will find an exact colouring of an amalgamation of H.

Sketch of proof

Six main steps:

1. Do some amalgamations to make the graph (almost) regular.
2. Do some more amalgamations to construct useful subgraphs (superleaves/special areas).
3. Colour the vertices where these subgraphs meet rest of graph.
4. Use a partly random technique to colour most of the graph.
5. Use special areas to sort out some problems.
6. Use the (highly structured) superleaves to finish the colouring.

Sketch of proof - Step 1 - Regularising

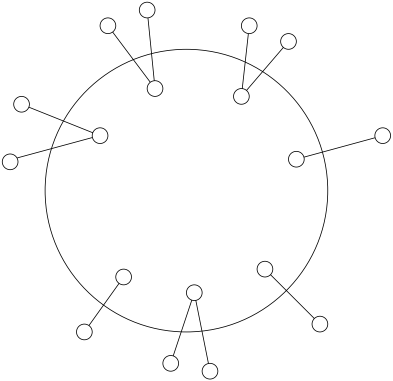
Do some amalgamations to make the graph (almost) regular.
Resulting graph has:

- n vertices of degree r'.
- All the rest of degree r.

We can do this provided the Necessary Conditions 2 are met.

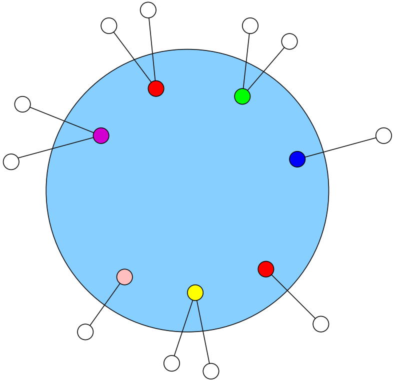
Sketch of proof - Step 2 - Superleaves

First suppose that the graph had lots of leaves.



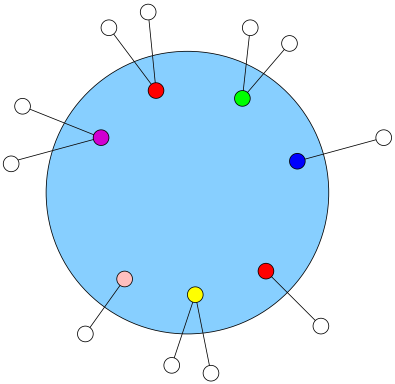
Sketch of proof - Step 2 - Superleaves

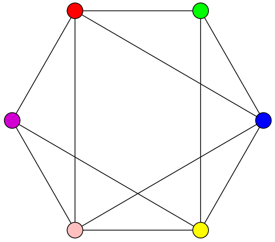
Suppose we can colour the rest of the graph, using almost all colour pairs.



Sketch of proof - Step 2 - Superleaves

We can form a graph from the colour pairs not yet used

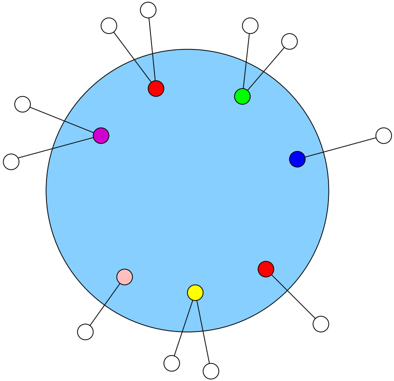


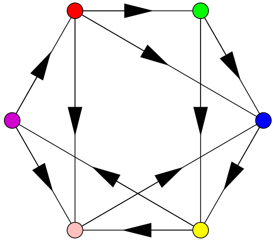


Unused colour pairs

Sketch of proof - Step 2 - Superleaves

To colour the leaves correctly, we need to orient the edges so that the outdegree of each colour equals the number of leaves attached to that colour.

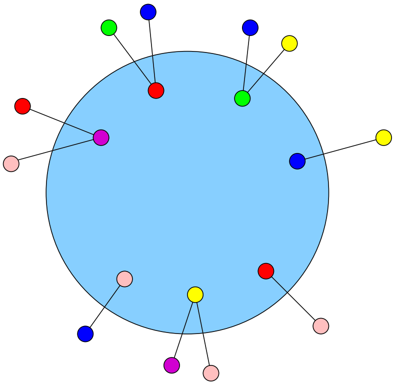


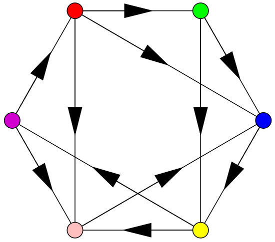


Unused colour pairs

Sketch of proof - Step 2 - Superleaves

For each colour, transfer its out-neighbours' colours to the leaves.

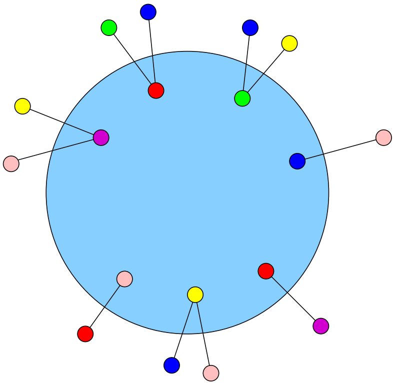


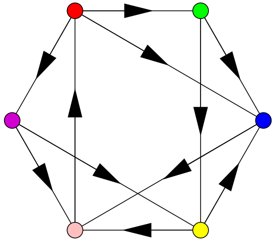


Unused colour pairs

Sketch of proof - Step 2 - Superleaves

This is easy to do as long as the degrees do not differ too much.

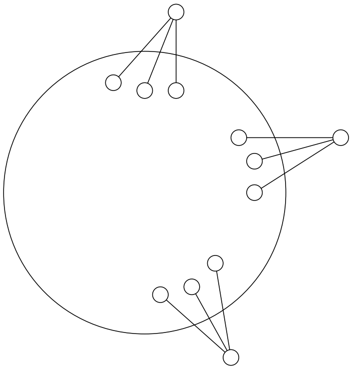




Unused colour pairs

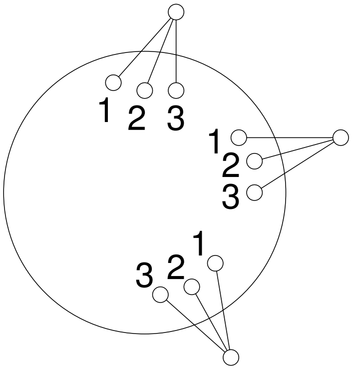
Sketch of proof - Step 2 - Superleaves

Unfortunately we don't actually have any leaves.



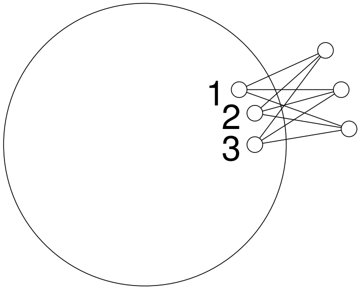
Sketch of proof - Step 2 - Superleaves

But we can make something similar. Identify (amalgamate) the vertices with same number.



Sketch of proof - Step 2 - Superleaves

This gives an complete bipartite graph attached to the graph by one part.



Sketch of proof - Step 2 - Superleaves

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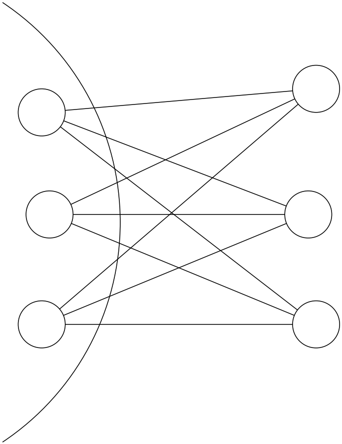
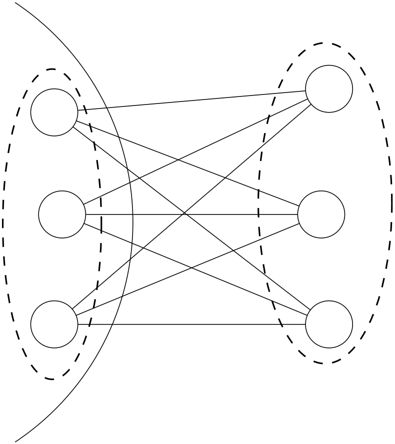


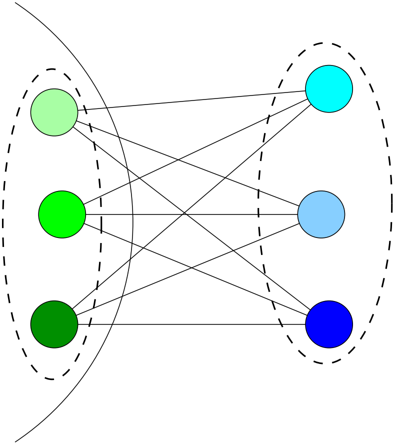
Sketch of proof - Step 2 - Superleaves

We think of each part as a supervertex . . .



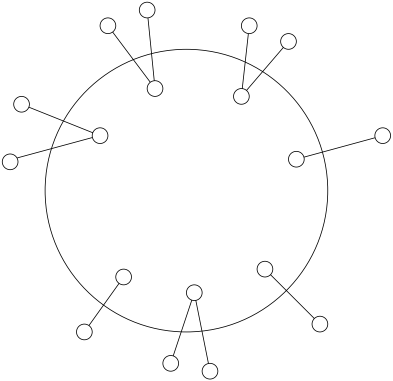
Sketch of proof - Step 2 - Superleaves

We think of each part as a supervertex . . .
. . . coloured by (shades of) a supercolour.



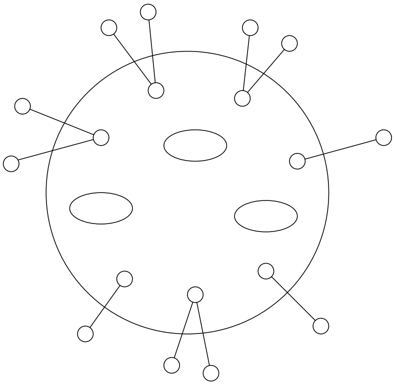
Sketch of proof - Step 2 - Superleaves

First construct the superleaves.



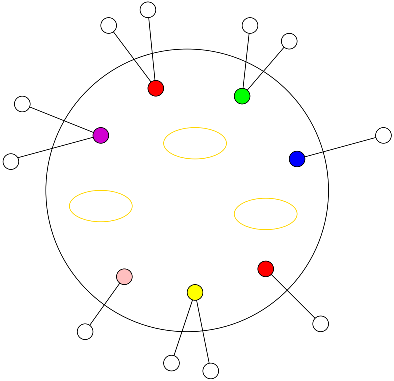
Sketch of proof - Step 2 - Superleaves

Also construct some “special” areas.



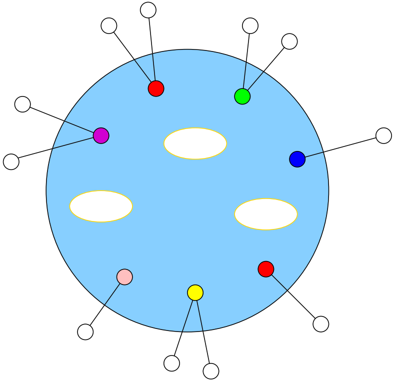
Sketch of proof - Step 3

Colour (with supercolours) the interfaces where the superleaves meet rest of graph. Also the boundary of the special areas.



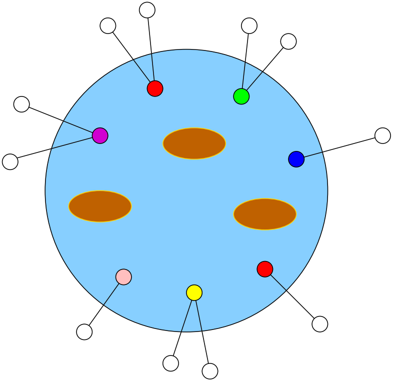
Sketch of proof - Step 4

Colour the main part of graph using semi-random technique, and new colours.



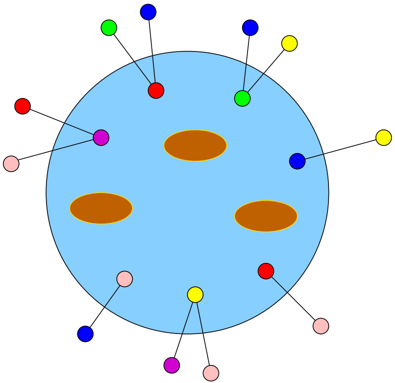
Sketch of proof - Step 5

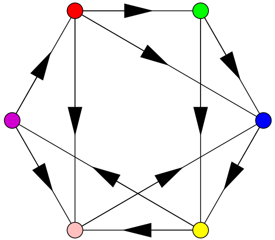
Colour the special areas, using up "difficult" colour pairs.



Sketch of proof - Step 6

Finally colour the superleaves as described earlier.





Unused colour pairs

What about directed graphs?

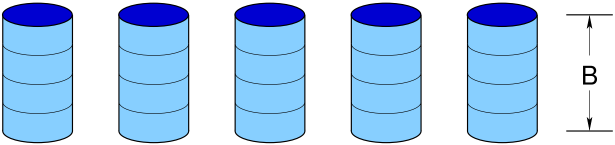
DIRECTED DETACHMENT OF COMPLETE GRAPH

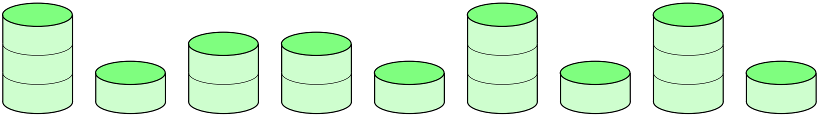
Instance	Directed graph H with maximum (in-,out-)degree d.
Question	Is H a detachment of a complete graph?

NP-complete for $d \geq 2$.

Bin Packing

Bin Packing: K BINS, each of capacity B





Set U of objects, with size $s(u)$ for each $u \in U$

Can assume B, $s(u)$ are even.

Directed detachments of complete graph

Bin packing:

K bins of size B, set U of objects, size $s(u)$ for each $u \in U$.

Detachment:

$G = K_{KB}$ (complete directed graph); H has three parts:

1. K directed cycles of length B.
2. Component H_u for each $u \in U$.
3. One undirected cycle; length so that $|E(H)| = |E(G)|$.

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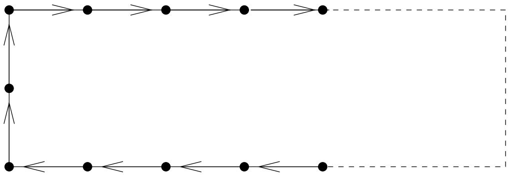

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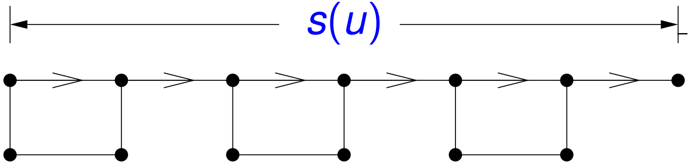

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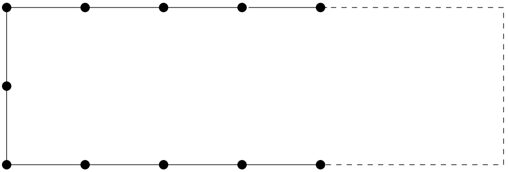
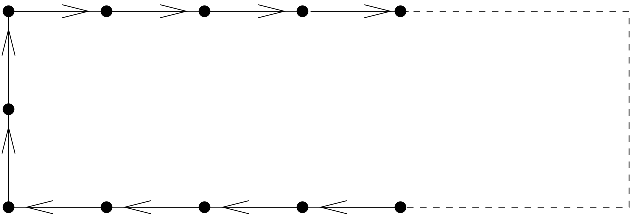


Directed detachments of complete graph

H can be amalgamated to give K_{KB} $\Leftrightarrow$ Bin Packing has a solution.

Look at degrees $(indegree, outdegree)$ of H.

1. K directed cycles of length B.
2. Component H_u for each $u \in U$.
3. One undirected cycle; length so that $|E(H)| = |E(G)|$.



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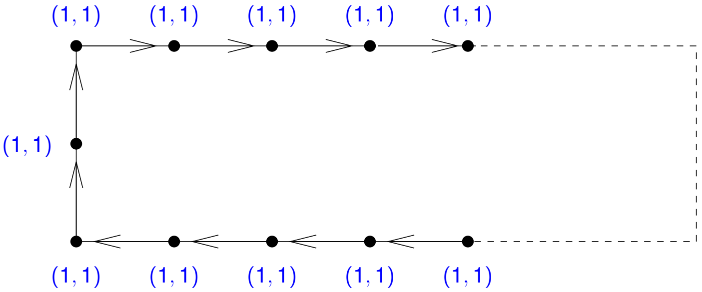

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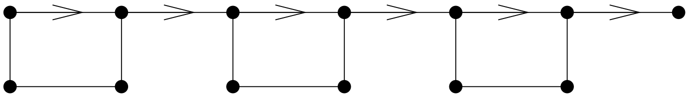

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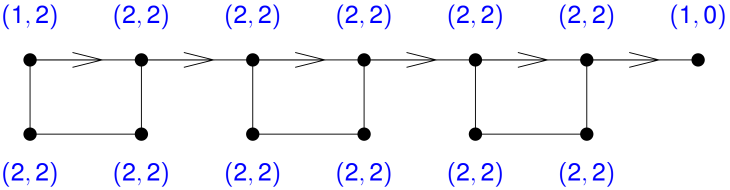

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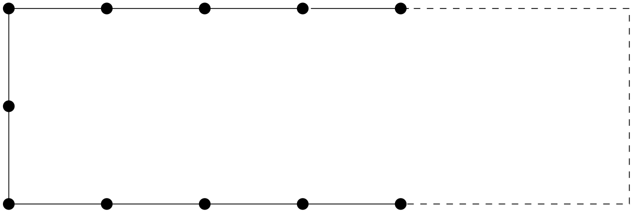

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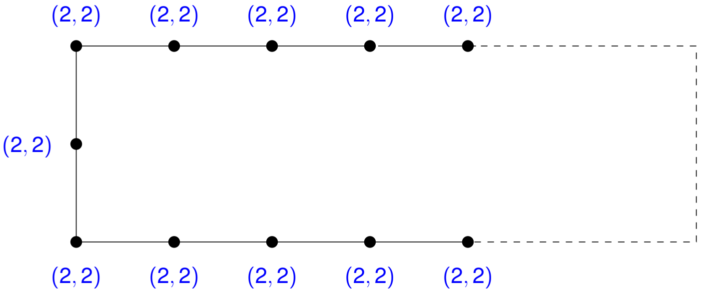
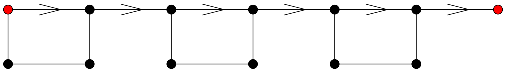


Directed detachments of complete graph

Can we amalgamate H to give K_{KB}?

1. Vertex of degree $(1, 2)$ must be paired with vertex of degree $(1, 0)$, so components H_u are linked together into directed "chains", with all degrees equal to $(2, 2)$.

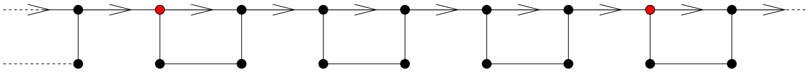


2. Degrees of K_{KB} are odd. H has only KB vertices of degree $(1, 1)$ (in the directed cycles), so these must not be amalgamated with each other.
3. So KB directed edges in cycles must be paired with the KB directed edges in the chains.

Directed detachments of complete graph

Can we amalgamate H to give K_{KB}?

1. Vertex of degree $(1, 2)$ must be paired with vertex of degree $(1, 0)$, so components H_u are linked together into directed "chains", with all degrees equal to $(2, 2)$.



2. Degrees of K_{KB} are odd. H has only KB vertices of degree $(1, 1)$ (in the directed cycles), so these must not be amalgamated with each other.
3. So KB directed edges in cycles must be paired with the KB directed edges in the chains.

Directed detachments of complete graph

Can we amalgamate H to give K_{KB}?

1. Each chain must have length B.
2. Conclusion: The components H_u must be linked to form K "cycles" of length B, so Bin Packing has a solution.
3. Converse is easy.

Directed detachments of complete graph

1. So general case with G complete, H bounded degree is NP-complete (not known in undirected case).
2. But this seems to rely on having just a few genuinely directed edges.
3. Cases with many directed edges can probably be solved in a similar way to the undirected case.
4. Natural extension of necessary conditions involves packing $(indegree, outdegree)$ pairs.

Open Problems

Consider a complete graph K_{td+1} for some t (d is fixed).

Detach each vertex (with degree td) into t vertices of degree d.
We obtain a d-regular graph H on $t(td+1)$ vertices.

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Open Problems

- More generally, can we remove the "triangle-free" condition and obtain necessary and sufficient conditions for the undirected case with G complete and H bounded degree.

  Or is this case NP-complete?
- Directed versions of these.
- Given a directed graph H (with degree bound), is H a detachment of *any* undirected graph?

  What is the complexity of this?